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Progress in Partial Differential Equations

Asymptotic Profiles, Regularity and Well-Posedness

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Editors

Progress in Partial Differential Equations

Asymptotic Profiles, Regularity and Well-Posedness

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Preface

The present volume is a collection of papers devoted to several topics in the theory of evolution equations. It originates from lectures given at the session devoted to partial differential equations at the 8th ISAAC Congress held in the period 22–27 August 2011 at the Peoples' Friendship University in Moscow, Russia. At the same time, it also includes papers originating from the ISAAC (International Society for Analysis, its Applications and Computation) Special Interest Group in Partial Differential Equations.

The papers collected in this volume are authored by participants of that meeting and by members of the special interest group. They focus on different aspects of the current research and are, in particular, centred around

- hyperbolic partial differential equations,
- p-evolution equations,
- boundary value problems,
- related optimisation problems,
- and non-linear aspects.

The aim of this volume is two-fold. On one hand it shall give an overview on a variety of problems in the field and, therefore, can serve as an introduction to some of the current research on different topics related to hyperbolic partial differential equations. On the other hand, all the papers are either full research papers presenting new results or surveys giving a broader overview of particular areas, thus also contributing to the advances in the area.

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Contents

Chapter 1
Global Existence and Energy Decay of Solutions for a Nondissipative Wave Equation with a Time-Varying Delay Term

Abbes Benaissa and Salim A. Messaoudi

Abstract We consider the energy decay for a nondissipative wave equation in a bounded domain with a time-varying delay term in the internal feedback. We use an approach introduced by Guesmia which leads to decay estimates (known in the dissipative case) when the integral inequalities method due to Haraux-Komornik (Haraux in Nonlinear Partial Differential Equations and Their Applications. Collège de France seminar, Vol. VII (Paris, 1983–1984), pp. 161–179, 1985; Komornik in Exact Controllability and Stabilization: The Multiplier Method, 1994) cannot be applied due to the lack of dissipativity. First, we study the stability of a nonlinear wave equation of the form

$$\begin{aligned} &u_{tt}(x,t)-\Delta_x u(x,t)+\mu_1\sigma(t)u_t(x,t)+\mu_2\sigma(t)u_t\big(x,t-\tau(t)\big)\\ &\quad+\theta(t)h\big(\nabla_x u(x,t)\big)=0 \end{aligned}$$

in a bounded domain. We consider the general case with a nonlinear function h satisfying a smallness condition and obtain the decay of solutions under a relation between the weight of the delay term in the feedback and the weight of the term without delay. We impose no control on the sign of the derivative of the energy related to the above equation. In the second case we take $\theta \equiv const$ and $h(\nabla u) = -\nabla\Phi \cdot \nabla u$. We prove an exponential decay result of the energy without any smallness condition on Φ.

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1.1 Introduction

In this paper we investigate the decay properties of solutions for the initial boundary value problem for the nonlinear wave equation of the form

$$\begin{cases} u_{tt}(x,t) - \Delta_x u(x,t) + \mu_1\sigma(t)u_t(x,t) \\ \quad + \mu_2\sigma(t)u_t(x,t-\tau(t)) + \theta(t)h(\nabla_x u(x,t)) = 0 & \text{in } \Omega\times]0,+\infty[, \\ u(x,t) = 0 & \text{on } \Gamma\times]0,+\infty[, \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x) & \text{on } \Omega, \\ u'(x,t-\tau(0)) = f_0(x,t-\tau(0)) & \text{on } \Omega\times]0,\tau(0)[, \end{cases} \qquad (P)$$

where Ω is a bounded domain in $\mathbb{R}^n$, $n \in \mathbb{N}^*$, with a smooth boundary $\partial\Omega = \Gamma$, $\tau > 0$ is a time delay, μ_1 and μ_2 are positive real numbers, and the initial data (u_0, u_1, f_0) belong to a suitable function space.

When $h \equiv 0$, it is well known that, in absence of delay ($\mu_2 = 0$), the energy of problem (P) is exponentially decaying to zero. See, for instance, [4, 5, 11, 12] and [15]. On the contrary, if $\mu_1 = 0$ and $\mu_2 > 0$, that is, there exits only the delay part in the internal, the system (P) becomes unstable (see, for instance [6]). In recent years, the PDEs with time delay effects have become an active area of research since they arise in many practical problems (see, for example, [1, 20]). In [6], the authors showed that a small delay in a boundary control could turn a well-behave hyperbolic system into a wild one and, therefore, delay becomes a source of instability. To stabilize a hyperbolic system involving input delay terms, additional control terms will be necessary (see [16, 17, 21]). For instance, in [16] the authors studied the wave equation with a linear internal damping term with constant delay ($\sigma(t) \equiv 1$, $\tau(t) = const$ in the problem (P)) and determined suitable relations between μ_1 and μ_2, for which the stability or alternatively instability takes place. More precisely, they showed that the energy is exponentially stable if $\mu_2 < \mu_1$ and they also found a sequence of delays for which the corresponding solution of (P) will be instable if $\mu_2 \geq \mu_1$. The main approach used in [16] is an observability inequality obtained with a Carleman estimate. The same results were obtained if both the damping and the delay are acting on the boundary. We also recall the result by Xu, Yung and Li [21], where the authors proved a result similar to the one in [16] for the one-space dimension by adopting the spectral analysis approach.

The case of time-varying delay in the wave equation has been studied recently by Nicaise, Valein and Fridman [18] in one-space dimension ($\sigma(t) \equiv 1$ in the problem (P)). They proved an exponential stability result under the condition

$$\mu_2 < \sqrt{1-d}\,\mu_1,$$

where the function τ satisfies

$$\tau'(t) \leq d, \quad \forall t > 0$$

for a constant $d < 1$.

In [19], Nicaise, Pignotti and Valein extended the above result to higher space dimensions and established an exponential decay.

When $h \not\equiv 0$, in the case $\mu_2 = 0$, very little is known in the literature (see [2, 3, 7–9]). In [7], Guesmia established well posedness and energy decay estimates in the case of constant coefficients ($\sigma \equiv 1$ and $\theta \equiv 1$). He used a new approach based on a combination of some ideas given in his paper [8] and the multiplier method. In [2], the authors proved the same result in the case of an unbounded domain and variable coefficients.

We note here that the gradient-like nonlinear term $h(\nabla u)$ makes the problem more delicate, because the system may not be dissipative.

Our purpose in this paper is to give an energy decay estimate of the solution of the problem (P) in the case when h is nonlinear and linear in the presence of a time-varying delay term in the feedback. We use the ideas given by Guesmia in [7–9] and the multiplier technique to prove our result.

1.2 Preliminaries and Main Results

First assume the following hypotheses:

(H1) $\sigma, \theta : \mathbb{R}_+ \to]0, +\infty[$ are non increasing functions of class $C^1(\mathbb{R}_+)$ satisfying

$$\int_0^{+\infty} \sigma(\tau)d\tau = +\infty, \tag{1}$$

$$\left|\sigma'(t)\right| \le c\sigma(t), \tag{2}$$

$$\left|\theta'(t)\right| \le c\theta(t), \tag{3}$$

$$\theta(t) \le c\sigma(t). \tag{4}$$

(H2) τ is a function such that

$$\tau \in W^{2,\infty}\big([0, T]\big), \quad \forall T > 0, \tag{5}$$

$$0 < \tau_0 \le \tau(t) \le \tau_1, \quad \forall t > 0, \tag{6}$$

$$\tau'(t) \le d < 1, \quad \forall t > 0, \tag{7}$$

where τ_0 and τ_1 are two positive constants.

(H3)

$$\mu_2 < \sqrt{1-d}\,\mu_1. \tag{8}$$

(H4) $h : \mathbb{R}^n \to \mathbb{R}$ is a C^1 function such that ∇h is bounded and there exists $\beta > 0$ such that

$$\left|h(\zeta)\right| \le \beta|\zeta|, \quad \forall \zeta \in \mathbb{R}^n. \tag{9}$$

We introduce, as in [16], the new variable

$$z(x, \rho, t) = u_t\big(x, t - \tau(t)\rho\big), \quad x \in \Omega, \rho \in (0, 1), t > 0. \tag{10}$$

Then, we have

$$\tau(t)z_t(x,\rho,t) + \big(1-\tau'(t)\rho\big)z_\rho(x,\rho,t) = 0, \quad \text{in } \Omega \times (0,1) \times (0,+\infty). \tag{11}$$

Therefore, problem (P) is equivalent to:

$$\begin{cases} u_{tt}(x,t) - \Delta_x u(x,t) + \mu_1\sigma(t)u_t(x,t) & \\ \quad + \mu_2\sigma(t)z(x,1,t) + \theta(t)h(\nabla_x u(x,t)) = 0, & x \in \Omega, t > 0, \\ \tau(t)z_t(x,\rho,t) + (1-\tau'(t)\rho)z_\rho(x,\rho,t) = 0, & x \in \Omega, \rho \in (0,1), t > 0, \\ u(x,t) = 0, & x \in \partial\Omega, t > 0, \\ z(x,0,t) = u_t(x,t) & x \in \Omega, t > 0, \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x) & x \in \Omega, \\ z(x,\rho,0) = f_0(x,-\tau(0)\rho) & x \in \Omega, \rho \in (0,1). \end{cases} \tag{12}$$

Let $\bar{\xi}$ be a positive constant such that

$$\frac{\mu_2}{\sqrt{1-d}} < \bar{\xi} < 2\mu_1 - \frac{\mu_2}{\sqrt{1-d}}. \tag{13}$$

We define the energy of the solution by:

$$E(t) = \frac{1}{2}\|u_t(t)\|_2^2 + \frac{1}{2}\|\nabla_x u(t)\|_2^2 + \frac{\xi(t)\tau(t)}{2}\int_\Omega \int_0^1 z^2(x,\rho,t)d\rho dx, \tag{14}$$

where

$$\xi(t) = \bar{\xi}\sigma(t).$$

We have the following theorem.

Theorem 1 *Let* $(u_0,u_1,f_0) \in (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega) \times H_0^1(\Omega; H^1(0,1))$ *satisfy the compatibility condition*

$$f_0(\cdot,0) = u_1.$$

Assume that the hypotheses (H1)–(H4) *hold with* β *small enough. Then problem* (P) *admits a unique weak solution*

$$u \in L^\infty_{loc}\big((-\tau(0),\infty); H^2(\Omega) \cap H_0^1(\Omega)\big),$$
$$u_t \in L^\infty_{loc}\big((-\tau(0),\infty); H_0^1(\Omega)\big), \qquad u_{tt} \in L^\infty_{loc}\big((-\tau(0),\infty); L^2(\Omega)\big).$$

Moreover, the energy satisfies for $t \geq 0$

$$E(t) \leq \frac{E(0)}{\omega(0)}\omega\big(h(t)\big)e^{\tilde{\lambda}(t)-\tilde{\lambda}(h(t))}e^{-\int_0^{h(t)}\omega(\tau)d\tau}, \tag{15}$$

where

$$\lambda(t) = 2\beta\frac{\theta \circ \tilde{\sigma}^{-1}}{\sigma \circ \tilde{\sigma}^{-1}} \quad \text{and} \quad \tilde{\sigma}(t) = \int_0^t \sigma(\tau)d\tau.$$

Next, we consider the case where $\theta \equiv 1$ and $h(\nabla u) = -\nabla \Phi \cdot \nabla u$, where

$$\Phi \in W^{1,\infty}(\Omega).$$

Let $E_\Phi(t)$ be the energy associated to the solution of problem (P):

$$\begin{aligned} E(t) &= E_\Phi(t) \\ &= \frac{1}{2}\left\| e^{\Phi/2} u_t(t)\right\|_2^2 + \frac{1}{2}\left\| e^{\Phi/2}\nabla_x u(t)\right\|_2^2 \\ &\quad + \frac{\xi(t)\tau(t)}{2}\int_\Omega \int_0^1 e^{\Phi} z^2(x,\rho,t)\,dx\,d\rho. \end{aligned} \tag{16}$$

Theorem 2 *Let* $(u_0, u_1, f_0) \in H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega \times (0,1))$ *satisfy the compatibility condition*

$$f_0(\cdot, 0) = u_1.$$

Then problem (P) *admits a unique weak solution*

$$u(t) \in C\big(\big[-\tau(0),\infty\big); H^1(\Omega)\big) \cap C^1\big(\big[-\tau(0),\infty\big); L^2(\Omega)\big).$$

In addition, we have the following decay estimate:

$$E(t) \le cE(0)e^{-\omega\tilde{\sigma}(t)}, \quad \forall t \ge 0, \tag{17}$$

while c *and* ω *are positive constants, independent of the initial data.*

Lemma 1 *Let* (u, z) *be a solution to the problem* (12). *Then, the energy functional defined by* (14) *satisfies*

$$\begin{aligned} E'(t) &\le -\sigma(t)\left(\mu_1 - \frac{\bar{\xi}}{2} - \frac{\mu_2}{2\sqrt{1-d}}\right)\|u_t\|_2^2 \\ &\quad - \sigma(t)\left(\frac{\bar{\xi}(1-\tau'(t))}{2} - \frac{\mu_2\sqrt{1-d}}{2}\right)\int_\Omega z^2(x,1,t)\,dx \\ &\quad - \theta(t)\int_\Omega u_t(x,t)h(\nabla_x u)\,dx \\ &\le 2\beta\theta(t)E(t). \end{aligned} \tag{18}$$

Proof Multiplying the first equation in (12) by u_t, integrating over Ω and using integration by parts, we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\big(\|u_t\|_2^2 + \|\nabla_x u\|_2^2\big) + \mu_1\sigma(t)\|u_t\|_2^2 + \mu_2\sigma(t)\int_\Omega z(x,1,t)u_t(x,t)\,dx \\ &\quad + \theta(t)\int_\Omega u'(t)h(\nabla_x u)\,dx = 0. \end{aligned} \tag{19}$$

We multiply the second equation in (12) by $\xi(t)z$ and integrate over $\Omega \times (0,1)$ to obtain:

$$\begin{aligned}\xi(t)\tau(t)\int_\Omega\int_0^1 z_t z(x,\rho,t)d\rho dx\\ =-\frac{\xi(t)}{2}\int_\Omega\int_0^1\big(1-\tau'(t)\rho\big)\frac{\partial}{\partial\rho}\big(z(x,\rho,t)\big)^2 d\rho dx.\end{aligned}\tag{20}$$

Consequently,

$$\begin{aligned}&\frac{d}{dt}\left(\frac{\xi(t)\tau(t)}{2}\int_\Omega\int_0^1 z^2(x,\rho,t)d\rho dx\right)\\ &=-\frac{\xi(t)}{2}\int_0^1\int_\Omega\frac{\partial}{\partial\rho}\big(\big(1-\tau'(t)\rho\big)z^2(x,\rho,t)\big)d\rho dx\\ &\quad+\frac{\xi'(t)\tau(t)}{2}\int_0^1\int_\Omega z^2(x,\rho,t)dxd\rho\\ &=\frac{\xi(t)}{2}\int_\Omega\big(z^2(x,0,t)-z^2(x,1,t)\big)dx+\frac{\xi(t)\tau'(t)}{2}\int_\Omega z^2(x,1,t)dx\\ &\quad+\frac{\xi'(t)\tau(t)}{2}\int_0^1\int_\Omega z^2(x,\rho,t)dxd\rho.\end{aligned}\tag{21}$$

From (14), (19) and (21) we obtain

$$\begin{aligned}E'(t)\le{}&-\sigma(t)\left(\mu_1-\frac{\bar{\xi}}{2}\right)\|u_t\|_2^2-\sigma(t)\left(\frac{\bar{\xi}(1-\tau'(t))}{2}\right)\int_\Omega z^2(x,1,t)dx\\ &-\mu_2\sigma(t)\int_\Omega z(x,1,t)u_t(x,t)dx-\theta(t)\int_\Omega u_t(x,t)h(\nabla_x u)dx.\end{aligned}\tag{22}$$

Due to Young's inequality, we have

$$\mu_2\int_\Omega z(x,1,t)u_t(x,t)dx\le\frac{\mu_2}{2\sqrt{1-d}}\|u_t\|_2^2+\frac{\mu_2\sqrt{1-d}}{2}\int_\Omega z^2(x,1,t)dx.\tag{23}$$

Inserting (23) into (22) we obtain

$$\begin{aligned}E'(t)\le{}&-\sigma(t)\left(\mu_1-\frac{\bar{\xi}}{2}-\frac{\mu_2}{2\sqrt{1-d}}\right)\|u_t\|_2^2\\ &-\sigma(t)\left(\frac{\bar{\xi}(1-\tau'(t))}{2}-\frac{\mu_2\sqrt{1-d}}{2}\right)\int_\Omega z^2(x,1,t)dx\\ &-\theta(t)\int_\Omega u_t(x,t)h(\nabla_x u)dx.\end{aligned}\tag{24}$$

From (9), the definition of $E(t)$ and the Cauchy-Schwarz inequality we obtain

$$\left| \int_\Omega u_t(x,t)h(\nabla_x u)dx \right| \leq \beta \|u_t\|_2 \|\nabla_x u\|_2 \leq 2\beta E(t). \tag{25}$$

Inserting (25) into (24), we obtain

$$\begin{aligned} E'(t) \leq & -\sigma(t)\left(\mu_1 - \frac{\bar{\xi}}{2} - \frac{\mu_2}{2\sqrt{1-d}}\right)\|u_t\|_2^2 \\ & - \sigma(t)\left(\frac{\bar{\xi}(1-\tau'(t))}{2} - \frac{\mu_2\sqrt{1-d}}{2}\right)\int_\Omega z^2(x,1,t)dx + 2\beta\theta(t)E(t). \end{aligned}$$

Then, by using (13) and (7), our assertion holds. □

1.3 Global Existence

Throughout this section we assume $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$ and $u_1 \in H_0^1(\Omega)$, $f_0 \in L^2(\Omega; H^1(0,1))$.

We employ the Galerkin method to construct a global solution. Let $T > 0$ be fixed and denote by V_k the space generated by $\{w_1, w_2, \ldots, w_k\}$, where the set $\{w_k, k \in \mathbb{N}\}$ is a basis of $H^2(\Omega) \cap H_0^1(\Omega)$.

Now, we define for $1 \leq j \leq k$ the sequence $\phi_j(x,\rho)$ as follows:

$$\phi_j(x,0) = w_j.$$

Then, we may extend $\phi_j(x,0)$ by $\phi_j(x,\rho)$ over $L^2(\Omega \times (0,1))$ such that $(\phi_j)_j$ form a basis of $L^2(\Omega; H^1(0,1))$ and denote by Z_k the space generated by $\{\phi_1, \phi_2, \ldots, \phi_k\}$.

We construct approximate solutions (u_k, z_k) $(k = 1, 2, 3, \ldots)$ in the form

$$u_k(t) = \sum_{j=1}^k g_{jk}(t) w_j,$$

$$z_k(t) = \sum_{j=1}^k h_{jk}(t) \phi_j,$$

where g_{jk} and h_{jk} $(j = 1, 2, \ldots, m)$ are determined by the following ordinary differential equations:

$$\begin{cases} (u_k''(t), w_j) + (\nabla_x u_k(t), \nabla_x w_j) + \mu_1\sigma(t)(u_k', w_j) + \mu_2\sigma(t)(z_k(\cdot,1), w_j) \\ \quad + \theta(t)(h(\nabla_x u_k), w_j) = 0, \\ 1 \leq j \leq k, \\ z_k(x,0,t) = u_k'(x,t), \end{cases} \tag{26}$$

$$u_k(0) = u_{0k} = \sum_{j=1}^{k} (u_0, w_j) w_j \to u_0 \text{ in } H^2(\Omega) \cap H_0^1(\Omega) \quad \text{as } k \to +\infty, \tag{27}$$

$$u_k'(0) = u_{1k} = \sum_{j=1}^{k} (u_1, w_j) w_j \to u_1 \text{ in } H_0^1(\Omega) \quad \text{as } k \to +\infty \tag{28}$$

and

$$\begin{cases} (\tau(t) z_{kt} + (1 - \tau'(t)\rho) z_{k\rho}, \phi_j) = 0, \\ 1 \leq j \leq k, \end{cases} \tag{29}$$

$$z_k(\rho, 0) = z_{0k} = \sum_{j=1}^{k} (f_0, \phi_j) \phi_j \to f_0 \text{ in } L^2\big(\Omega; H^1(0,1)\big) \quad \text{as } k \to +\infty. \tag{30}$$

By virtue of the theory of ordinary differential equations, the system (26)–(30) has a unique local solution which is extended to a maximal interval $[0, T_k[$ (with $0 < T_k \leq +\infty$) by Zorn lemma since the nonlinear terms in (26) are locally Lipschitz continuous. Note that $u_k(t)$ is of class C^2.

In the next step, we obtain a priori estimates for the solution of the system (26)–(30), so that it can be extended beyond $[0, T_k[$ to obtain a single solution defined for all $t > 0$.

We will utilize a standard compactness argument for the limiting procedure and it suffices to derive some a priori estimates for (u_k, z_k).

The First Estimate Since the sequences u_{0k}, u_{1k} and z_{0k} converge, then from (18) we can find a positive constant C independent of k such that

$$E_k(t) + a_1 \int_0^t \sigma(s) \|u_k'\|_2^2 ds + a_2 \int_0^t \sigma(s) \|z_k(x,1,t)\|_2^2 ds \leq C E_k(0) e^{CT}, \tag{31}$$

where

$$E_k(t) = \frac{1}{2}\|u_k'(t)\|_2^2 + \frac{1}{2}\|\nabla_x u_k(t)\|_2^2 + \frac{\xi(t)\tau(t)}{2} \int_\Omega \int_0^1 z_k^2(x,\rho,t) d\rho dx,$$

$$a_1 = \mu_1 - \frac{\bar{\xi}}{2} - \frac{\mu_2}{2\sqrt{1-d}} \quad \text{and} \quad a_2 = \frac{\bar{\xi}(1-d)}{2} - \frac{\mu_2\sqrt{1-d}}{2}.$$

These estimates imply that the solution (u_k, z_k) exists globally in $[0, +\infty[$.

Estimate (31) yields

$$(u_k) \text{ is bounded in } L^\infty_{loc}\big(0, \infty; H_0^1(\Omega)\big), \tag{32}$$

$$\big(u_k'\big) \text{ is bounded in } L^\infty_{loc}\big(0, \infty; L^2(\Omega)\big), \tag{33}$$

$$\big(\sigma(t) u_k'^2(t)\big) \text{ is bounded in } L^1\big(\Omega \times (0,T)\big), \tag{34}$$

$$\big(\sigma(t)z_k^2(x,\rho,t)\big) \text{ is bounded in } L^\infty_{loc}\big(0,\infty; L^1\big(\Omega\times(0,1)\big)\big), \tag{35}$$

$$\big(\sigma(t)z_k^2(x,1,t)\big) \text{ is bounded in } L^1\big(\Omega\times(0,T)\big). \tag{36}$$

The Second Estimate We first estimate $u_k''(0)$. Replacing w_j by $u_k''(t)$ in (26) and taking $t=0$, we obtain:

$$\begin{aligned}\big\|u_k''(0)\big\|_2 &\le \|\Delta_x u_{0k}\|_2 + \mu_1\sigma(0)\|u_{1k}\|_2 + \mu_2\sigma(0)\|z_{0k}\|_2 + \beta\theta(0)\|\nabla_x u_{0k}\|_2\\ &\le \|\Delta_x u_0\|_2 + \mu_1\sigma(0)\|u_1\|_2 + \mu_2\sigma(0)\|z_0\|_2 + \beta\theta(0)\|\nabla_x u_0\|_2\\ &\le C.\end{aligned}$$

Differentiating (26) with respect to t, we get

$$\begin{aligned}&\big(u_k'''(t) + \Delta_x u_k'(t) + \mu_1\sigma(t)u_k''(t) + \mu_1\sigma'(t)u_k' + \mu_2\sigma(t)z_k' + \mu_2\sigma'(t)z_k\\ &\quad + \theta(t)\nabla_x u_k' h'(\nabla_x u_k) + \theta'(t)h(\nabla_x u_k), w_j\big) = 0.\end{aligned}$$

Multiplying by $g_{jk}''(t)$, summing over j from 1 to k, it follows that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\big(\big\|u_k''(x,t)\big\|_2^2 + \big\|\nabla_x u_k'(t)\big\|_2^2\big)\\ &\qquad + \mu_1\sigma(t)\int_\Omega u_k''^2(x,t)dx + \mu_1\sigma'(t)\int_\Omega u_k''(x,t)u_k'(x,t)dx\\ &\qquad + \mu_2\sigma(t)\int_\Omega u_k''(x,t)z_k'(x,1,t)dx + \mu_2\sigma'(t)\int_\Omega u_k''(x,t)z_k(x,1,t)dx\\ &\qquad + \theta(t)\int_\Omega u_k''(x,t)\nabla_x u_k'(x,t)h'(\nabla_x u_k)dx + \int_\Omega u_k''(t)\theta'(t)h(\nabla_x u_k)dx\\ &= 0.\end{aligned} \tag{37}$$

Differentiating (29) with respect to t, we get

$$\left(\left(\frac{\tau(t)}{1-\tau'(t)\rho}\right)' z_k' + \frac{\tau(t)}{1-\tau'(t)\rho} z_k''(t) + \frac{\partial}{\partial\rho} z_k', \phi_j\right) = 0.$$

Multiplying by $h_{jk}'(t)$, summing over j from 1 to k, it follows that

$$\left(\frac{\tau(t)}{1-\tau'(t)\rho}\right)' \big\|z_k'(t)\big\|_2^2 + \frac{1}{2}\frac{\tau(t)}{1-\tau'(t)\rho}\frac{d}{dt}\big\|z_k'(t)\big\|_2^2 + \frac{1}{2}\frac{d}{d\rho}\big\|z_k'(t)\big\|_2^2 = 0. \tag{38}$$

Then, we have

$$\begin{aligned}&\frac{1}{2}\left(\frac{\tau(t)}{1-\tau'(t)\rho}\right)' \big\|z_k'(t)\big\|_2^2 + \frac{1}{2}\frac{d}{dt}\left(\frac{\tau(t)}{1-\tau'(t)\rho}\big\|z_k'(t)\big\|_2^2\right)\\ &\quad + \frac{1}{2}\frac{d}{d\rho}\big\|z_k'(t)\big\|_2^2 = 0.\end{aligned} \tag{39}$$

Taking the sum of (37) and (39), we obtain that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\left\|u_k''(t)\right\|_2^2+\left\|\nabla_x u_k'(t)\right\|_2^2+\int_0^1\frac{\tau(t)}{1-\tau'(t)\rho}\left\|z_k'(x,\rho,t)\right\|_{L^2(\Omega)}^2 d\rho\right)\\
&\quad+\mu_1\sigma(t)\int_\Omega u_k''^2(x,t)dx+\frac{1}{2}\int_\Omega\left|z_k'(x,1,t)\right|^2 dx\\
&\quad=-\frac{1}{2}\int_0^1\left(\frac{\tau(t)}{1-\tau'(t)\rho}\right)'\left\|z_k'(x,\rho,t)\right\|_2^2 d\rho-\mu_2\sigma(t)\int_\Omega u_k''(x,t)z_k'(x,1,t)dx\\
&\quad-\mu_1\sigma'(t)\int_\Omega u_k''(x,t)u_k'(x,t)dx-\mu_2\sigma'(t)\int_\Omega u_k''(x,t)z_k(x,1,t)dx\\
&\quad+\frac{1}{2}\left\|u_k''(x,t)\right\|_2^2\\
&\quad-\theta(t)\int_\Omega u_k''(x,t)\nabla_x u_k'(x,t)h'(\nabla_x u_k)dx-\theta'(t)\int_\Omega u_k''(x,t)h(\nabla_x u_k)dx.
\end{aligned}$$

Using (H2), (H4), Cauchy-Schwarz and Young's inequalities, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\left\|u_k''(t)\right\|_2^2+\left\|\nabla_x u_k'(t)\right\|_2^2+\int_0^1\frac{\tau(t)}{1-\tau'(t)\rho}\left\|z_k'(x,\rho,t)\right\|_{L^2(\Omega)}^2 d\rho\right)\\
&\quad+\mu_1\sigma(t)\int_\Omega {u''}_k^2(t)dx+c\int_\Omega\left|z_k'(x,1,t)\right|^2 dx\\
&\quad\le c'\left\|u_k''(t)\right\|_2^2+c''\int_0^1\frac{\tau(t)}{1-\tau'(t)\rho}\left\|z_k'(x,\rho,t)\right\|_{L^2(\Omega)}^2 d\rho\\
&\quad\quad+c\left|\sigma'(t)\right|\left\|u_k'(t)\right\|_2^2+\left|\sigma'(t)\right|\left\|z_k(x,1,t)\right\|_2^2+c\left\|\nabla_x u_k'(t)\right\|_2^2+c\left\|\nabla_x u_k(t)\right\|_2^2\\
&\quad\le c'\left\|u_k''(t)\right\|_2^2+c''\int_0^1\frac{\tau(t)}{1-\tau'(t)\rho}\left\|z_k'(x,\rho,t)\right\|_{L^2(\Omega)}^2 d\rho\\
&\quad\quad+c\left|\sigma(t)\right|\left\|u_k'(t)\right\|_2^2+\left|\sigma(t)\right|\left\|z_k(x,1,t)\right\|_2^2+c\left\|\nabla_x u_k'(t)\right\|_2^2+c\left\|\nabla_x u_k(t)\right\|_2^2.
\end{aligned}$$

Integrating the last inequality over $(0,t)$ and using (31), we get

$$\begin{aligned}
&\left(\left\|u_k''(t)\right\|_2^2+\left\|\nabla_x u_k'(t)\right\|_2^2+\int_0^1\frac{\tau(t)}{1-\tau'(t)\rho}\left\|z_k'(x,\rho,t)\right\|_{L^2(\Omega)}^2 d\rho\right)\\
&\quad\le\Bigg(\left\|u_k''(0)\right\|_2^2+\left\|\nabla_x u_k'(0)\right\|_2^2\\
&\quad\quad+\int_0^1\frac{\tau(0)}{1-\tau'(0)\rho}\left\|z_k'(x,\rho,0)\right\|_{L^2(\Omega)}^2 d\rho+\left(cT+c'\right)e^{cT}\Bigg)\\
&\quad\quad+c\int_0^t\left(\left\|u_k''(s)\right\|_2^2+\left\|\nabla_x u_k'(s)\right\|_2^2+\int_0^1\frac{\tau(s)}{1-\tau'(s)\rho}\left\|z_k'(x,\rho,s)\right\|_{L^2(\Omega)}^2 d\rho\right)ds.
\end{aligned}$$

Using Gronwall's lemma, we deduce that

$$\begin{aligned}
&\|u_k''(t)\|_2^2 + \|\nabla_x u_k'(t)\|_2^2 + \int_0^1 \frac{\tau(t)}{1-\tau'(t)\rho}\|z_k'(x,\rho,t)\|_{L^2(\Omega)}^2 d\rho \\
&\leq e^{cT}\Bigg(\|u_k''(0)\|_2^2 + \|\nabla_x u_k'(0)\|_2^2 \\
&\quad + \int_0^1 \frac{\tau(0)}{1-\tau'(0)\rho}\|z_k'(x,\rho,0)\|_{L^2(\Omega)}^2 d\rho + \big(cT + c'\big)e^{cT}\Bigg)
\end{aligned}$$

for all $t \in \mathbb{R}^+$, therefore, we conclude that

$$\big(u_k''\big) \text{ is bounded in } L^\infty_{loc}\big(0,\infty; L^2(\Omega)\big), \tag{40}$$

$$\big(u_k'\big) \text{ is bounded in } L^\infty_{loc}\big(0,\infty; H_0^1(\Omega)\big), \tag{41}$$

$$\big(\tau(t)z_k'\big) \text{ is bounded in } L^\infty_{loc}\big(0,\infty; L^2\big(\Omega\times(0,1)\big)\big). \tag{42}$$

1.3.1 Analysis of the Nonlinear Term

From the assumption (9) we obtain

$$\int_\Omega \big|h\big(\nabla_x u_k(t)\big)\big|^2 dx \leq \beta \int_\Omega \big|\nabla_x u_k(t)\big|^2 dx \leq C',$$

where C' is a positive constant and, consequently, we conclude that

$$h\big(\nabla_x u_k(t)\big) \text{ is bounded in } L^2\big(0,T; L^2(\Omega)\big). \tag{43}$$

Applying Dunford-Pettis theorem, we deduce from (32), (33), (34), (35), (36), (40), (41), (42) and (43), replacing the sequence u_k with a subsequence, if necessary, that

$$u_k \to u \text{ weak-star in } L^\infty_{loc}\big(0,\infty; H^2(\Omega)\cap H_0^1(\Omega)\big), \tag{44}$$

$$u_k' \to u' \text{ weak-star in } L^\infty_{loc}\big(0,\infty; H_0^1(\Omega)\big),$$

$$u_k'' \to u'' \text{ weak-star in } L^\infty_{loc}\big(0,\infty; L^2(\Omega)\big), \tag{45}$$

$$u_k' \to \chi \text{ weak in } L^2\big(\Omega\times(0,T);\sigma\big),$$

$$z_k \to z \text{ weak-star in } L^\infty_{loc}\big(0,\infty; H_0^1\big(\Omega; L^2(0,1)\big)\big),$$

$$F\big(\nabla_x u_k(t)\big) \to \zeta \text{ weak-star in } L^\infty_{loc}\big(0,\infty; L^2(\Omega)\big), \tag{46}$$

$$z_k' \to z' \text{ weak-star in } L^\infty_{loc}\big(0,\infty; L^2\big(\Omega\times(0,1)\big)\big), \tag{47}$$

$$z_k(x,1,t) \to \psi \text{ weak in } L^2\big(\Omega\times(0,T);\sigma\big)$$

for suitable functions $u \in L^\infty(0,T;H^2(\Omega)\cap H_0^1(\Omega))$, $z \in L^\infty(0,T;L^2(\Omega\times(0,1)))$, $\chi \in L^2(\Omega\times(0,T);\sigma)$, $\psi \in L^2(\Omega\times(0,T);\sigma)$, $\zeta \in L^\infty(0,T;L^2(\Omega))$, for all $T \geq 0$. ($L^2(\Omega\times(0,T);\sigma)$ is the space of square-integrable functions with weight σ.) We have to show that u is a solution of (P).

From (41) we have that (u_k') is bounded in $L^\infty(0,T;H_0^1(\Omega))$. Then (u_k') is bounded in $L^2(0,T;H_0^1(\Omega))$. Since (u_k'') is bounded in $L^\infty(0,T;L^2(\Omega))$, then (u_k'') is bounded in $L^2(0,T;L^2(\Omega))$. Consequently, (u_k') is bounded in $H^1(\Omega\times(0,T))$.

Since the embedding $H^1(\Omega\times(0,T))\hookrightarrow L^2(\Omega\times(0,T))$ is compact, using Aubin-Lions theorem [13], we can extract a subsequence (u_ς) of (u_k) such that

$$\frac{\partial}{\partial t}u_\varsigma \to \frac{\partial}{\partial t}u \text{ strongly in } L^2\big(\Omega\times(0,T)\big). \tag{48}$$

Therefore

$$\frac{\partial}{\partial t}u_\varsigma \to \frac{\partial}{\partial t}u \text{ strongly and a.e. in } \Omega\times(0,T). \tag{49}$$

Similarly we obtain

$$z_\varsigma \to z \text{ strongly in } L^2\big(\Omega\times(0,1)\times(0,T)\big) \tag{50}$$

and

$$z_\varsigma \to z \text{ strongly and a.e. in } \Omega\times(0,1)\times(0,T). \tag{51}$$

It follows at once, from (44), (45), (46), (47), (48) and (50), that for each fixed $v \in L^2(0,T;L^2(\Omega))$ and $w \in L^2(0,T;L^2(\Omega)\times(0,1))$

$$\begin{aligned}&\int_0^T\int_\Omega\left(\frac{\partial^2}{\partial t^2}u_\varsigma - \Delta_x u_\varsigma + \mu_1\sigma(t)\frac{\partial}{\partial t}u_\varsigma + \mu_2\sigma(t)z_\varsigma + h(\nabla_x u_\varsigma)\right)v\,dx\,dt\\ &\quad\to \int_0^T\int_\Omega\big(u_{tt} - \Delta_x u + \mu_1\sigma(t)u_t + \mu_2\sigma(t)z + \zeta\big)v\,dx\,dt,\end{aligned} \tag{52}$$

$$\begin{aligned}&\int_0^T\int_0^1\int_\Omega\left(\tau(t)\frac{\partial}{\partial t}z_\varsigma + \big(1-\tau'(t)\rho\big)\frac{\partial}{\partial\rho}z_\varsigma\right)w\,dx\,d\rho\,dt\\ &\quad\to \int_0^T\int_0^1\int_\Omega\left(\tau(t)z_t + \big(1-\tau'(t)\rho\big)\frac{\partial}{\partial\rho}z\right)w\,dx\,d\rho\,dt\end{aligned} \tag{53}$$

as $\varsigma \to +\infty$.

On the other hand, multiplying the approximate problem (26)–(29) by g_{jk} and h_{jk} and summing over j from 1 to k and integrating the result over $[0,T]$, we infer

$$\begin{aligned}&\int_0^T\left(\frac{\partial^2}{\partial t^2}u_\varsigma, u_\varsigma\right)dt + \int_0^T|\nabla_x u_\varsigma|^2dt + \mu_1\int_0^T\sigma(t)\left(\frac{\partial}{\partial t}u_\varsigma, u_\varsigma\right)dt\\ &\quad + \mu_2\int_0^T\sigma(t)(z_\varsigma,u_\varsigma)dt + \int_0^T\theta(t)h(\nabla_x u_\varsigma,u_\varsigma)dt = 0,\end{aligned} \tag{54}$$

$$\int_0^T \tau(t)\left(\frac{\partial}{\partial t}z_\varsigma, z_\varsigma\right)dt + \int_0^T \big(1-\tau'(t)\rho\big)\left(\frac{\partial}{\partial \rho}z_\varsigma, z_\varsigma\right)dt = 0. \tag{55}$$

Considering (48) and (50), we are able to pass to the limit in (54) and (55), observing the weak convergence related to this identity. More precisely, we obtain

$$\lim_{\varsigma\to\infty}\int_0^T \|\nabla_x u_\varsigma\|^2 dt = -\int_0^T (u_{tt}, u)dt - \mu_1\int_0^T \sigma(t)(u_t, u)dt$$
$$- \mu_2\int_0^T \sigma(t)(z, u)dt - \int_0^T \theta(t)(\zeta, u)dt, \tag{56}$$

$$\int_0^T \tau(t)(z_t, z)dt + \int_0^T \big(1-\tau'(t)\rho\big)\left(\frac{\partial}{\partial \rho}z, z\right)dt = 0. \tag{57}$$

Recalling (52) and (53), we easily see that

$$\lim_{\varsigma\to\infty}\int_0^T \|\nabla_x u_\varsigma\|^2 dt = \int_0^T \|\nabla_x u\|^2 dt. \tag{58}$$

Now, taking into account that

$$\int_0^T \|\nabla_x u_\varsigma - \nabla_x u\|^2 dt = \int_0^T \|\nabla_x u_\varsigma\|^2 dt - \int_0^T (\nabla_x u_\varsigma, \nabla_x u)dt + \int_0^T \|\nabla_x u\|^2 dt,$$

we obtain

$$\nabla_x u_\varsigma \to \nabla_x u \text{ in } L^2\big(0, T; L^2(\Omega)\big)$$

and consequently

$$\nabla_x u_\varsigma \to \nabla_x u \text{ a.e. in } Q. \tag{59}$$

From (59), we get

$$h(\nabla_x u_\varsigma) \to h(\nabla_x u) \text{ a.e. in } Q. \tag{60}$$

From (43), (46) and (60), we get

$$h(\nabla_x u_\varsigma) \rightharpoonup h(\nabla_x u) \text{ weakly in } L^2\big(0, T; L^2(\Omega)\big).$$

Therefore,

$$\int_0^T\int_\Omega \big(u_{tt} - \Delta_x u + \mu_1\sigma(t)u_t + \mu_2\sigma(t)z + h(\nabla_x u)\big)v\,dx\,dt = 0,$$
$$v \in L^2\big(0, T; L^2(\Omega)\big),$$
$$\int_0^T\int_0^1\int_\Omega \left(\tau(t)u_t + \big(1-\tau'(t)\rho\big)\frac{\partial}{\partial\rho}z\right)w\,dx\,d\rho\,dt = 0,$$
$$w \in L^2\big(0, T; L^2(\Omega)\times(0,1)\big).$$

Thus the problem (P) admits a global weak solution u.

1.4 Asymptotic Behavior

From now on, we denote by c various positive constants which may be different at different occurrences. We multiply the first equation of (12) by $\phi' E^q u$, where ϕ is a bounded function satisfying all the hypotheses of Lemma 4. We obtain

$$\begin{aligned}
0 &= \int_S^T E^q\phi' \int_\Omega u\big(u_{tt} - \Delta u + \mu_1\sigma(t)u_t + \mu_2\sigma(t)z(x,1,t) + \theta(t)h(\nabla_x u)\big)dxdt \\
&= \left[E^q\phi' \int_\Omega uu_t dx\right]_S^T - \int_S^T \big(qE'E^{q-1}\phi' + E^q\phi''\big)\int_\Omega uu_t dxdt \\
&\quad - 2\int_S^T E^q\phi' \int_\Omega u'^2 dxdt + \int_S^T E^q\phi' \int_\Omega \big(u_t^2 + |\nabla u|^2\big)dxdt \\
&\quad + \mu_1 \int_S^T E^q\phi'\sigma(t)\int_\Omega uu_t dxdt + \mu_2\int_S^T E^q\phi'\sigma(t)\int_\Omega uz(x,1,t)dxdt \\
&\quad + \int_S^T E^q\phi'\theta(t)\int_\Omega uh(\nabla_x u)dxdt.
\end{aligned}$$

Similarly, we multiply the second equation of (12) by $E^q\phi'\xi(t)e^{-2\tau\rho}z(x,\rho,t)$ and get

$$\begin{aligned}
0 &= \int_S^T E^q\phi' \int_\Omega\int_0^1 e^{-2\tau\rho}\xi(t)z\big(\tau z_t + \big(1-\tau'(t)\rho\big)z_\rho\big)dxd\rho dt \\
&= \left[\frac{1}{2}E^q\phi'\xi(t)\tau(t)\int_\Omega\int_0^1 \tau e^{-2\tau(t)\rho}z^2 dxd\rho\right]_S^T \\
&\quad - \frac{1}{2}\int_S^T\int_\Omega\int_0^1 \big(E^q\phi'\xi(t)\tau(t)e^{-2\tau\rho}\big)' z^2 dxd\rho dt \\
&\quad + \int_S^T E^q\phi'\int_\Omega\int_0^1 \xi(t)\bigg(\frac{1}{2}\frac{\partial}{\partial\rho}\big(e^{-2\tau(t)\rho}\big(1-\tau'(t)\rho\big)z^2\big) \\
&\quad + \tau(t)\big(1-\tau'(t)\rho\big)e^{-2\tau\rho}z^2 + \frac{1}{2}\tau'(t)e^{-2\tau\rho}z^2\bigg)dxd\rho dt \\
&= \left[\frac{1}{2}E^q\phi'\xi(t)\tau(t)\int_\Omega\int_0^1 \tau e^{-2\tau(t)\rho}z^2 dxd\rho\right]_S^T \\
&\quad - \frac{1}{2}\int_S^T \big(E^q\phi'\xi(t)\big)'\tau(t)\int_\Omega\int_0^1 e^{-2\tau\rho}z^2 dxd\rho dt \\
&\quad + \frac{1}{2}\int_S^T E^q\phi'\xi(t)\int_\Omega \big(e^{-2\tau(t)}\big(1-\tau'(t)\big)z^2(x,1,t) - z^2(x,0,t)\big)dxdt \\
&\quad + \int_S^T E^q\phi'\xi(t)\tau(t)\int_0^1\int_\Omega e^{-2\tau\rho}z^2 dxd\rho dt.
\end{aligned}$$

Taking their sum, we obtain

$$
\begin{aligned}
&A\int_S^T E^{q+1}\phi' dt\\
&\leq -\Big[E^q\phi'\int_\Omega uu_t dx\Big]_S^T + \int_S^T \big(qE'E^{q-1}\phi' + E^q\phi''\big)\int_\Omega uu_t dxdt\\
&\quad + 2\int_S^T E^q\phi'\int_\Omega u_t^2 dxdt - \mu_1\int_S^T E^q\phi'\sigma(t)\int_\Omega uu_t dxdt\\
&\quad - \mu_2\int_S^T E^q\phi'\sigma(t)\int_\Omega uz(x,1,t)dxdt - \int_S^T E^q\phi'\theta(t)\int_\Omega uh(\nabla_x u)dxdt\\
&\quad - \Big[\frac{1}{2}E^q\phi'\xi(t)\tau(t)\int_\Omega\int_0^1 e^{-2\tau(t)\rho}z^2 dxd\rho\Big]_S^T\\
&\quad + \frac{1}{2}\int_S^T \big(E^q\phi'\xi(t)\big)'\tau(t)\int_\Omega\int_0^1 e^{-2\tau\rho}z^2 dxd\rho dt\\
&\quad - \frac{1}{2}\int_S^T E^q\phi'\xi(t)\int_\Omega\big(e^{-2\tau(t)}\big(1-\tau'(t)\big)z^2(x,1,t) - z^2(x,0,t)\big)dxdt,
\end{aligned}
\tag{61}
$$

where $A = 2\min\{1, e^{-2\tau_1}\}$. Using the Cauchy-Schwarz and Poincaré's inequalities and the definition of E and assuming that ϕ' is a bounded non-negative function on $\mathbb{R}^+$, we get

$$
\left|E^q(t)\phi'\int_\Omega uu' dx\right| \leq cE(t)^{q+1}.
$$

By recalling (18), we have

$$
\begin{aligned}
&\left|qE'E^{q-1}\phi'\int_\Omega uu_t dx\right|\\
&\leq cE^q(t)\big|E'(t)\big|\\
&\leq cE^q(t)\left(\left|E' + \theta(t)\int_\Omega u_t(t)h(\nabla_x u)dx\right| + \theta(t)\left|\int_\Omega u_t(t)h(\nabla_x u)dx\right|\right)\\
&= cE^q(t)\left(-E' - \theta(t)\int_\Omega u_t(t)h(\nabla_x u)dx + \theta(t)\left|\int_\Omega u_t(t)h(\nabla_x u)dx\right|\right)\\
&\leq cE^q(t)\big(-E'(t) + c\beta\theta(t)E(t)\big)
\end{aligned}
$$

and

$$
\begin{aligned}
\int_S^T E^q\phi'\int_\Omega u_t^2 dxdt &\leq \int_S^T E^q\phi'\frac{1}{\sigma(t)}\int_\Omega \sigma(t)u_t^2 dxdt\\
&\leq \int_S^T E^q\phi'\frac{1}{\sigma(t)}\left(-E' - \theta(t)\int_\Omega u_t h(\nabla_x u)dx\right)dt.
\end{aligned}
\tag{62}
$$

Define

$$\phi(t) = \int_0^t \sigma(\tau)d\tau. \tag{63}$$

It is clear that ϕ is a non-decreasing function of class C^1 on $\mathbb{R}^+$. Then, hypothesis (1) ensures that

$$\phi(t) \to +\infty \quad \text{as } t \to +\infty. \tag{64}$$

So, we deduce, from (62), that

$$\begin{aligned}\int_S^T E^q \phi' \int_\Omega u_t^2 dxdt &\le c\int_S^T E^q\big(-E'\big)dt + c\int_S^T E^q \theta(t)\int_\Omega |u_t|\big|h(\nabla_x u)\big|dx \\ &\le cE^{q+1}(S) + c\beta\int_S^T E^{q+1}\theta(t)dt\end{aligned} \tag{65}$$

and from (65) and (4) that

$$\int_S^T E^q \phi' \int_\Omega u_t^2 dxdt \le cE^{q+1}(S) + c\beta\int_S^T E^{q+1}\phi'(t)dt. \tag{66}$$

By the hypothesis (H1), Young's and Poincaré's inequality and (66), we have

$$\begin{aligned}&\left|\int_S^T E^q \phi'' \int_\Omega uu_t dxdt\right| \\ &\quad\le \int_S^T E^q\big|\phi''\big|\|u\|_2\|u_t\|_2 dt \\ &\quad\le \varepsilon'\int_S^T E^q\big|\phi''\big|\|u\|_2^2 dt + c\big(\varepsilon'\big)\int_S^T E^q\big|\phi''\big|\big\|u'\big\|_2^2 dt \\ &\quad\le \varepsilon' c_*\int_S^T E^q\phi'\|\nabla_x u\|_2^2 dt + c\big(\varepsilon'\big)\int_S^T E^q\phi'\|u_t\|_2^2 dt \\ &\quad\le cc\big(\varepsilon'\big)E^{q+1}(S) + \big(\varepsilon' c_* + cc\big(\varepsilon'\big)\beta\big)\int_S^T E^{q+1}\phi' dt, \\ &\left|\int_S^T E^q \phi' \int_\Omega uu_t dxdt\right| \\ &\quad\le c\int_S^T E^q\phi'\|u\|_2\|u_t\|_2 dt \\ &\quad\le c\varepsilon'\int_S^T E^q\phi'\|u\|_2^2 dt + c\big(\varepsilon'\big)\int_S^T E^q\phi'\|u_t\|_2^2 dt \\ &\quad\le \varepsilon' c_*\int_S^T E^q\phi'\|\nabla_x u\|_2^2 dt + c\big(\varepsilon'\big)\int_S^T E^q\phi'\|u_t\|_2^2 dt\end{aligned} \tag{67}$$

$$\leq cc\big(\varepsilon'\big)E^{q+1}(S) + \big(\varepsilon' c_* + cc\big(\varepsilon'\big)\beta\big)\int_S^T E^{q+1}\phi' dt,$$

$$\left|\int_S^T E^q \phi' \theta(t)\int_\Omega uh(\nabla_x u)dxdt\right|$$

$$\leq c\beta \int_S^T E^{q+1}\phi' dt,$$

$$-\left[E^q\phi'\xi(t)\tau(t)\int_\Omega\int_0^1 e^{-2\tau(t)\rho}z^2 dxd\rho\right]_S^T$$

$$= E^q(S)\phi'(S)\xi(S)\tau(S)\int_\Omega\int_0^1 e^{-2\tau(S)\rho}z^2(x,\rho,S)dxd\rho$$

$$- E^q(T)\phi'(T)\sigma(T)\xi(T)\tau(T)\int_\Omega\int_0^1 e^{-2\tau(T)\rho}z^2(x,\rho,T)dxd\rho$$

$$\leq CE^{q+1}(S) + C'E^{q+1}(T).$$

Recalling that $\xi' \leq 0$ and the definition of E we have

$$\int_S^T \big(E^q\phi'\xi(t)\big)'\tau(t)\int_\Omega\int_0^1 e^{-2\tau(t)\rho}z^2 dxd\rho dt$$

$$\leq \int_S^T \big(E^q\phi'\big)'\xi(t)\tau(t)\int_\Omega\int_0^1 e^{-2\tau(t)\rho}z^2 dxd\rho dt$$

$$\leq c\int_S^T E^q\big|E'\big|\phi' dt$$

$$\leq c\int_S^T E^q\phi'\big(-E'(t) + c\beta\theta(t)E(t)\big)dt$$

$$\leq cE^{q+1}(S) + c\beta\int_S^T E^{q+1}\phi' dt,$$

$$\int_S^T E^q\phi'\xi(t)\int_\Omega e^{-2\tau}\big(1-\tau'(t)\big)z^2(x,1,t)dxdt$$

$$\leq c\int_S^T E^q\phi'\int_S^T E^q\phi'\frac{1}{\sigma(t)}\int_\Omega \sigma(t)z^2(x,1,t)dxdt$$

$$\leq c\int_S^T E^q\phi'\frac{1}{\sigma(t)}\left(-E' - \theta(t)\int_\Omega u_t h(\nabla_x u)dx\right)dt$$

$$\leq cE^{q+1}(S) + c\beta\int_S^T E^{q+1}\phi' dt,$$

$$\begin{aligned}
&\int_S^T E^q \phi' \xi(t) \int_\Omega z^2(x,0,t)dxdt \\
&\quad = \int_S^T E^q \phi' \int_\Omega u_t^2(x,t)dxdt \\
&\quad \le cE^{q+1}(S) + c\beta \int_S^T E^{q+1}\phi' dt,
\end{aligned}$$

where we have also used the Cauchy-Schwarz inequality. Combining these estimates and choosing ε', β sufficiently small, we conclude from (61) that

$$\int_S^T E^{q+1}\phi' dt \le CE^{q+1}(S) + C' E^{q+1}(T),$$
$$E'(t) \le 2\beta\theta(t)E(t).$$

Let $E_1 = E \circ \phi^{-1}$ (note that ϕ^{-1} is a bijection from $\mathbb{R}^+$ to $\mathbb{R}^+$). Then

$$\int_S^T E_1^{q+1} dt \le CE_1^{q+1}(S) + C' E_1^{q+1}(T),$$
$$E_1'(t) \le \lambda(t)E_1(t),$$

where

$$\lambda(t) = 2\beta \frac{\theta \circ \phi^{-1}}{\phi' \circ \phi^{-1}}.$$

We obtain (15) and (17), where $\tilde{\lambda}(t) = \int_0^t \lambda(\tau)d\tau$ and $\omega = \frac{1}{a}$ such that

$$a(s) = \frac{a_1(s) + a_2(s)(d(s))^p(s) + a_3(s)(d(s))^r(s)}{1 - a_3(r+1)\sup_{t\ge 0}\{\lambda(t)\}},$$

where $a_1(s) = const$, $a_2(s) = const$, $a_3(s) = const$ and $q = 0$.

This ends the proof of Theorem 1. □

Proof of Theorem 2

Lemma 2 *Let* (u, z) *be a solution to the problem* (12). *Then, the energy functional defined by* (16) *satisfies*

$$\begin{aligned}
E'(t) &\le -\sigma(t)\left(\mu_1 - \frac{\bar{\xi}}{2} - \frac{\mu_2}{2\sqrt{1-d}}\right)\|e^{\Phi/2}u'\|_2^2 \\
&\quad - \sigma(t)\left(\frac{\bar{\xi}(1-\tau'(t))}{2} - \frac{\mu_2\sqrt{1-d}}{2}\right)\int_\Omega e^{\Phi} z^2(x,1,t)dx \\
&\le 0. \qquad (68)
\end{aligned}$$

For the proof of Lemma 2 we follow the same steps as in the proof of Lemma 1.

Now, we shall derive the decay estimate for the solutions of (P). We also denote here by c various positive constants which may be different at different occurrences. We multiply the first equation of (12) by $\phi' E^q e^{\Phi} u$ and obtain

$$\begin{aligned}
0 &= \int_S^T E^q \phi' \int_\Omega e^{\Phi} u\big(u_{tt} - \Delta u + \mu_1 \sigma(t) u_t + \mu_2 \sigma(t) z(x,1,t) - \nabla\Phi \nabla u\big) dx dt \\
&= \left[E^q \phi' \int_\Omega e^{\Phi} u u_t dx \right]_S^T - \int_S^T \left(q E' E^{q-1} \phi' + E^q \phi'' \right) \int_\Omega e^{\Phi} u u_t dx dt \\
&\quad - 2 \int_S^T E^q \phi' \int_\Omega e^{\Phi} u_t^2 dx dt + \int_S^T E^q \phi' \int_\Omega e^{\Phi} \left(u_t^2 + |\nabla u|^2 \right) dx dt \\
&\quad + \mu_1 \int_S^T E^q \phi' \int_\Omega e^{\Phi} u u_t dx dt + \mu_2 \int_S^T E^q \phi' \int_\Omega e^{\Phi} u z(x,1,t) dx dt.
\end{aligned}$$

Similarly, we multiply the second equation of (12) by $E^q \phi' \xi(t) e^{-2\tau\rho} e^{\Phi} z(x,\rho,t)$ and get

$$\begin{aligned}
0 &= \int_S^T E^q \phi' \xi(t) \int_\Omega \int_0^1 e^{\Phi} e^{-2\tau\rho} z\big(\tau z' + \big(1 - \tau'(t)\rho\big) z_\rho\big) dx d\rho dt \\
&= \left[\frac{1}{2} E^q \phi' \xi(t) \tau(t) \int_\Omega \int_0^1 e^{-2\tau(t)\rho} e^{\Phi} z^2 dx d\rho \right]_S^T \\
&\quad - \frac{1}{2} \int_S^T \int_\Omega \int_0^1 \left(E^q \phi' \xi(t) \tau(t) e^{-2\tau\rho} \right)' e^{\Phi} z^2 dx d\rho dt \\
&\quad + \int_S^T E^q \phi' \xi(t) \int_\Omega \int_0^1 e^{\Phi} \bigg(\frac{1}{2} \frac{\partial}{\partial \rho} \left(e^{-2\tau(t)\rho} \big(1 - \tau'(t)\rho\big) z^2 \right) \\
&\quad + \tau(t) \big(1 - \tau'(t)\rho\big) e^{-2\tau\rho} z^2 + \frac{1}{2} \tau'(t) e^{-2\tau\rho} z^2 \bigg) dx d\rho dt \\
&= \left[\frac{1}{2} E^q \phi' \xi(t) \tau(t) \int_\Omega \int_0^1 e^{\Phi} e^{-2\tau(t)\rho} z^2 dx d\rho \right]_S^T \\
&\quad - \frac{1}{2} \int_S^T \left(E^q \phi' \xi(t) \right)' \tau(t) \int_\Omega \int_0^1 e^{-2\tau\rho} e^{\Phi} z^2 dx d\rho dt \\
&\quad + \frac{1}{2} \int_S^T E^q \phi' \xi(t) \int_\Omega e^{\Phi} \left(e^{-2\tau(t)} \big(1 - \tau'(t)\big) z^2(x,1,t) - z^2(x,0,t) \right) dx dt \\
&\quad + \int_S^T E^q \phi' \xi(t) \tau(t) \int_0^1 \int_\Omega e^{-2\tau\rho} e^{\Phi} z^2 dx d\rho dt.
\end{aligned}$$

Taking their sum, we obtain

$$\begin{aligned}
&A\int_S^T E^{q+1}\phi' dt\\
&\le -\left[E^q\phi'\int_\Omega e^{\Phi}uu_t dx\right]_S^T\\
&\quad+\int_S^T \left(qE'E^{q-1}\phi'+E^q\phi''\right)\int_\Omega e^{\Phi}uu_t dxdt\\
&\quad+2\int_S^T E^q\phi'\int_\Omega e^{\Phi}u_t^2 dxdt-\mu_1\int_S^T E^q\phi'\int_\Omega e^{\Phi}uu_t dxdt\\
&\quad-\mu_2\int_S^T E^q\phi'\int_\Omega e^{\Phi}uz(x,1,t)dxdt\\
&\quad-\left[\frac{1}{2}E^q\phi'\xi(t)\tau(t)\int_\Omega\int_0^1 e^{-2\tau(t)\rho}e^{\Phi}z^2 dxd\rho\right]_S^T\\
&\quad+\frac{1}{2}\int_S^T \left(E^q\phi'\xi(t)\right)'\tau(t)\int_\Omega\int_0^1 e^{-2\tau\rho}e^{\Phi}z^2 dxd\rho dt\\
&\quad-\frac{1}{2}\int_S^T E^q\phi'\xi(t)\int_\Omega e^{\Phi}\left(e^{-2\tau(t)}\left(1-\tau'(t)\right)z^2(x,1,t)-z^2(x,0,t)\right)dxdt,
\end{aligned} \tag{69}$$

where $A=2\min\{1,2\tau e^{-2\tau}/\xi\}$. Since E is non-increasing and taking ϕ' to be bounded and non-negative on $\mathbb{R}^+$ (and we denote by μ its supremum), we find that

$$\begin{aligned}
&-\left[E^q\phi'\int_\Omega e^{\Phi}uu_t dx\right]_S^T\\
&\quad=E^q(S)\phi'(S)\int_\Omega e^{\Phi}u(S)u_t(S)dx-E^q(T)\phi'(T)\int_\Omega e^{\Phi}u(T)u_t(T)dx\\
&\quad\le CE^{q+1}(S),\\
&\left|\int_S^T \left(qE'E^{q-1}\phi'+E^q\phi''\right)\int_\Omega e^{\Phi}uu_t dxdt\right|\\
&\quad\le c\mu\int_S^T \left(-E'\right)E^q dt+c'\int_S^T E^{q+1}\left(-\phi''\right)dt\\
&\quad\le cE^{q+1}(S),\\
&\left|\frac{1}{2}E^q\phi'\xi(t)\tau(t)\int_\Omega\int_0^1 e^{-2\tau(t)\rho}e^{\Phi}z^2 dxd\rho\right|\le c\mu E(S)^{q+1}\quad \forall t\ge S,\\
&\int_S^T \left(E^q\phi'\xi(t)\right)'\tau(t)\int_\Omega\int_0^1 e^{-2\tau\rho}e^{\Phi}z^2 dxd\rho dt\le 0,
\end{aligned}$$

where we have also used the Cauchy-Schwarz inequality. Combining these estimates we conclude from (69) that

$$
\begin{aligned}
A & \int_S^T E^{q+1} \phi' dt \\
& \leq 2 \int_S^T E^q \phi' \int_\Omega e^{\Phi} u_t^2 dx dt - \mu_2 \int_S^T E^q \phi' \int_\Omega e^{\Phi} u z(x,1,t) dx dt \\
& \quad - \mu_1 \int_S^T E^q \phi' \int_\Omega e^{\Phi} u u_t dx dt + \frac{1}{2} \int_S^T E^q \phi' \xi(t) \int_\Omega e^{\Phi} z^2(x,0,t) dx dt \\
& \quad - \frac{1}{2} \int_S^T E^q \phi' \xi(t) \int_\Omega e^{\Phi} e^{-2\tau(t)} \big(1 - \tau'(t)\big) z^2(x,1,t) dx dt. \qquad (70)
\end{aligned}
$$

Now, we estimate the terms of the right-hand side of (70) in order to apply the results of Lemma 4.

Define

$$
\phi(t) = \int_0^t \sigma(s) ds.
$$

It is clear that ϕ is a non-decreasing function of class C^2 on $\mathbb{R}^+$. Hypothesis (1) ensures that

$$
\phi(t) \to +\infty \quad \text{as } t \to +\infty. \qquad (71)
$$

Using the Cauchy-Schwarz and Poincaré's inequalities and the energy inequality from Lemma 2 we get

$$
\begin{aligned}
\int_S^T E^q \phi' \int_\Omega e^{\Phi} u_t^2 dx dt & \leq c \int_S^T E^q \phi' \left(\frac{-E'}{\sigma(t)} \right) dt \\
& \leq c E^{q+1}(S),
\end{aligned}
$$

$$
\begin{aligned}
\int_S^T E^q \phi' \xi(t) \int_\Omega e^{\Phi} e^{-2\tau(t)} \big(1 - \tau'(t)\big) z^2(x,1,t) dx dt & \leq c \int_S^T E^q \phi' \left(\frac{-E'}{\sigma(t)} \right) dt \\
& \leq c E^{q+1}(S),
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{2} \int_S^T E^q \phi' \xi(t) \int_\Omega e^{\Phi} z^2(x,0,t) dx dt & = \frac{1}{2} \int_S^T E^q \phi' \xi(t) \int_\Omega e^{\Phi} u'^2 dx dt \\
& \leq c E^{q+1}(S),
\end{aligned}
$$

$$
\begin{aligned}
& \int_S^T E^q \phi' \int_\Omega e^{\Phi} u u_t dx dt \\
& \quad \leq \varepsilon \int_S^T E^q \phi' \int_\Omega e^{\Phi} u^2 dx dt + c(\varepsilon) \int_S^T E^q \phi' \int_\Omega e^{\Phi} u_t^2 dx dt
\end{aligned}
$$

$$\leq \varepsilon c \int_\Omega E^{q+1}\phi' dt + c(\varepsilon) \int_S^T E^q \phi' \int_\Omega e^\Phi u_t^2 dxdt$$
$$\leq \varepsilon c \int_\Omega E^{q+1}\phi' dt + c(\varepsilon) \int_S^T E^q \left(-E'\right) dt$$
$$\leq \varepsilon c \int_\Omega E^{q+1}\phi' dt + c(\varepsilon) E(S)^{q+1}, \tag{72}$$

and

$$\int_S^T E^q \phi' \int_\Omega e^\Phi uz(x,1,t) dxdt$$
$$\leq \varepsilon_1 \int_S^T E^q \phi' \int_\Omega e^\Phi u^2 dxdt + c(\varepsilon_1) \int_S^T E^q \phi' \int_\Omega e^\Phi z(x,1,t)^2 dxdt$$
$$\leq \varepsilon_1 c \int_{\mathbb{R}^n} E^{q+1}\phi' dt + c(\varepsilon_1) \int_S^T E^q \phi' \int_\Omega e^\Phi z(x,1,t)^2 dxdt$$
$$\leq \varepsilon_1 c \int_\Omega E^{q+1}\phi' dt + c(\varepsilon_1) \int_S^T E^q \left(-E'\right) dt$$
$$\leq \varepsilon_1 c \int_\Omega E^{q+1}\phi' dt + cE^{q+1}(S). \tag{73}$$

Choosing ε and ε_1 small enough, we deduce from (70), (72) and (73) that

$$\int_S^T E^{q+1}\phi' dt \leq cE^{q+1}(S),$$

where c is a positive constant independent of $E(0)$. Hence, we deduce from Lemma 4 that

$$E(t) \leq cE(0)e^{-\omega\tilde{\sigma}(t)}, \quad t \geq 0.$$

This ends the proof of Theorem 2. □

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Appendix

We now state some lemmas that we previously used (see proofs in [8, 14]).

Lemma 3 (Sobolev-Poincaré's inequality) *Let q be a number with $2 \leq q < +\infty$ ($n = 1, 2$) or $2 \leq q \leq 2n/(n-2)$ ($n \geq 3$). Then there is a constant $c_* = c_*(\Omega, q)$*

such that

$$\|u\|_q \leq c_* \|\nabla u\|_2 \quad \textit{for } u \in H_0^1(\Omega).$$

Lemma 4 ([14]) *Let* $E : \mathbb{R}_+ \to \mathbb{R}_+$ *be a non increasing function and* $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ *be an increasing* C^1 *function such that*

$$\phi(0) = 0 \quad \textit{and} \quad \phi(t) \to +\infty \quad \textit{as } t \to +\infty.$$

Assume that there exist $\sigma \geq 0$ *and* $\omega > 0$ *such that*

$$\int_S^{+\infty} E^{1+\sigma}(t)\phi'(t)dt \leq \frac{1}{\omega} E^{\sigma}(0)E(S), \quad 0 \leq S < +\infty. \tag{74}$$

Then

$$E(t) \leq E(0)\left(\frac{1+\sigma}{1+\omega\sigma\phi(t)}\right)^{1/\sigma} \quad \forall t \geq 0, \ \textit{if } \sigma > 0, \tag{75}$$

$$E(t) \leq cE(0)e^{1-\omega\phi(t)} \quad \forall t \geq 0, \ \textit{if } \sigma = 0. \tag{76}$$

In order to state the last lemma, we follow [8, 9] to introduce the function $h : \mathbb{R}^+ \to \mathbb{R}^+$. Let r be a non-negative real number, α a strictly positive real number, $\omega : \mathbb{R}^+ \to \mathbb{R}^{+*}$ and $\lambda : \mathbb{R}^+ \to \mathbb{R}^+$ two continuous functions. We set $\tilde{\lambda}(t) = \int_0^t \lambda(\tau)d\tau$ and find

$$\int_0^{+\infty} e^{(r+1)\tilde{\lambda}(t)}dt = +\infty. \tag{77}$$

For fixed $s \in \mathbb{R}^+$, we define the function $I_s : \mathbb{R}^+ \to \mathbb{R}$ by

$$I_s(t) = \big(\omega(s)\big)^{r+1} \int_s^t e^{(r+1)\tilde{\lambda}(\tau)}d\tau - e^{(r+1)\tilde{\lambda}(s)}\left(\big(\alpha\omega(0)\big)^r + r\int_0^s \big(\omega(\tau)\big)^{r+1}d\tau\right).$$

We have: $I_s \in C^1(\mathbb{R}^+)$, $I_s'(t) = (\omega(s))^{r+1}e^{(r+1)\tilde{\lambda}(t)} > 0$,

$$I_s(0) = \big(\omega(s)\big)^{r+1} \int_s^0 e^{(r+1)\tilde{\lambda}(\tau)}d\tau - e^{(r+1)\tilde{\lambda}(s)}\left(\big(\alpha\omega(0)\big)^r + r\int_0^s \big(\omega(\tau)\big)^{r+1}d\tau\right) < 0$$

and from (77) $\lim_{t\to+\infty} I_s(t) = +\infty$. Therefore I_s has a unique root in $\mathbb{R}^{+*}$ which will be noted $g(s)$ whence we define $g : \mathbb{R}^+ \to \mathbb{R}^{+*}$ by

$$I_s\big(g(s)\big) = 0, \quad \forall s \geq 0. \tag{78}$$

On the other hand, g is continuous due to the continuity of ω. Also we have

$$I_s(s) = -e^{(r+1)\tilde{\lambda}(s)}\left(\big(\alpha\omega(0)\big)^r + r\int_0^s \big(\omega(\tau)\big)^{r+1}d\tau\right) < 0,$$

hence $g(s) > s$, and $\lim_{s\to+\infty} g(s) = +\infty$. Therefore, g is surjective from $\mathbb{R}^+$ to $[g(0), +\infty[$. Now let $t \in]g(0), +\infty[$ be fixed. We define the function $J_t : [0,t] \to \mathbb{R}^+$ by

$$J_t(s) = \left(\int_s^t e^{\tilde{\lambda}(\tau)} d\tau\right) e^{\int_0^s \omega(\tau)d\tau} \quad \text{if } r = 0,$$

$$J_t(s) = \left(\int_s^t e^{(r+1)\tilde{\lambda}(\tau)} d\tau\right)\left(\big(\alpha\omega(0)\big)^r + r\int_0^s \big(\omega(\tau)\big)^{r+1} d\tau\right)^{1/r} \quad \text{if } r > 0.$$

The function J_t is positive and differentiable on $[0,t]$ and we have:

$$J_t'(s) = I_s(t) e^{\int_0^s \omega(\tau)d\tau} \quad \text{if } r = 0,$$

$$J_t'(s) = I_s(t)\left(\big(\alpha\omega(0)\big)^r + r\int_0^s \big(\omega(\tau)\big)^{r+1} d\tau\right)^{1/r-1} \quad \text{if } r > 0.$$

Since $J_t'(s)$ has the same sign as $I_s(t)$, then $J_t' > 0$ holds on the right of 0 (because $t > g(0)$) and on the left of 0 (because $g(s) > s$). Then J_t has a maximum on $[0,t]$ at least in one point $s_0 \in]0,t[$ satisfying $I_{s_0}(t) = 0$, hence $s_0 \in g^{-1}(\{t\})$.

Now, we define the function $h : \mathbb{R}^+ \to \mathbb{R}^+$ by:

$$h(t) = \begin{cases} 0 & \text{if } t \in [0, g(0)], \\ \max g^{-1}(\{t\}) & \text{if } t \in]g(0), +\infty]. \end{cases} \tag{79}$$

We have, for all $t > g(0) : h(t) \in g^{-1}(\{t\})$ and $I_{h(t)}(t) = 0$.

If ω is a constant, then g is an increasing function (it suffices to derive the equality (78)) and in this case

$$h(t) = \begin{cases} 0 & \text{if } t \in \left[0, D^{-1}\left(\dfrac{\alpha^r}{\omega}\right)\right], \\ g^{-1}\big(\{t\}\big) = K^{-1}\big(D(t)\big) & \text{if } t \in \left]D^{-1}\left(\dfrac{\alpha^r}{\omega}\right), +\infty\right], \end{cases}$$

where K and D are two functions defined on $\mathbb{R}^+$ by

$$K(t) = D(t) + e^{(r+1)\tilde{\lambda}(t)}\left(rt + \frac{\alpha^r}{\omega}\right), \quad D(t) = \int_0^t e^{(r+1)\tilde{\lambda}(\tau)} d\tau.$$

Lemma 5 ([8]) *Let $E : \mathbb{R}_+ \to \mathbb{R}_+$ be a differentiable function, $\lambda \in \mathbb{R}^+$, $a_3 \in \mathbb{R}^+$, $a_1, a_2 : \mathbb{R}^+ \to \mathbb{R}^{+*} = (0, +\infty)$ and $\lambda : \mathbb{R}^+ \to \mathbb{R}^+$ three continuous functions. Assume that there exist $r, p \geq 0$ such that*

$$a_3(r+1)\sup_{t\geq 0}\big\{\lambda(t)\big\} < 1$$

and for all $0 \le s \le T < +\infty$

$$\begin{cases} \int_s^T E^{r+1}(t)dt \le a_1 E(s) + a_2 E^{p+1}(s) + a_3 E^{r+1}(T), & \forall 0 \le s \le T, \\ E'(t) \le \lambda(t) E(t), & \forall t \ge 0. \end{cases} \tag{80}$$

Then E satisfies the following estimate:

$$E(t) \le \frac{E(0)}{\omega(0)} \omega\big(h(t)\big) e^{\tilde{\lambda}(t) - \tilde{\lambda}(h(t))} e^{-\int_0^{h(t)} \omega(\tau) d\tau} \quad \textit{if } r = 0, \tag{81}$$

$$E(t) \le \omega\big(h(t)\big) e^{\tilde{\lambda}(t) - \tilde{\lambda}(h(t))} \times \left(\left(\frac{E(0)}{\omega(0)} \right)^r + r \int_0^{h(t)} \big(\omega(\tau)\big)^{r+1} d\tau \right)^{-1/r} \quad \textit{if } r > 0, \tag{82}$$

where $\tilde{\lambda}(t) = \int_0^t \lambda(\tau) d\tau$, *h is defined by* (79) *with* $\alpha = \frac{1}{E(0)}$ *and* $\omega = \frac{1}{a}$ *such that*

$$a(s) = \frac{a_1(s) + a_2(s)(d(s))^p + a_3(s)(d(s))^r}{1 - a_3(r+1)\sup_{t \ge 0}\{\lambda(t)\}} \tag{83}$$

with

$$d(s) = \min\left\{ E(0) e^{\tilde{\lambda}(s)}, \left(\frac{b(0)E(0)}{f_0(s)} \right)^{1/(r+1)} \right\},$$

$$f_0(s) = e^{-(r+1)\tilde{\lambda}(s)} \int_0^s e^{(r+1)\tilde{\lambda}(\tau)} d\tau$$

and

$$b(s) = \frac{a_1(s) + a_2(s) E^p(s) + a_3(s) E^r(s)}{1 - a_3(r+1)\sup_{t \ge 0}\{\lambda(t)\}}, \quad \forall s \ge 0.$$

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Chapter 2
Non-uniqueness and Uniqueness in the Cauchy Problem of Elliptic and Backward-Parabolic Equations

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Abstract In this paper we consider the non-uniqueness and the uniqueness property for the solutions to the Cauchy problem for the operators

$$\mathcal{E}u = \partial_t^2 u + \sum_{k,l=1}^{n} \partial_{x_k}\big(a_{kl}(t,x)\partial_{x_l}u\big) + \beta(t,x)\partial_t u + \sum_{m=1}^{n} b_m(t,x)\partial_{x_m}u + c(t,x)u$$

and

$$\mathcal{P}u = \partial_t u + \sum_{k,l=1}^{n} \partial_{x_k}\big(a_{kl}(t,x)\partial_{x_l}u\big) + \sum_{m=1}^{n} b_m(t,x)\partial_{x_m}u + c(t,x)u,$$

where $\sum_{k,l=1}^{n} a_{kl}(t,x)\xi_k\xi_l|\xi|^{-2} \geq a_0 > 0$. We study non-uniqueness and uniqueness in dependence of global and local regularity properties of the coefficients of the principal part. The global regularity will be ruled by the modulus of continuity of a_{kl} on $[0,T]$ while the local regularity will concern a bound on $|\partial_t a_{kl}(t,x)|$ on every interval $[\varepsilon,T] \subseteq (0,T]$. By suitable counterexamples we show that our conditions seem to be sharp in many cases and we compare our statements with known results in the theory of hyperbolic Cauchy problems. We make also some remarks on continuous dependence for $\mathcal{P}$.

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2.1 Introduction

In this paper we collect some results on non-uniqueness and uniqueness for the solutions to the Cauchy problem for elliptic and backward-parabolic operators.

The subject of uniqueness and non-uniqueness in the Cauchy problem has a fairly long history and, from the pioneering works of Carleman [4] up to today, a huge number of results have been obtained and many different aspects of this topic have been developed (see e.g. [45] for a not so recent bibliography).

Here we are interested in two restricted classes of differential operators, precisely second order elliptic operators of the type

$$\mathcal{E}u = \partial_t^2 u + \sum_{k,l=1}^{n} \partial_{x_k}\big(a_{kl}(t,x)\partial_{x_l}u\big) + \beta(t,x)\partial_t u + \sum_{m=1}^{n} b_m(t,x)\partial_{x_m}u + c(t,x)u$$

and backward-parabolic operators of the type

$$\mathcal{P}u = \partial_t u + \sum_{k,l=1}^{n} \partial_{x_k}\big(a_{kl}(t,x)\partial_{x_l}u\big) + \sum_{m=1}^{n} b_m(t,x)\partial_{x_m}u + c(t,x)u.$$

In both cases we will suppose that $\sum_{k,l=1}^{n} a_{kl}(t,x)\xi_k\xi_l \geq a_0|\xi|^2$, with $a_0 > 0$.

We say that $\mathcal{E}$ or $\mathcal{P}$ have the *uniqueness* property (in the space X, with respect to the oriented surface $\{t \geq 0\}$, in the point 0) if for every function $u \in X$, from the fact that u is a solution to $\mathcal{E}u = 0$ or $\mathcal{P}u = 0$ with $\mathrm{supp}(u) \subseteq \{t \geq 0\}$, it follows that $u = 0$ in a neighborhood of 0.

In turn, by *non-uniqueness* for $\mathcal{E}$ or $\mathcal{P}$ we mean that we are able to find a non-zero solution $u \in X$ to $\mathcal{E}u = 0$ or $\mathcal{P}u = 0$ such that $0 \in \mathrm{supp}(u) \subseteq \{t \geq 0\}$.

Our aim is to study the connections between the properties of non-uniqueness and uniqueness with the regularity of the coefficients of the principal part of the operators under consideration. The results of Hörmander [26, 27] and J.-L. Lions and Malgrange [34] guarantee that Lipschitz regularity for a_{kl} is sufficient for uniqueness for $\mathcal{E}$ and $\mathcal{P}$ respectively, while the counterexample of Pliš [39] and some easy modifications of it (see for instance [19]) show that non-uniqueness can occur for some particular $\mathcal{E}$ and $\mathcal{P}$ having $a_{kl} \in \bigcap_{0<\alpha<1} C^\alpha$.

The investigation we want to develop will be in the narrow interval between these two bounds and the regularity of the coefficients will be measured from two points of view: using the notion modulus of continuity (we will call it *global regularity* property) and controlling the oscillation of the coefficients (this will be called *local regularity* property).

The idea to control the oscillations of the coefficients of the principal part originates from the technique of construction of all of the known counterexamples to uniqueness. In all these constructions the uniqueness property is destroyed by sufficiently fast oscillations of the principal part coefficients around a single point. Away from this point the coefficients are smooth. Hence it is natural to think that a bound on the oscillations of the coefficients might restore the uniqueness property.

In the case of the hyperbolic Cauchy problem, the interaction of global and local regularity conditions have been extensively studied with respect to the well-posedness of the problem. The Cauchy problem for elliptic and backward-parabolic operators is severally ill-posed but, anyhow, besides this difference, we would like to compare briefly the conditions for uniqueness and non-uniqueness for elliptic and backward-parabolic to those for well-posedness in the hyperbolic theory.

Considering the hyperbolic Cauchy problem

$$\begin{cases} \mathcal{L}u = \partial_t^2 u - a(t)\partial_x^2 u = 0 \\ u(0,x) = u_0(x), u_t(0,x) = u_1(x), \end{cases} \tag{1}$$

where $a(t) \geq a_0 > 0$, it is well known (see e.g. Chap. IX in [28]) that (1) is C^∞-well-posed if one supposes $a \in \mathrm{Lip}[0,T]$. In [10] the Cauchy problem (1) was studied under the condition

$$\sup_{\substack{0<|t-s|<1 \\ t,s\in[0,T]}} \frac{|a(t)-a(s)|}{|t-s||\log(|t-s|)|} \leq C < +\infty \tag{2}$$

and (2) was proved to be sufficient for C^∞-well-posedness with the so called *loss of derivatives* (this means that there is a shifting between the Sobolev norms in the energy estimates for $\mathcal{L}$, see [32] for a detailed explanation of this phenomenon). Condition (2) means that a is *globally regular* with respect to the Log-Lipschitz modulus of continuity, for short $a \in \mathrm{LogLip}[0,T]$. A counterexample in [9] shows that one cannot weaken this condition without further assumptions (see also [42] for a recent interesting improvement of [10]).

A second possibility to weaken the Lipschitz property of a in (1) goes back to [12] where the notion of *local regularity* was first explicitly introduced (see also [43]). Precisely in [12] the coefficient a was in $C^0[0,T] \cap C^1(0,T]$ with

$$\sup_{t\in[0,T]} \left|ta'(t)\right| \leq C < +\infty. \tag{3}$$

Under this hypothesis there is again the C^∞-well-posedness with loss of derivatives and some counterexamples similar to those of [9] show that this assumption can be considered optimal.

Supposing more regularity for a away from $t = 0$, some other interesting results have been obtained. It has been proved (see e.g. [25, 43]) that one gets C^∞-well-posedness without loss of derivatives under the condition $a \in C^0[0,T] \cap C^1(0,T] \cap C^2(0,T]$ with

$$\sup_{t\in[0,T]} \left|ta'(t)\right| + \left|t^2 a''(t)\right| \leq C < +\infty,$$

while the hypothesis (see [13]) for the same result with loss of derivatives is

$$\sup_{t\in[0,T]} \left|\big(t\log(t)\big)a'(t)\right| + \left|\big(t\log(t)\big)^2 a''(t)\right| \leq C < +\infty.$$

A list of counterexamples shows that these results are sharp and, at the present, even if it should be reasonable that supposing more regularity on a (e.g. a is in $C^m(0, T]$ for $m \geq 3$) some different and weaker conditions on derivatives of a should ensure C^∞-well-posedness, only the C^2-theory has been developed.

The effect that one can weaken the assumption on a' by a logarithm by assuming a condition on a'' is called the *Log-effect*. This is connected to the classification of oscillations introduced by Reissig and Yagdjian in [40] and also studied for other types of operators, as p-evolution operators in [5]. The reader may also consult [44] for more on the matter and related questions.

In [12, 22, 32] the authors have studied the possible couplings between the global regularity and the local regularity. They have, for example, proved that a coefficient with a modulus of continuity $f(s) = s\frac{\mu(s)}{\eta(s)}$, worse than Log-Lipschitz, needs a control of oscillations precisely by $-C\frac{d}{dt}\mu(\eta^{-1}(t))$ to guarantee well-posedness in some scales of Sobolev spaces. For further information we refer to the cited papers and the references therein.

Finally we refer to [30] for a more exhaustive comparison of the hyperbolic theory to the elliptic and backward parabolic theory with respect to the question of uniqueness in the Cauchy problem.

The paper is organized as follows; in Sect. 2.2 we state several non-uniqueness theorems for $\mathcal{E}$ and $\mathcal{P}$ modeled on the well-known Pliš example in [39]. We will state theorems with global and local assumptions on the principal part coefficients and we will give an example of non-uniqueness for coefficients with non-Osgood global regularity and a certain control of the oscillations, similar to the one in [32] for hyperbolic operators.

The non-uniqueness theorems under local conditions on the principal part coefficients will show that the Log-effect does not appear in the case of elliptic and backward-parabolic operators, i.e. it is not possible to weaken the condition $\sup_{t\in[0,T]} |ta'(t)| \leq C$ by adding a condition on the second derivative. Hence, under local conditions only C^1-theory is interesting for the uniqueness of the Cauchy problem for our operators.

At the end of the section we give an outline of the construction of those counterexamples with the various changes according to the different types of conditions.

Section 2.3 contains the uniqueness counterpart to Sect. 2.2. In the first sub-part of this section we recall some important results about the uniqueness in the Cauchy problem for $\mathcal{E}$ and $\mathcal{P}$ under global regularity conditions.

The next part contains the statement and the proof of a uniqueness result for $\mathcal{E}$ and $\mathcal{P}$ under a local condition like (3) with a smallness condition on the constant. We also note how one can slightly weaken this smallness condition if one restricts the uniqueness results to solutions in certain Gevrey classes, where the Gevrey-index depends on the size of the constant.

In the third sub-section of this section we state some uniqueness theorems for $\mathcal{P}$ under the assumption $\sum_{k,l=1}^{n} a_{kl}(t,x)\xi_k\xi_l|\xi|^{-2} \geq 0$. They complement in some sense the results under local conditions in the case of degenerate backward-parabolicity. The corresponding theorems for degenerate elliptic operators are

Table 2.1 Comparison between hyperbolic and elliptic/backward-parabolic operators with respect to uniqueness in the Cauchy problem

Hyperbolic theory	Elliptic/backward-parabolic theory
$\mu(s) = s^{1-\alpha}$, $\alpha \in (0,1)$	
$\lvert a'(t)\rvert \le Ct^{-(1/2)(1+1/\alpha)}$	$\lvert a'(t)\rvert \le Ct^{-1}$
$\lvert a''(t)\rvert \le Ct^{-(1+1/\alpha)}$	$\lvert a^{(k)}(t)\rvert \le Ct^{-((k-1)(1/\alpha)+k)}$, for all $k \ge 1$
$\Rightarrow$ In the elliptic/backward-parabolic theory only C^1-theory is interesting	
Log-effect	
$\lvert a'(t)\rvert \le C\frac{1}{t}\log(\frac{1}{t})$	No Log-effect
$\lvert a''(t)\rvert \le C\frac{1}{t^2}(\log(\frac{1}{t}))^2$	
Constant in the estimate of a'	
$C \in \mathbb{R}_{>0}$	$C_{ell} \in (0, 2a_0)$ and $C_{bp} \in (0, a_0)$

proved in [6, 7, 11, 36]. The last part of this section is devoted to some open problems and an out-view to possible further developments.

Table 2.1 summarizes very shortly the main differences between the hyperbolic and the elliptic/backward-parabolic operators concerning uniqueness in the Cauchy problem which are proved in Sects. 2.2 and 2.3.

The first part of the table shows that for elliptic and backward-parabolic operators the additional assumption of Hölder regularity for the principal part coefficients brings nothing with respect to the allowed oscillations of the coefficient in contrast to the hyperbolic theory. For this see Sect. 2.2.2. Besides the Log-effect, for which the reader finds a more detailed discussion in Sect. 2.2.2, it illustrates that only in the elliptic and backward-parabolic regime appears a restriction on the size of the constant in the control of the oscillations, see also Sects. 2.2.2 and 2.3.1.

In the last section of the present work we summarize some results about continuous dependence of solutions to $\mathcal{P}$ and $\mathcal{E}$ on the Cauchy data in the sense of John (see [31]). These results are only concerned with global regularity.

2.1.1 Modulus of Continuity and Related Oscillation Conditions

In this section we state some definitions which we need in the subsequent sections.

Definition 1.1 (Modulus of continuity, C^μ) We call a continuous, concave, increasing function $\mu : [0, s_0] \to [0, 1]$, $s_0 > 0$, a modulus of continuity. A function $f \in C^0(Q)$, $Q \subseteq \mathbb{R}^n$ belongs to $C^\mu(Q)$ iff

$$\exists C > 0 : \sup_{\substack{0<|x-y|\le s_0 \\ x,y\in Q}} \frac{|f(x) - f(y)|}{\mu(|x - y|)} \le C < +\infty.$$

The next definition introduces the notion of the Osgood condition. The Osgood condition first appeared in [38] where Osgood studied the uniqueness of solutions of ordinary differential equations without the Lipschitz condition.

Definition 1.2 Osgood condition A modulus of continuity is said to satisfy the Osgood condition if there exists an $s_0 > 0$ such that

$$\int_0^{s_0} \frac{ds}{\mu(s)} = +\infty. \tag{4}$$

If there exist an $s_0 > 0$ such that condition (4) fails to hold we will call μ a non-Osgood modulus of continuity.

For the sake of brevity we introduce some symbols for moduli of continuity which we are going to use in the subsequent part:

$$\begin{aligned}
&\mathrm{Log}^{-1} : \mu(s) = \left(\log\left(\frac{1}{s}\right)\right)^{-1},\\
&C^{\alpha} : \mu(s) = s^{\alpha}, \quad \alpha \in (0,1),\\
&\mathrm{Lip} : \mu(s) = s,\\
&\mathrm{Log}^{1+\varepsilon}\,\mathrm{Lip} : \mu(s) = s\left(\log\left(\frac{1}{s}\right)\right)^{1+\varepsilon},\\
&\mathrm{Log}^{[m,1+\varepsilon]}\,\mathrm{Lip} : \mu(s) = s\left(\prod_{i=1}^{m-1} \log^{[i]}\left(\frac{1}{s}\right)\right)\left(\log^{[m]}\left(\frac{1}{s}\right)\right)^{1+\varepsilon}.
\end{aligned}$$

We define $\log^{[i]}(s) := \log(\log^{[i-1]}(s))$ with $\log^{[1]}(s) = \log(s)$. The last three lines of the list above are, with $\varepsilon = 0$, examples for Osgood moduli of continuity. If one takes $\varepsilon > 0$ they are examples for non-Osgood moduli of continuity.

Definition 1.3 (Osgood distance function) For a non-Osgood modulus of continuity μ we associate a function

$$\eta(t) := \int_0^t \frac{ds}{\mu(s)}. \tag{5}$$

Remark 1.1 The function η measures essentially how far the modulus of continuity is from an Osgood modulus of continuity. The *velocity* of the function $\eta(t)$ going to 0 for $t \to 0+$ carries this information. The slower this function converges to 0 the closer is μ to an Osgood modulus of continuity.

Remark 1.2 Another way to illustrate how the function η, defined by (5) for a non-Osgood modulus of continuity, measures the difference between μ and an Osgood modulus of continuity is to say that $\sigma(s) := \mu(s)\eta(s)$ is an Osgood modulus of continuity. This can be seen as follows:

$$\int_0^{s_0} \frac{ds}{\sigma(s)} = \lim_{\epsilon\to 0+} \int_\varepsilon^{s_0} \frac{ds}{\mu(s)\eta(s)} = \lim_{\epsilon\to 0+} \int_\varepsilon^{s_0} \frac{\eta'(s)}{\eta(s)} ds$$
$$= \lim_{\epsilon\to 0+} \left(\log\big(\eta(s_0)\big) + \log\left(\frac{1}{\eta(\varepsilon)}\right)\right) = +\infty,$$

where we have used the fact that $\eta'(t) = \frac{1}{\mu(t)}$ for $t > 0$.

Throughout the paper we denote all $C^\infty(Q)$ functions bounded with all their derivatives by $B^\infty(Q)$. A function defined on the n-dimensional torus $\mathbb{T}^n$ will as usual be considered as a periodic function on $\mathbb{R}^n$.

2.2 Non-uniqueness

In this section we state some counterexamples to uniqueness in the Cauchy problem for elliptic and backward-parabolic operators. Actually we will state them just for elliptic operators but they are literally also true if one replaces the elliptic by a backward-parabolic operator. In the last part of this section we will show the general scheme how to construct such counterexamples of Pliš-type.

The counterexamples will also show that the so-called Log-effect, known from the hyperbolic theory or the theory of p-evolution operators, does not occur in the Cauchy problem for elliptic and backward-parabolic operators.

2.2.1 Non-uniqueness Under Global Conditions

The first counterexample to uniqueness in the Cauchy problem for elliptic operators was presented by Pliš in [39] and it came along as quite a surprise. It shows that certain amount of global regularity is necessary for the uniqueness in the Cauchy problem for elliptic operators (and apparently also for backward-parabolic operators), at least if one regards the problem in more than two dimensions. In two dimensions the situation is different. For more information on this matter the reader may consult [3, 45] and the references therein.

The original result of Pliš is

Theorem 2.1 (Theorem 1 in [39]) *There exist five real-valued functions u, a, f, g, h such that the PDE*

$$\mathcal{E}u = \frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial x^2} + a(t)\frac{\partial^2 u}{\partial y^2} + f(t,x,y)\frac{\partial u}{\partial x} + g(t,x,y)\frac{\partial u}{\partial y} + h(t,x,y)u$$
$$= 0 \qquad (6)$$

is satisfied on $\mathbb{R}^3$. *Furthermore, the solution* u *vanishes identically for* $t \geq 0$ *but does not vanish identically in any neighborhood of* $t = 0$. *The coefficient* a *satisfies* $\frac{1}{2} \leq a(t) \leq \frac{3}{2}$ *belongs to* $C^\infty(\mathbb{R} \setminus \{0\}) \cap \bigcap_{0<\alpha<1} C^\alpha(\mathbb{R})$ *and the functions* u, f, g, *and* h *belong to* $B^\infty(\mathbb{R}^3)$.

Remark 2.1 In [39] Pliš proved in fact a little more than he claimed. The coefficient a which he constructed in his proof is not just in $\bigcap_{0<\alpha<1} C^\alpha(\mathbb{R})$ but in $\mathrm{Log}^2\mathrm{Lip}(\mathbb{R})$. Additionally the first derivative of a satisfies the bound $|t^2 a'(t)| \leq C$ for $t < 0$ and some $C > 0$.

Later Tarama proved in [41] local uniqueness for elliptic operators whose principal coefficients are not Lipschitz-continuous (see Sect. 2.3.1). In fact they have a modulus of continuity μ satisfying the Osgood condition (see Definition 1.2). A counterexample in [15] shows that this condition cannot be weakened from the point of view of global regularity. Precisely it states:

Theorem 2.2 (Theorem 2 in [15]) *Let* μ *be a non-Osgood modulus of continuity. Then there exist five real-valued functions* u, a, f, g, h *such that the PDE*

$$\mathcal{E}u = \frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial x^2} + a(t)\frac{\partial^2 u}{\partial y^2} + f(t,x,y)\frac{\partial u}{\partial x} + g(t,x,y)\frac{\partial u}{\partial y} + h(t,x,y)u = 0$$

is satisfied on $\mathbb{R}^3$. *Furthermore, the solution* u *vanishes for* $t \geq 0$ *but does not vanish identically in any neighborhood of* $t = 0$. *The coefficient* a *satisfies* $1 \leq a(t) \leq 2$, *belongs to* $C^\mu(\mathbb{R}) \cap C^\infty(\mathbb{R} \setminus \{0\})$ *and the functions* u, f, g *and* h *belong to* $B^\infty(\mathbb{R})$.

2.2.2 Non-uniqueness Under Local Conditions

In this section we state a non-uniqueness example under a local condition on the derivatives of the principal part coefficients. This is given by

Theorem 2.3 *There exist five real-valued functions* u, a, f, g, h *such that the PDE*

$$\mathcal{E}u = \frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial x^2} + a(t)\frac{\partial^2 u}{\partial y^2} + f(t,x,y)\frac{\partial u}{\partial x} + g(t,x,y)\frac{\partial u}{\partial y} + h(t,x,y)u = 0$$

is satisfied on $\mathbb{R}^3$. *Furthermore, the solution* u *vanishes for* $t \geq 0$ *but does not vanish identically in any neighborhood of* $t = 0$. *The coefficient* a *satisfies* $1 \leq a(t) \leq 2$, *belongs to* $C^0(\mathbb{R}) \cap C^\infty(\mathbb{R} \setminus \{0\})$ *and satisfies*

$$\forall k \geq 1\ \exists C_k > 0 : \left| \frac{d^k a}{dt^k}(t) \right| \leq C_k |t|^{-k} \quad \forall t < 0.$$

The functions u, f, g, *and* h *belong to* $B^\infty(\mathbb{R}^3)$.

Remark 2.2 From Theorem 2.3 we detect some interesting differences between the elliptic and backward-parabolic case on one side and the hyperbolic case on the other side.

Firstly in the hyperbolic case a control of the type Ct^{-1} on the first time-derivatives of the coefficients second order terms gives the well-posedness (see [12]). Here if the constant in front of t^{-1} is sufficiently large we can construct a counterexample to uniqueness (see the Theorems 3.5 and 3.6 for an estimate of this constant).

Secondly in the hyperbolic case a control of the type $C_1 t^{-1}\log(t^{-1})$ on the first time-derivatives of the second order terms and of $Ct^{-2}(\log(t^{-1}))^2$ on the second time-derivatives ensure the well-posedness (it is the so called log-*effect*, see [13, 25]). Here we can construct a counterexample to uniqueness with a control of $C_1 t^{-1}$ and $C_2 t^{-2}$ respectively. So we cannot hope for a Log-*effect*, and it is not only a matter of the choice of the constants.

2.2.3 Non-uniqueness Under a Mixed Condition

The result we are going to state in this section mixes the two kinds of conditions we have focused on in the last two sections. The interesting point is how the regularity and the oscillations interact. See also Sect. 2.3.4 for further explanations and the discussion of a uniqueness counterpart for this theorem.

Theorem 2.4 *Let μ be a non-Osgood modulus of continuity. Then there exist five real-valued functions u, a, f, g, h such that the PDE*

$$\mathcal{E}u = \frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial x^2} + a(t)\frac{\partial^2 u}{\partial y^2} = f(t,x,y)\frac{\partial u}{\partial x} + g(t,x,y)\frac{\partial u}{\partial y} + h(t,x,y)u = 0$$

is satisfied on $\mathbb{R}^3$. Furthermore, the solution u vanishes identically for $t \geq 0$ but does not vanish identically in any neighborhood of $t = 0$. The coefficient a satisfies $\frac{1}{2} \leq a(t) \leq \frac{3}{2}$, belongs to $C^\mu(\mathbb{R}) \cap C^\infty(\mathbb{R} \setminus \{0\})$ and satisfies

$$\forall k \geq 1\ \exists C_k > 0 : \left|\frac{d^k a}{dt^k}(t)\right| \leq C_k \frac{(\mu(\eta^{-1}(|t|)))^k}{(\eta^{-1}(|t|))^{2k-1}} \quad \forall t < 0, \tag{7}$$

where η is the Osgood distance function (5). The functions u, f, g and h belong to $B^\infty(\mathbb{R}^3)$.

Remark 2.3 Formula (7) gives for all moduli of continuity defining spaces beneath $\bigcap_{0<\alpha<1} C^\alpha$ just the control t^{-1} for the first derivative. Hence, to think about a possible positive result with a weaker control on a' we need more regularity. The Theorems 3.5 and 3.6 show, as a counterpart to Theorem 2.3, that a control of t^{-1} ensures uniqueness without any additional regularity if the constant in the estimate is sufficiently small.

Table 2.2 Some examples of oscillation conditions related to moduli of continuity

$a \in$	$\lvert a'(t)\rvert \lesssim$	$\lvert a^{(k)}(t)\rvert \lesssim$
C^0	t^{-1}	t^{-k}
Log^{-1}	t^{-1}	$t^{-(2k-1)}$
C^α	t^{-1}	$t^{-((k(2-\alpha)-1)/(1-\alpha))}$
$Log^{1+\varepsilon}Lip$	$t^{-(1+1/\varepsilon)}$	$t^{-k(1+1/\varepsilon)}(\exp(t^{-1/\varepsilon}))^{k-1}$
$Log^{[m,1+\varepsilon]}Lip$	$t^{-k(1+1/\varepsilon)}\prod_{i=1}^{m-1}\exp^{[i]}(t^{-1/\varepsilon})$	$t^{-(1+1/\varepsilon)}(\prod_{i=1}^{m-1}\exp^{[i]}(t^{-1/\varepsilon})^k)$ $\times(\exp^{[m]}(t^{-1/\varepsilon}))^{k-1}$

Table 2.2 shows some moduli of continuity and the associated control of oscillations: $\alpha \in (0,1)$, $\varepsilon > 0$

2.2.4 Scheme of the Construction of the Counterexamples

In this section we present the scheme how to construct an operator

$$\mathcal{E}u = \frac{\partial^2}{\partial t^2}u + \frac{\partial^2}{\partial x^2}u + a(t)\frac{\partial^2}{\partial y^2}u + f(t,x,y)\frac{\partial}{\partial x}u + g(t,x,y)\frac{\partial}{\partial y}u + c(t,x,y)u$$
$$= 0 \tag{8}$$

with the properties stated in the theorems of the last three sections. We will not present every detail and the reader may consult the original paper of Pliš [39] or [30].

Step 1: Auxiliary Functions and Sequences Let $A(x)$, $B(x)$, $C(x)$, and $J(x)$ be elements of $B^\infty(\mathbb{R})$ with the following properties:

$$A(x) = 1 \quad \text{for } x \le \frac{1}{5}, \qquad A(x) = 0 \quad \text{for } x \ge \frac{1}{4},$$

$$B(x) = 0 \quad \text{for } x \le 0 \text{ or } x \ge 1, \qquad B(x) = 1 \quad \text{for } \frac{1}{6} \le x \le \frac{1}{2},$$

$$C(x) = 0 \quad \text{for } x \le \frac{1}{4}, \qquad C(x) = 1 \quad \text{for } x \ge \frac{1}{3},$$

$$J(x) = -2 \quad \text{for } x \le \frac{1}{6} \text{ or } x \ge \frac{1}{2}, \qquad J(x) = 2 \quad \frac{1}{5} \le x \le \frac{1}{3}.$$

In order to control the behavior of the solution and the coefficients of (8) we need two sequences $(a_n)_n$ and $(z_n)_n$ with the properties

$$-1 < a_n < a_{n+1} \quad \text{for all } n \ge 1, \qquad \lim_{n\to+\infty} a_n = 0,$$

$$1 < z_n < z_{n+1} \quad \text{for all } n \ge 1, \qquad \lim_{n\to+\infty} z_n = +\infty.$$

Furthermore, we define $r_n := a_{n+1} - a_n$, $q_1 := 0$, $q_n := \sum_{k=2}^{n} z_k r_{k-1}$ for all $n \geq 2$, and $p_n := (z_{n+1} - z_n) r_n$, where we suppose $p_n > 1$ for all $n \geq 1$. We transport the behavior of our auxiliary functions to the intervals $[a_n, a_{n+1}]$ where we shall construct our solution and the coefficients. We introduce

$$A_n(t) = A\left(\frac{t - a_{n+1}}{r_n}\right), \qquad B_n(t) = B\left(\frac{t - a_{n+1}}{r_n}\right), \qquad C_n(t) = C\left(\frac{t - a_{n+1}}{r_n}\right),$$
$$J_n(t) = J\left(\frac{t - a_{n+1}}{r_n}\right), \quad n \geq 1.$$

Step 2: Construction of a Solution u on $[a_n, a_{n+1}]$ We define the auxiliary functions

$$\begin{aligned} v_n(t, x) &= \exp\bigl(-q_n - z_n(t - a_{n+1})\bigr)\cos(z_n x), \\ w_n(t, y) &= \exp\bigl(-q_n - z_n(t - a_{n+1}) + J_n(t) p_n\bigr)\cos(z_n y) \end{aligned} \tag{9}$$

which are solutions of $u_{tt} + u_{xx} + a(t)u_{yy} = 0$ with a suitable coefficient a, which will be determined in Step 3 of the proof. To construct a solution of (8) we define

$$u(t, x, y) = \begin{cases} v_1(t, x) : t < a_1, \\ A_n(t) v_n(t, x) + B_n(t) w_n(t, y) + C_n(t) v_{n+1}(t, x) : t \in [a_n, a_{n+1}], \\ 0 : t \geq 0. \end{cases}$$

This function is obviously in $B^\infty(\mathbb{R}^3 \setminus \{0\})$. For u to be in $C^\infty(\mathbb{R}^3)$ the condition

$$\forall \alpha, \beta, \gamma \in \mathbb{N} : \left|\partial_t^\alpha \partial_x^\beta \partial_y^\gamma u(t, x, y)\right| \overset{t \to 0+}{\longrightarrow} 0$$

is necessary and sufficient. This will be implied by the condition

$$\lim_{n \to \infty} \exp(-q_n + 2p_n) z_{n+1}^\alpha p_n^\beta r_n^{-\gamma} = 0 \quad \forall \alpha, \beta, \gamma \in \mathbb{N}. \tag{10}$$

Step 3: Construction of a Suitable $a(t)$ We denote $\tilde{\mathcal{E}} u = \partial_t^2 u + \partial_x^2 u + a(t)\partial_y u$ and we get from (9) that $\tilde{\mathcal{E}} v_n = 0$ for all $n \geq 1$. In order to get $\tilde{\mathcal{E}} w_n = 0$ on $[a_n, a_{n+1}]$ for $n \geq 1$ we have to set

$$a(t) = \begin{cases} 1 : t < a_1 \text{ or } t \geq 0, \\ 1 - 2J_n'(t) p_n z_n^{-1} + [J_n'(t)]^2 p_n^2 z_n^{-2} \\ \qquad + J_n''(t) p_n z_n^{-2} : t \in [a_n, a_{n+1}] \end{cases} \tag{11}$$

The condition

$$\sup_{n \in \mathbb{N}} \bigl(p_n r_n^{-1} z_n^{-1} + p_n^2 r_n^{-2} z_n^{-2}\bigr) \leq \frac{1}{2(\|J'\|_{L^\infty} + \|J''\|_{L^\infty})} \tag{12}$$

ensures that $\mathcal{E}$ is elliptic and that $1 \leq a(t) \leq 2$.

Step 4: The Regularity of $a(t)$ *Global regularity*: In this step of the construction we show how one ensures that a has the modulus of continuity μ. By the mean value theorem we have to control $|a'(t)|$ globally. From (11) we get on $[a_n, a_{n+1}]$:

$$a'(t) = -2J_n''(t)p_n z_n^{-1} + 2J_n'(t)J_n''(t)p_n^2 z_n^{-2} + J_n'''(t)p_n z_n^{-2}$$

which we can estimate as

$$\left|a'(t)\right| \lesssim r_n^{-2} p_n z_n^{-1} + r_n^{-3} p_n^2 z_n^{-2} + r_n^{-3} p_n z_n^{-2} \lesssim r_n^{-2} p_n z_n^{-1} + r_n^{-3} p_n^2 z_n^{-2}.$$

In order to get the μ-continuity for a we need $|a'(t)| \lesssim \frac{\mu(r_n)}{r_n}$ on $[a_n, a_{n+1}]$ because in this case we will be able to establish the inequality

$$\left|a(s) - a(t)\right| \lesssim \frac{\mu(r_n)}{r_n}|t-s| \lesssim \mu\big(|t-s|\big) \quad \forall s, t \in [a_n, a_{n+1}],$$

where we use $|t-s| \le r_n$ for $s, t \in [a_n, a_{n+1}]$ and the fact that $\sigma \mapsto \frac{\mu(\sigma)}{\sigma}$ is decreasing. This will be implied by the condition

$$\sup_{n\in\mathbb{N}} \frac{r_n^{-1} p_n z_n^{-1}}{\mu(r_n)} \le C < +\infty. \tag{13}$$

Local regularity: Here we want to control the oscillations of a near $t = 0$ like in Theorem 2.3. First we derive from (11) an estimate for the behavior of the k-th derivative of a:

$$\left|a^{(k)}(t)\right| \lesssim \frac{p_n}{r_n^{k+1} z_n} + \frac{p_n}{r_n^{k+2} z_n^2} + k\frac{p_n^2}{r_n^{k+2} z_n^2} \lesssim \frac{p_n}{r_n^{k+1} z_n} + k\frac{p_n^2}{r_n^{k+2} z_n^2}.$$

It is enough to analyze the term $p_n r_n^{-(k+1)} z_n^{-1}$. The other term has a better behavior and can be handled in the same way. Our goal is now to ensure that a relation like

$$\left|F(t)a'(t)\right| = \mathcal{O}(1) \quad (t \to 0-) \tag{14}$$

holds for a certain function F. In order to do that, we have to express t in terms of our sequences. By the definition of our intervals, we can conclude that

$$t \sim -\sum_{k=n}^{+\infty} r_k.$$

Now condition (14) reads as follows

$$\sup_{n\in N}\left(F\left(-\sum_{k=n}^{+\infty} r_k\right)\frac{p_n}{r_n^{k+1} z_n}\right) \le C < +\infty, \quad \forall k \ge 1. \tag{15}$$

Remark 2.4 For Theorem 2.3 we have to take $F(t) = |t|^k$ and we have to find sequences such that

$$\sup_{n\in\mathbb{N}} \left(\sum_{j=n}^{+\infty} r_j\right)^k \frac{p_n}{r_n^{k+1} z_n} \leq C < +\infty, \quad \forall k \geq 1.$$

Remark 2.5 For Theorem 2.4 we have to take $F(t) = \frac{(\eta^{-1}(t))^{2k-1}}{(\mu(\eta^{-1}(|t|)))^k}$ with $\eta(t) := \int_0^t \frac{ds}{\mu(s)}$ and we have to find sequences such that

$$\sup_{n\in\mathbb{N}} \frac{(\eta^{-1}(t_n))^{2k-1}}{(\mu(\eta^{-1}(t_n)))^k} \frac{p_n}{r_n^{k+1} z_n} \leq C < +\infty, \quad \forall k \geq 1, \tag{16}$$

where $t_n := \sum_{j=n}^{+\infty} r_j$.

Step 5: Definition of Lower-Order Coefficients As lower order coefficients we define

$$f(t,x,y) := -\frac{\tilde{\mathcal{E}} u}{u^2 + (\partial_x u)^2 + (\partial_y u)^2} \frac{\partial}{\partial x} u,$$

$$g(t,x,y) := -\frac{\tilde{\mathcal{E}} u}{u^2 + (\partial_x u)^2 + (\partial_y u)^2} \frac{\partial}{\partial y} u,$$

$$h(t,x,y) := -\frac{\tilde{\mathcal{E}} u}{u^2 + (\partial_x u)^2 + (\partial_y u)^2} u.$$

This coefficients will belong to $C^\infty(\mathbb{R}^3)$ if

$$\forall \alpha, \beta, \gamma \in \mathbb{N} : \lim_{n\to+\infty} \exp(-p_n) z_{n+1}^\alpha p_n^\beta r_n^{-\gamma} = 0. \tag{17}$$

To finish the construction we give examples of sequences which fulfill the conditions (10), (12), (17) and (13) and/or (15) for a sufficiently large j:

- Pliš example: $a_n := (\log(n+j))^{-1}$, $z_n := (n+j)^3$,
- Non-Osgood regularity: $a_n := \sum_{l=n}^{+\infty} ((l+j)^2 \mu(\frac{1}{(l+j)}))^{-1}$, $z_n := (n+j)^3$,
- Oscillation control: $r_n := \rho^{-(n+j)}$, $z_n := \rho^{(n+j)}(n+j)\log(n+j)$,
- Mixing situation: $a_n := \sum_{l=n}^{+\infty} ((l+j)^2 \mu(\frac{1}{(l+j)}))^{-1}$, $z_n := (n+j)^3$.

With the choice of these sequences the construction of the counterexample is finished. As already mentioned the same construction (with small changes) also work for backward-parabolic operators, see [19, 30].

Remark 2.6 To prove (16) in the mixing situation it is essential to use the relation $t_n := \sum_{j=n}^{+\infty} r_j \sim \eta(\frac{1}{n})$. This reflects precisely the non-Osgood condition.

2.3 Uniqueness

In this section we present some complementary theorems to the theorems of Sect. 2.2. The history of uniqueness in the Cauchy problem for elliptic equations is fairly long and we attempt by no means to give a survey about this development. Here we focus on theorems which are more or less direct complements of our non-uniqueness theorems.

2.3.1 Uniqueness Under Global Conditions

First, we state results which are counterparts to the results in Sect. 2.2.1. The theorems, as proved in the original papers, hold mostly for more general solutions than stated here. But for our purpose the formulations presented here are sufficient.

In [26, 27] Hörmander has deeply investigated the question of uniqueness for the Cauchy problem for partial differential operators. One of the results is the uniqueness for solutions to elliptic operators with Lipschitz continuous coefficients in the principal part. It can be stated as

Theorem 3.1 *Suppose that* $a_{kl} \in \mathrm{Lip}([0,T] \times \mathbb{R}^n)$ *and* $\beta, b_m, c \in L^\infty([0,T] \times \mathbb{R}^n)$. *Then* $\mathcal{E}$ *has the* C^∞*-uniqueness property.*

In [41] Tarama proved that uniqueness in the Cauchy problem for elliptic operators still holds true if one weakens the Lipschitz-condition on the principal part coefficients and replaces it with the Osgood-condition. As the counterexamples of Sect. 2.2.1 show, this cannot be weakened without further assumptions. We remark again that the result of Pliš is in fact a result about the sharpness of the Osgood condition (see Remark 2.1). The result of Tarama can be stated as

Theorem 3.2 (Theorem 1.2 in [41]) *Suppose that* $a_{kl} \in C^\mu([0,T] \times \mathbb{R}^n)$ *with an Osgood modulus of continuity* μ; *suppose* $\beta, b_m, c \in L^\infty([0,T] \times \mathbb{R}^n)$. *Then* $\mathcal{E}$ *has the* C^∞*-uniqueness property.*

Remark 3.1 In comparison to the hyperbolic theory it is unknown weather the Osgood condition is sufficient for the uniqueness of the Cauchy problem or not. However, a counterexample to uniqueness in [8] shows at least that one cannot consider coefficients with regularity beneath the Osgood condition.

Similar results to Theorems 3.1 and 3.2 have been proved for the backward parabolic operator $\mathcal{P}$. The first paper in this direction was perhaps [34] where J.-L. Lions and Malgrange proved uniqueness for the solutions of the Cauchy problem for $\mathcal{P}$ under the condition that the principal part coefficients Lipschitz-continuous with respect to t and L^∞ with respect to x. Similar results can be found in [1, 2, 23, 35]. The result of J.-L. Lions and Malgrange can be stated as

Theorem 3.3 *Suppose, for $\mathcal{P}$, that $a_{kl} \in \mathrm{Lip}([0,T], L^\infty(\mathbb{R}^n))$; suppose $\beta, b_m, c \in L^\infty([0,T]\times\mathbb{R}^n)$. Then, $\mathcal{P}$ has the $\mathcal{H}$-uniqueness property, where*

$$\mathcal{H} := H^1\big([0,T], L^2\big(\mathbb{R}^n\big)\big) \cap L^2\big([0,T], H^2\big(\mathbb{R}^n\big)\big). \tag{18}$$

In [16, 19, 21] Del Santo and Prizzi proved that one can weaken the Lipschitz-regularity at least in the time variable to an Osgood modulus of continuity.

Theorem 3.4 (Theorem 1 in [19]) *Let μ be a Osgood modulus of continuity, suppose that the coefficients $a_{kl} \in C^\mu([0,T], \mathrm{Lip}(\mathbb{R}^n))$ and $b_m, c \in L^\infty([0,T]\times\mathbb{R}^n)$. Then $\mathcal{P}$ has the $\mathcal{H}$-uniqueness property, where $\mathcal{H}$ is defined by* (18).

Recently in [17] the uniqueness has been proved under a regularity in x which is below Lipschitz and the modulus of continuity in x is connected with the modulus of continuity in time.

2.3.2 Uniqueness Under Local Conditions

In this section we will prove uniqueness results for backward-parabolic and elliptic operators under a local condition as a counterpart to Sect. 2.2.2. We will not give all the details of the proofs. We perform some of the calculations for the backward-parabolic operators and the proofs for the elliptic case follow exactly the same lines. Furthermore, to be as close to the counterexamples as possible, we will state the theorems mainly for solutions which are periodic in x. We consider the operators

$$\mathcal{E}u = \partial_t^2 u + \sum_{k,l=1}^{n} \partial_{x_k}\big(a_{kl}(t,x)\partial_{x_l}u\big) + \beta(t,x)\partial_t u + \sum_{m=1}^{n} b_m(t,x)\partial_{x_m}u + c(t,x)u \tag{19}$$

and

$$\mathcal{P}u = \partial_t u + \sum_{k,l=1}^{n} \partial_{x_k}\big(a_{kl}(t,x)\partial_{x_l}u\big) + \sum_{m=1}^{n} b_m(t,x)\partial_{x_m}u + c(t,x)u \tag{20}$$

under the following assumptions:

(A1) For all $k, l = 1, \dots, n$ one has $a_{kl}(t, x) = a_{lk}(t, x)$.
(A2) There exist a constant a_0 such that

$$\sum_{k,l=1}^{n} a_{kl}(t, x) \frac{\xi_k \xi_l}{|\xi|^2} \geq a_0 > 0 \quad \forall (t, x, \xi) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\}.$$

(A3) Let $a_{kl} = a_{kl}(t, x) \in C^0([0, T], L^\infty(\mathbb{R}^n)) \cap C^1((0, T], L^\infty(\mathbb{R}^n))$ and

$$\exists C \in (0, a_0) : \left| \sum_{k,l=1}^{n} \frac{\partial}{\partial t} a_{kl}(t, x) \frac{\xi_k \xi_l}{|\xi|^2} \right| \leq \frac{C}{t}$$
$$\forall (t, x, \xi) \in (0, T] \times \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\}.$$

(A4) Let b_k, β and c belong to $L^\infty([0, T] \times \mathbb{R}^n, \mathbb{C})$ for all $k = 1, \dots, n$.

Remark 3.2 For the Cauchy problem for the elliptic operator $\mathcal{E}$ the constant C in (A3) can be chosen from the interval $(0, 2a_0)$. The conclusions of Theorem 3.5 and Theorem 3.6 are the same.

Remark 3.3 The restriction on the size of the constant in (A3) is unavoidable as long as uniqueness in C^∞ classes is concerned. A simple computation in the counterexample in Sect. 2.2.2 shows that the size is sharp.

Even if the uniqueness results will be stated for C^∞ solutions, the important property to be satisfied by the solutions under consideration is that for all $N \in \mathbb{N}$ it holds that $\lim_{t \to 0+} t^{-N} |u(t, x)| = 0$ for all $x \in \mathbb{T}^n$ (or $\mathbb{R}^n$) or similar conditions expressed in an integral form.

We set

$$\mathcal{H}_{per} := \left\{ u \in C^\infty\big([0, T], C^\infty(\mathbb{T}^n)\big) : \forall N \in \mathbb{N} : \lim_{t \to 0+} t^{-N} \big|u(t, x)\big| = 0 \ \forall x \in \mathbb{T}^n \right\},$$

where $\mathbb{T}^n$ denotes the n dimensional torus $[0, 2\pi]^n$. Furthermore, we define

$$\mathcal{H} := \left\{ u \in C^\infty\big([0, T], C^\infty(\mathbb{R}^n)\big) : \forall N \in \mathbb{N} : \lim_{t \to 0+} t^{-N} \big|u(t, x)\big| = 0 \ \forall x \in \mathbb{R}^n \right\}.$$

With this preparations we state

Theorem 3.5 (Periodic case) *Let $\mathcal{P}$ be the operator defined by* (20), *assume* (A1)–(A4) *and, moreover, assume that the coefficients a_{kl} are periodic in x. Then $\mathcal{P}$ has the $\mathcal{H}_{per}$-uniqueness property.*

Theorem 3.6 (Non-periodic case) *Let $\mathcal{P}$ be the operator defined by* (20) *and assume* (A1)–(A4). *Then $\mathcal{P}$ has the $\mathcal{H}$-compact uniqueness property, i.e. if $u \in \mathcal{H}$,*

$\operatorname{supp}(u) \subseteq \{t \geq 0\}$, $\operatorname{supp}(u) \cap (\{0\} \times \mathbb{R}^n) = \{(0,0)\}$ *and* $\mathcal{P}u = 0$ *on* $[0,T] \times \mathbb{R}^n$, *then* $u \equiv 0$ *on* $[0,T] \times \mathbb{R}^n$.

Both theorems follow from an appropriate Carleman estimate. The arguments are quite standard and we refer the reader to [45] and [30] for more details. We state the Carleman estimate only for the periodic case. The changes for the general case are easy.

Theorem 3.7 *Suppose the assumptions* (A1)–(A3) *and that the coefficients* a_{kl} *are periodic in* x. *Then there exist positive constants* $C, \gamma_0 > 0$ *and* $\sigma \in (0, \frac{1}{2})$ *such that*

$$\int_0^{T/2} t^{2(\sigma-\gamma)} \left\| \partial_t u + \sum_{k,l=1}^{n} \partial_{x_k}\big(a_{kl}(t,x)\partial_{x_l} u\big) \right\|^2_{L^2(\mathbb{T}^n)} dt$$
$$\geq C\left(\gamma \int_0^{T/2} t^{2(\sigma-\gamma-1)} \|u\|^2_{L^2(\mathbb{T}^n)} dt + \sum_{i=1}^{n} \int_0^{T/2} t^{2(\sigma-\gamma-1/2)} \|\partial_{x_i} u\|^2_{L^2(\mathbb{T}^n)} dt \right) \tag{21}$$

holds for all $u \in \mathcal{H}_{per}$ *with* $\operatorname{supp}(u) \subseteq [0, T/2] \times \mathbb{R}^n$ *and for all* $\gamma \geq \gamma_0$.

Proof To prove the Carleman estimate we put $u(t,x) = t^\gamma v(t,x)$ and we obtain

$$u_t(t,x) = \gamma t^{\gamma-1} v(t,x) + t^\gamma v_t(t,x).$$

This leads to the new equation

$$\mathcal{P}_\gamma v = v_t + \sum_{k,l=1}^{n} \partial_{x_k}\big(a_{kl}(t,x)\partial_{x_l} v\big) + \gamma t^{-1} v. \tag{22}$$

To be able to control some sign during our calculations we need to introduce an auxiliary weight. We multiply the operator $\mathcal{P}_\gamma$ by t^σ, where $\sigma > 0$ will be specified later, and take L^2-norms. We get

$$\begin{aligned}\int_0^{T/2} t^{2\sigma} \|\mathcal{P}_\gamma v\|^2_{L^2(\mathbb{T}^n)} dt = &\int_0^{T/2} t^{2\sigma} \left\| \sum_{k,l=1}^{n} \partial_{x_i}\big(a_{kl}(t,x)\partial_{x_l} v\big) + \gamma t^{-1} v \right\|^2_{L^2(\mathbb{T}^n)} dt \\ &+ 2\,\mathrm{Re} \int_0^{T/2} \left\langle v_t \middle| t^{2\sigma} \sum_{k,l=1}^{n} \partial_{x_k}\big(a_{kl}(t,x)\partial_{x_l} v\big) \right\rangle_{L^2(\mathbb{T}^n)} dt \\ &+ 2\,\mathrm{Re} \int_0^{T/2} \big\langle v_t \big| t^{2\sigma-1} v \big\rangle_{L^2(\mathbb{T}^n)}.\end{aligned}$$

By integration by parts we obtain

$$2\,\mathrm{Re}\int_0^{T/2}\big\langle v_t|t^{2\sigma-1}v\big\rangle_{L^2(\mathbb{T}^n)} = \gamma(1-2\sigma)\int_0^{T/2} t^{2(\gamma-\sigma)}\|v\|^2_{L^2(\mathbb{T}^n)}dt.$$

Here, in order to ensure the positivity of this term, we put, for an $\varepsilon > 0$, $\sigma := \frac{1}{2} - \varepsilon > 0$. Again by integration by parts we obtain

$$\begin{aligned}&2\,\mathrm{Re}\int_0^{T/2}\left\langle v_t|t^{2\sigma}\sum_{k,l=1}^n \partial_{x_k}\big(a_{kl}(t,x)\partial_{x_l}v\big)\right\rangle_{L^2(\mathbb{T}^n)} dt\\ &\quad= \sum_{k,l=1}^n\int_0^{T/2} t^{2\sigma-1}\big\langle\partial_{x_k}v|\big(2\sigma a_{kl}(t,x)+t\partial_t a_{kl}(t,x)\big)\partial_{x_l}v\big\rangle_{L^2(\mathbb{T}^n)}.\end{aligned}\tag{23}$$

To get this integral we approximate the left hand side of (23) by

$$2\,\mathrm{Re}\int_\delta^{T/2}\left\langle v_t|t^{2\sigma}\sum_{k,l=1}^n \partial_{x_k}\big(a_{kl}(t,x)\partial_{x_l}v\big)\right\rangle_{L^2(\mathbb{T}^n)} dt,\tag{24}$$

integrate by parts and take the limit $\delta \to 0+$. The boundary terms vanish since $v \in \mathcal{H}_{per}$ and $\mathrm{supp}(v) \subseteq [0,T]\times\mathbb{R}^n$. Using (A2), (A3) and choosing ε small enough there exist a $C > 0$ such that

$$\begin{aligned}&\sum_{k,l=1}^n\int_0^{T/2} t^{2\sigma-1}\big\langle\partial_{x_k}v|\big(2\sigma a_{kl}(t,x)+t\partial_t a_{kl}(t,x)\big)\partial_{x_l}v\big\rangle_{L^2(\mathbb{T}^n)}\\ &\quad\geq C\sum_{k,l=1}^n\int_0^{T/2} t^{2\sigma-1}\langle\partial_{x_k}v|\partial_{x_l}v\rangle_{L^2(\mathbb{T}^n)}dt.\end{aligned}$$

From that we get

$$\begin{aligned}\int_0^{T/2} t^{2\sigma}\|\mathcal{P}_\gamma v\|^2_{L^2(\mathbb{T}^n)}dt \geq{}& \gamma(1-2\sigma)\int_0^{T/2} t^{2(\gamma-\sigma)}\|v\|^2_{L^2(\mathbb{T}^n)}dt\\ &+C\sum_{k,l=1}^n\int_0^{T/2} t^{2\sigma-1}\langle\partial_{x_k}v|\partial_{x_l}v\rangle_{L^2(\mathbb{T}^n)}dt\end{aligned}$$

from which we, going back to u, reach estimate (21). □

For elliptic operators $\mathcal{E}$ the Carleman estimate is essentially the same. One gets just one term more to control also lower order derivatives in t. The precise statement is

Theorem 3.8 *Suppose assumptions* (A1)–(A3). *Then there exist positive constants* $C, \gamma_0 > 0$ *and* $\sigma \in (0, 1)$ *such that*

$$\int_0^{T/2} t^{2(\sigma-\gamma)} \left\| \partial_t^2 u + \sum_{k,l=1}^n \partial_{x_k} \big(a_{kl}(t,x)\partial_{x_l} u\big) \right\|_{L^2(\mathbb{T}^n)}$$

$$\geq C\gamma \left(\sum_{i=1}^n \int_0^{T/2} t^{2(\sigma-\gamma-1)} \|\partial_{x_i} u\|^2_{L^2(\mathbb{T}^n)} dt + \int_0^{T/2} \gamma t^{2(\sigma-\gamma-1)} \|\partial_t u\|^2_{L^2(\mathbb{T}^n)} dt \right.$$

$$\left. + \int_0^{T/2} \gamma^2 t^{2(\sigma-\gamma-1)} \|u\|^2_{L^2(\mathbb{T}^n)} dt \right)$$

holds for all $u \in \mathcal{H}_{per}$ *with* $\operatorname{supp}(u) \subseteq [0, T/2] \times \mathbb{R}^n$ *and all* $\gamma \geq \gamma_0$.

As already remarked one cannot weaken the assumption on the constant in oscillation control condition (A3). To use a bigger constant there we have to shrink the space (with respect to t) in which we are proving uniqueness. It turns out that Gevrey spaces provide us the possibility to weaken condition (A3) with respect to the constant C. First, we introduce the Gevrey spaces under consideration.

For all $s \geq 1$ we define $\gamma_t^{(s)}$ to be the space of all $C^\infty((-\infty, T], C^\infty(\mathbb{R}^n))$-functions u with $\operatorname{supp}(u) \subseteq [0, T] \times \mathbb{R}^n$ which are in the Gevrey class of index s with respect to t, uniformly in x. This means $u \in \gamma_t^{(s)}$ if and only if for all compact subsets $K \subseteq (-\infty, T] \times \mathbb{R}^n$ and for all multi-indices $\alpha \in \mathbb{N}_0^n$, there exist positive constants $C = C(u, \alpha, K)$ and $M = M(u, \alpha, K)$ such that, for all $k \in \mathbb{N}_0$

$$\sup_{(t,x)\in K} \left| \partial_x^\alpha \partial_t^k u(t,x) \right| \leq C M^k (k!)^s$$

holds true.

With this we define the spaces

$$\mathcal{H}_{per}^{(s)} := \gamma_t^{(s)} \cap C^0\big((-\infty, T], C^\infty\big(\mathbb{T}^n\big)\big)$$

and

$$\mathcal{H}^{(s)} := \gamma_t^{(s)}$$

for which we are going to state the uniqueness theorems.

With this preparations we can state our new local condition

(A3$_\alpha$) Let $a_{kl} = a_{kl}(t,x) \in C^0([0,T], L^\infty(\mathbb{R}^n)) \cap C^1((0,T], L^\infty(\mathbb{R}^n))$, $\alpha > 0$ and

$$\exists C \in \big(0, (1+2\alpha)a_0\big) : \left| \sum_{k,l=1}^n \frac{\partial}{\partial t} a_{kl}(t,x) \frac{\xi_k \xi_l}{|\xi|^2} \right| \leq \frac{C}{t}.$$

for all $(t, x, \xi) \in (0, T] \times \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\}$.

Theorem 3.9 (Periodic case) *Let $\mathcal{P}$ be the operator defined by* (20), *assume* (A1), (A2), (A3$_\alpha$), (A4) *with an* $\alpha > 0$ *and, moreover, that the coefficients* a_{kl} *are periodic in* x. *Then* $\mathcal{P}$ *has the* $\mathcal{H}^{(s)}_{per}$*-uniqueness property for* $s < 1 + \frac{1}{\alpha}$.

It also holds

Theorem 3.10 (Non-periodic case) *Let $\mathcal{P}$ be the operator defined by* (20), *assume* (A1), (A2), (A3$_\alpha$), (A4) *with an* $\alpha > 0$. *Then* $\mathcal{P}$ *has the* $\mathcal{H}^{(s)}$*-compact uniqueness property for* $s < 1 + \frac{1}{\alpha}$, *i.e. if* $u \in \mathcal{H}^{(s)}$, $\text{supp}(u) \subseteq [0,T] \times \mathbb{R}^n$, $\text{supp}(u) \cap (\{0\} \times \mathbb{R}^n) = \{(0,0)\}$ *and* $\mathcal{P}u = 0$ *on* $[0,T] \times \mathbb{R}^n$, *then* $u \equiv 0$ *on* $[0,T] \times \mathbb{R}^n$.

Both theorems follow again from a suitable Carleman estimate and as in the former case we state the estimate just for the periodic case. To obtain a suitable Carleman estimate for our uniqueness result in the Gevrey frame we need to use a weight function which is connected with the way of going to zero for such functions. From Lemma 2 of [33] we know, that we can write every $u \in \mathcal{H}^{(s)}_{per}$ as a product of a function $v \in C_0^\infty((-\infty, T], C^\infty(\mathbb{T}^n))$ with $\text{supp}(u) \subseteq [0,T] \times \mathbb{R}^n$ and the function $\exp(-\gamma t^{-\alpha})$, where s and α satisfy the relation $s < 1 + \frac{1}{\alpha}$.

With this we can state

Theorem 3.11 *Suppose assumptions* (A1), (A2), (A3$_\alpha$) *with an* $\alpha > 0$ *and assume, moreover, that the coefficients* a_{kl} *are periodic in* x. *Then there exist constants* C, γ_0 *and* $\sigma \in (0, \frac{1}{2}(\alpha+1))$ *such that*

$$\int_0^{T/2} t^{2\sigma} e^{2\gamma t^{-\alpha}} \left\| \partial_t u + \sum_{k,l=1}^{n} \partial_{x_k}\big(a_{kl}(t,x)\partial_{x_l} u\big) \right\|^2_{L^2(\mathbb{T}^n)} dt$$

$$\geq C\Bigg(\gamma \int_0^{T/2} t^{2\sigma-\alpha-2} e^{2\gamma t^{-\alpha}} \|w\|^2_{L^2(\mathbb{T}^n)} dt$$

$$+ \sum_{m=1}^{n} \int_0^{T/2} t^{2\sigma} e^{2\gamma t^{-\alpha}} \|\partial_{x_m} u\|^2_{L^2(\mathbb{T}^n)} dt\Bigg)$$

holds for all $\gamma \geq \gamma_0$ *and* $u \in \mathcal{H}^{(s)}_{per}$ *with* $\text{supp}(u) \subseteq [0, T/2] \times \mathbb{R}^n$ *and* $s < 1 + \frac{1}{\alpha}$.

In the elliptic case the same result as stated in Theorems 3.9 and 3.10 hold true. The constant C in the oscillation condition (A3$_\alpha$) can be chosen from the interval $(0, 2(1+\alpha)a_0)$. The Carleman estimate has again one term more and we state it for the sake of completeness.

Theorem 3.12 *Suppose assumptions* (A1), (A2), (A3$_\alpha$) *with an* $\alpha > 0$ *and that the principal part coefficients* a_{kl} *are periodic in* x. *Then there exist constants* C, $\gamma_0 > 0$

and $\sigma \in (0, \frac{3}{2}(\alpha+1))$ *such that*

$$\int_0^{T/2} t^{2\sigma} e^{2\gamma t^{-\alpha}} \left\| \partial_t^2 u + \sum_{k,l=1}^n \partial_{x_k}\big(a_{kl}(t,x)\partial_{x_l} u\big) \right\|^2_{L^2(\mathbb{T}^n)} dt$$

$$\geq C\Bigg(\gamma^3 \int_0^{T/2} t^{2\sigma-3\alpha-4} e^{2\gamma t^{-\alpha}} \|u\|^2_{L^2(\mathbb{T}^n)} dt$$

$$+ \gamma^2 \int_0^{T/2} t^{2\sigma-2\alpha-2} e^{2\gamma t^{-\alpha}} \|\partial_t u\|^2_{L^2(\mathbb{T}^n)} dt$$

$$+ \gamma \sum_{i=1}^n \int_0^{T/2} t^{2\sigma-\alpha-1} e^{2\gamma t^{-\alpha}} \|\partial_{x_i} u\|^2_{L^2(\mathbb{T}^n)} dt \Bigg)$$

holds for all $\gamma \geq \gamma_0$ *and* $u \in \mathcal{H}^{(s)}$ *with* $\operatorname{supp}(u) \subseteq [0, T/2] \times \mathbb{R}^n$ *and* $s < 1 + \frac{1}{\alpha}$.

2.3.3 Degenerate Operators and Local Conditions

In this section we would like to complement the results of the last two sections with some results about degenerate elliptic and backward-parabolic operators. Since, global regularity does in general not help to ensure uniqueness, which may fail even for C^∞ coefficients (see [11, 14]) we ask about the situation for local conditions.

This section gives an overview about results which seem to be new in the literature, as far as backward-parabolic operators are concerned. The proofs follow very closely the lines of the corresponding results for elliptic operators in [7, 11, 36] and therefore we omit them here. We refer also to [30]. We suppose the assumptions (A1) and (A4) and we replace condition (A2) by

(A2′) For the principal part coefficients of $\mathcal{E}$ and $\mathcal{P}$ holds

$$\sum_{k,l=1}^n a_{kl}(t,x)\xi_k\xi_l \geq 0 \quad \forall \xi \in \mathbb{R}^n.$$

In [36] Nirenberg proved compact uniqueness for C^2-solutions of degenerate elliptic operators whose coefficients satisfy the Oleinik condition from [37]. Similar to his approach one can prove compact uniqueness for $C^{1,2}_{t,x}$-solutions of degenerate backward-parabolic operators:

Theorem 3.13 *Suppose assumptions* (A1), (A2′) *and that there exist* $C', C > 0$ *such that*

$$\sum_{k,l=1}^n \big(C' a_{kl}(t,x) + \partial_t a_{kl}(t,x)\big)\xi_k\xi_l \geq C \left| \sum_{m=1}^n b_m(t,x)\xi_m \right|^2$$

holds. Then the operator $\mathcal{P}$ has the $\mathcal{H}$-compact uniqueness property.

Inspired by the work of Nirenberg, Colombini and Del Santo have investigated this kind of condition further and have proved several (compact) uniqueness theorems for C^∞ and Gevrey solutions of the Cauchy problem for degenerate elliptic operators. As already mentioned we want to state similar results for backward-parabolic operators.

Theorem 3.14 *Suppose there exist an $\varepsilon > 0$ and a $C > 0$ such that*

$$\sum_{k,l=1}^{n} \big((1-\varepsilon)a_{kl}(t,x) + t\partial_t a_{kl}(t,x)\big)\xi_k\xi_l \geq Ct^2 \left|\sum_{m=1}^{n} b_m(t,x)\xi_m\right|^2$$

for all $(t,x) \in [0,T] \times \mathbb{R}^n$ and for all $\xi \in \mathbb{R}^n$. Then the backward-parabolic operator $\mathcal{P}$ has the $C^{1,2}_{t,x}$-compact uniqueness property.

Using Gevrey solutions the last condition can we weakened, analogue to Theorem 3.10.

Theorem 3.15 *Let $s > 1$. Suppose there exist an $\varepsilon > 0$ and $C > 0$ such that*

$$\sum_{k,l=1}^{n} \left(\left(\frac{s}{1-s} - \varepsilon\right)a_{kl}(t,x) + t\partial_t a_{kl}(t,x)\right)\xi_k\xi_l \geq Ct^{2+s/(s-1)} \left|\sum_{m=1}^{n} b_m(t,x)\xi_m\right|^2$$

for all $(t,x) \in [0,T] \times \mathbb{R}^n$ and for all $\xi \in \mathbb{R}^n$. Then the backward-parabolic operator $\mathcal{P}$ has the $\mathcal{H}^{(s)}$-compact uniqueness property.

2.3.4 Open Problems and Further Developments

Here we would like to sketch briefly some open questions and possible further developments. Unfortunately we have no counterpart to Theorem 2.4. We expect that one can prove uniqueness in this situation if one chooses a constant in the oscillation control condition which is sufficiently small. Such a result would be analogous (of course only concerned about uniqueness) to those in [32] and [22].

Conjecture 2.3.1 *Suppose that the principal part coefficients of $\mathcal{E}$ or $\mathcal{P}$ are in $C^\mu([0,T],\mathbb{R}) \cap C^1((0,T],\mathbb{R})$ with a non-Osgood modulus of continuity μ. Furthermore, we suppose*

$$\left|\sum_{k,l=1}^{n} \frac{\partial}{\partial t} a_{kl}(t) \frac{\xi_k\xi_l}{|\xi|^2}\right| \leq C\frac{\mu(\eta^{-1}(t))}{\eta^{-1}(t)} \quad \forall (t,x,\xi) \in (0,T] \times \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\}$$

holds for a sufficiently small $C > 0$, *where* $\eta(t) := \int_0^t \frac{ds}{\mu(s)}$. *Then the operators* $\mathcal{P}$ *and* $\mathcal{E}$ *have the* $\mathcal{H}$*-compact uniqueness property.*

If this can be proved one can expect a similar result for degenerate operators under a condition like

$$\sum_{k,l=1}^{n} \left(C a_{kl}(t) + \frac{\eta^{-1}(t)}{\mu(\eta^{-1}(t))} \partial_t a_{kl}(t) \right) \xi_k \xi_l \geq K f_\mu(t) \left| \sum_{m=1}^{n} b_m(t,x) \xi_m \right|^2$$

for a sufficiently small constant $C > 0$. In both cases one might also consider x-dependent coefficients.

It is clear that then similar improvements as described before can be expected if one considers Gevrey classes.

2.4 Continuous Dependence for Backward-Parabolic Operators

Since the Cauchy problem for elliptic and backward-parabolic operators is severely ill-posed one cannot expect the usual stability properties. However, for applications it is important to have some quantitative information about the nature of the dependency of solution on the Cauchy data. In his celebrated paper *Continuous dependence on data for solutions of partial differential equations with a prescribed bound* [31] John attempted this problem and introduced the notion of a well-behaved problem. In the notion of John a problem is well-behaved if *only a fixed percentage of the significant digits need be lost in determining the solution from the data*. To be a little bit more precise, that means that the solution in a space $\mathcal{H}$ depends Hölder continuously on the data in some space $\mathcal{K}$, provided they satisfy a *prescribed bound*.

In [1] Agmon and Nirenberg proved well-behavedness of the Cauchy problem for $\mathcal{P}$ in the space

$$\mathcal{H} := C^0\big([0,T], L^2(\mathbb{R}^n)\big) \cap C^0\big([0,T), H^1(\mathbb{R}^n)\big) \cap C^1\big([0,T), L^2(\mathbb{R}^n)\big)$$

with data in $L^2(\mathbb{R}^n)$. They assumed the coefficients to be Lip with respect to t and $L^\infty(\mathbb{R}^n)$ with respect to x. At about the same time Glagoleva obtained in [24] almost the same result by a different technique and time-independent coefficients. In [29] Hurd developed the technique of Glagoleva further to cover also the case where the coefficients also depend Lipschitz continuously on time. This result has been partially improved by Del Santo and Prizzi in [20]. They considered the operator $\mathcal{P}$ with coefficients depending Log-Lipschitz continuously on time. But, due to some technical difficulties arising from a commutator estimate, they had to require C^2-regularity in x. The result they got can be summarized as follows: *For every* $T' \in (0,T)$ *and* $D > 0$ *there exist* $M, N, \rho > 0$ *and* $\delta \in (0,1)$ *such that if* $u \in \mathcal{H}$ *is a solution of* $\mathcal{P}u = 0$ *on* $[0,T] \times \mathbb{R}^n$ *with* $\|u(0,\cdot)\|_{L^2(\mathbb{R}^n)} \leq \rho$ *and* $\|u(t,\cdot)\|_{L^2(\mathbb{R}^n)} \leq D$

on $[0, T]$*, then*

$$\sup_{t\in[0,T']} \|u(t,\cdot)\|^2_{L^2(\mathbb{R}^n)} \le M \exp\bigl(-N\bigl|\log\bigl(\|u(0,\cdot)\|_{L^2(\mathbb{R}^n)}\bigr)\bigr|^{\delta}\bigr).$$

As one sees this dependence is weaker than Hölder continuous dependence. A counterexample in [20] shows that this result is sharp in the sense that one can in general not expect Hölder continuous dependence if the coefficients depend only Log-Lipschitz continuously on time.

The C^2 regularity with respect to x has recently been removed by use of Bony's paraproduct and could be replaced by Lipschitz continuity which is more natural in this context (see [18]).

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Chapter 3
On Internal Regularity of Solutions to the Initial Value Problem for the Zakharov–Kuznetsov Equation

A.V. Faminskii and A.P. Antonova

Abstract The initial value problem is considered for the Zakharov–Kuznetsov equation in two spatial dimensions, which generalizes the Korteweg–de Vries equation for description of wave propagation in dispersive media on the plane. An initial function is assumed to be irregular, namely, from the spaces L_2 or H^1. Results on gain of internal regularity for corresponding weak solutions depending on the decay rate of the initial function at infinity are established. Existence of both Sobolev and continuous derivatives of any prescribed order is proved. One of important items of the study is the investigation of the fundamental solution to the corresponding linearized equation. The obtained properties are to some extent similar to the ones of the Airy function.

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3.1 Introduction. Description of Main Results

The initial value problem for the Zakharov–Kuznetsov equation

$$u_t + u_{xxx} + u_{xyy} + uu_x = f(t,x,y) \tag{1}$$

($u = u(t,x,y)$) with an initial condition

$$u|_{t=0} = u_0(x,y) \tag{2}$$

is considered in a layer $\Pi_T = \{(t,x,y) : 0 < t < T, (x,y) \in \mathbb{R}^2\}$ ($T > 0$—arbitrary) and internal regularity of its solutions is studied.

The equation of type (1) was derived in [14] for the description of propagation of nonlinear ion-acoustic waves in plasma placed in a magnetic field. Further this equation was named Zakharov–Kuznetsov equation. Equation (1) is one of the vari-

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ants of the $(2+1)$-dimensional generalizations of the Korteweg–de Vries equation (KdV) $u_t + u_{xxx} + uu_x = 0$.

Equation (1) (for $f \equiv 0$) possesses two conservation laws

$$\iint u^2 dxdy \equiv \text{const}, \qquad \iint \left(u_x^2 + u_y^2 - \frac{1}{3}u^3\right) dxdy \equiv \text{const}. \tag{3}$$

In [4] with the use of these equalities global well-posedness of the problem (1), (2) was established for the initial function from $H^m(\mathbb{R}^2)$ and the right-hand side from $L_1(0,T;H^m(\mathbb{R}^2))$, m—natural, in a certain special functional class $K_m(0,T) \subset C([0,T];H^m(\mathbb{R}^2))$. In the case $m=1$ this class is the following one:

$$K_1(0,T) = \bigl\{u \in C\bigl([0,T];H^1\bigl(\mathbb{R}^2\bigr)\bigr), u_{xx}, u_{xy}, u_{yy} \in L_\infty\bigl(\mathbb{R}^x; L_2\bigl((0,T)\times\mathbb{R}^y\bigr)\bigr),$$
$$u \in L_3\bigl(0,T;W^1_\infty\bigl(\mathbb{R}^2\bigr)\bigr), u \in L_2\bigl(\mathbb{R}^x; L_\infty\bigl((0,T)\times\mathbb{R}^y\bigr)\bigr)\bigr\}.$$

Previously similar function classes for the KdV equation were introduced in [8].

Note that for less regular data $u_0 \in L_2(\mathbb{R}^2)$, $f \in L_1(0,T;L_2(\mathbb{R}^2))$ existence of global solutions to the considered problem from the space $L_\infty(0,T;L_2(\mathbb{R}^2))$ follows from results of [3]. These solutions also possess additional smoothness in comparison with the initial data:

$$\sup_{x_0\in\mathbb{R}} \int_0^T \int_{x_0}^{x_0+1} \int_{\mathbb{R}} \bigl(u_x^2 + u_y^2\bigr) dydxdt < +\infty.$$

Moreover, if the initial data and the right-hand side as $x \to +\infty$ satisfy additional decay assumptions $(1+x)^\alpha u_0 \in L_2(\mathbb{R}^2_+)$, $(1+x)^\alpha f \in L_1(0,T;L_2(\mathbb{R}^2_+))$ for certain $\alpha > 0$, then

$$\begin{gathered}(1+x)^\alpha u \in L_\infty\bigl(0,T;L_2\bigl(\mathbb{R}^2_+\bigr)\bigr),\\ (1+x)^{\alpha-1/2}\bigl(|u_x|+|u_y|\bigr) \in L_2\bigl((0,T)\times\mathbb{R}^2_+\bigr)\end{gathered} \tag{4}$$

(here and further $\mathbb{R}^2_+ = \{(x,y): x>0\} = \mathbb{R}_+ \times \mathbb{R}$). An analog of the first of conservation laws (3) was used to obtain these results. However, uniqueness of such solutions was not established.

In [9] and [2] the gain of internal regularity of weak solutions with respect to the decay rate of irregular initial function $u_0(x)$ as $x \to +\infty$ was found for the initial value problem for the KdV equation. In particular, it was shown that if $u_0 \in L_2(\mathbb{R})$ and $x^\alpha u_0 \in L_2(\mathbb{R}_+)$ for certain $\alpha > 0$ then the corresponding solution $u(t,x)$ possessed generalized (in the Sobolev sense) derivatives $\partial_x^n u$ for $n \le 2\alpha + 1$. For $n < 2\alpha - 1/2$ these derivatives were continuous. If, additionally, u_0' had the same properties as the initial function itself, then the orders of all aforementioned derivatives could be enlarged by one.

In [7] a result of internal regularity for weak solutions to the initial value problem for the KdV equation was obtained in the case of initial functions from $L_2(\mathbb{R})$ decaying exponentially at $+\infty$. The solutions became infinitely smooth for $t > 0$.

The first result on internal regularity of solutions to the initial value problem for the Zakharov–Kuznetsov equation was established in [10] (in fact, more general equations of the third order on the plane were considered). There the result on the gain of regularity was obtained for solutions having a priori generalized spatial derivatives up to the sixth order from the space $L_2(\mathbb{R}^2)$ with a certain spatial weight as $x \to +\infty$. Existence of such solutions was proved only locally in time for initial functions from the space $H^6(\mathbb{R}^2)$ with a corresponding weight.

In the present paper for problem (1), (2) the results similar to the ones for KdV from [9] and [2] are established. Solutions are considered for initial functions from the spaces $L_2(\mathbb{R}^2)$ and $H^1(\mathbb{R}^2)$ with a certain power weight as $x \to +\infty$. In order to present the main results we introduce the following notations.

Let $\nu = (k, n)$ be a integer-valued multi-index, $|\nu| = k + n$, define $D^\nu = \partial^{|\nu|}/\partial x^k \partial y^n$. Let $L_p = L_p(\mathbb{R}^2)$.

For $\alpha \geq 0$ we define the following function spaces

$$L_2^\alpha = L_2^\alpha\big(\mathbb{R}^2\big) = \big\{\phi \in L_2\big(\mathbb{R}^2\big) : (1+x)^\alpha \phi \in L_2\big(\mathbb{R}^2_+\big)\big\},$$
$$H^{1,\alpha} = H^{1,\alpha}\big(\mathbb{R}^2\big) = \big\{\phi \in H^1\big(\mathbb{R}^2\big) : \phi, \phi_x, \phi_y \in L_2^\alpha\big(\mathbb{R}^2\big)\big\}$$

with natural norms.

Theorem 1 *Let $u_0 \in L_2^\alpha$, $f \in L_1(0, T; L_2^\alpha)$ for a certain $\alpha \geq 1/2$ and, in addition, there exists a natural $m \leq 2\alpha$ such that $D^\nu f \in L_1(0, T; L_2^{\alpha-|\nu|/2})$ for $1 \leq |\nu| \leq m$ and also if $m = 2$ then $\alpha > 1$. Then there exists a solution $u(t, x, y)$ to problem (1), (2) from the space $L_\infty(0, T; L_2^\alpha)$ possessing in Π_T generalized (Sobolev) derivatives $D^\nu u$ of the orders $|\nu| \leq m + 1$. Moreover, for all $\delta \in (0, T)$ and $x_0 \in \mathbb{R}$*

$$(x - x_0 + 1)^{\alpha-|\nu|/2} D^\nu u \in L_\infty\big(\delta, T; L_2\big((x_0, +\infty) \times \mathbb{R}\big)\big), \quad 1 \leq |\nu| \leq m; \tag{5}$$

$$D^\nu u \in L_2\big((\delta, T) \times (x_1, x_1 + 1) \times \mathbb{R}\big) \quad \forall x_1 \geq x_0, |\nu| = m + 1; \tag{6}$$

and, if $m < 2\alpha$, then for $|\nu| = m + 1$

$$(x - x_1 + 1)^{\alpha-(m+1)/2} D^\nu u \in L_2\big((\delta, T) \times (x_1, +\infty) \times \mathbb{R}\big) \quad \forall x_1 \geq x_0, \tag{7}$$

where in the last two cases the norms are estimated uniformly with respect to x_1.

Theorem 2 *Let $u_0 \in H^{1,\alpha}$, $f \in L_1(0, T; H^{1,\alpha})$ for a certain $\alpha > 0$. Then the solution $u(t, x, y)$ to problem (1), (2) from the space $K_1(0, T)$ also belongs to the space $L_\infty(0, T; H^{1,\alpha})$ and if $|\nu| = 2$ then for all $x_0 \in \mathbb{R}$*

$$(x - x_1 + 1)^{\alpha-1/2} D^\nu u \in L_2\big((0, T) \times (x_1, +\infty) \times \mathbb{R}\big) \quad \forall x_1 \geq x_0, \tag{8}$$

where the norm is estimated uniformly with respect to x_1.

Remark 1 If $\alpha \geq 1/2$, then (8) is equivalent to the property $(x + 1)^{\alpha-1/2} D^\nu u \in L_2((0, T) \times \mathbb{R}^2_+)$.

Theorem 3 *Let the hypothesis of Theorem 2 be satisfied for $\alpha \geq 1/2$ and, in addition, there exists a natural $m \in [2, 2\alpha + 1]$ such that $D^\nu f \in L_1(0, T; L_2^{\alpha-|\nu|/2+1/2})$*

for $2 \le |\nu| \le m$. *Then the solution* $u(t,x,y)$ *to problem* (1), (2) *from the space* $K_1(0,T)$ *possesses in* Π_T *generalized* (*Sobolev*) *derivatives* $D^\nu u$ *of the orders* $|\nu| \le m+1$. *Moreover, for all* $\delta \in (0,T)$ *and* $x_0 \in \mathbb{R}$

$$(x-x_0+1)^{\alpha-|\nu|/2+1/2} D^\nu u \in L_\infty\big(\delta,T; L_2\big((x_0,+\infty)\times\mathbb{R}\big)\big),$$
$$2 \le |\nu| \le m; \tag{9}$$

$$D^\nu u \in L_2\big((\delta,T)\times(x_1,x_1+1)\times\mathbb{R}\big) \quad \forall x_1 \ge x_0, |\nu| = m+1; \tag{10}$$

and, if $m < 2\alpha+1$, *then for* $|\nu| = m+1$

$$(x-x_1+1)^{\alpha-m/2} D^\nu u \in L_2\big((\delta,T)\times(x_1,+\infty)\times\mathbb{R}\big) \quad \forall x_1 \ge x_0, \tag{11}$$

where in the last two cases the norms are estimated uniformly with respect to x_1.

Remark 2 Since with the use of (1) itself the time derivative of the solution can be expressed via the spatial derivatives then under additional assumptions on smoothness of the function f with respect to t the solution itself also possesses corresponding time smoothness.

These results are completely similar to the ones obtained in [2] for KdV (of course, without taking into account y).

Note also that properties (8)–(11) are similar to the ones established in [10] with the only difference that in the latter the original space $H^{1,\alpha}$ is substituted by $H^{6,\alpha}$ defined in a similar way.

The proof of properties (5)–(11) is performed in Sect. 3.2. It is based on integral estimates and develops the methods of [9] and [2] for the two-dimensional case.

In [9] and [2] the idea of the inversion of the linear part of the KdV equation and the application of properties of the fundamental solution to the operator $\partial_t + \partial^3_{xxx}$ is used to prove continuity of derivatives of the considered solutions to KdV itself. This fundamental solution is well-known and can be expressed via the Airy function.

In contrast to KdV the linearized Zakharov–Kuznetsov equation is considerably less studied. In Sect. 3.3 of the present paper properties of the fundamental solution to the operator $\partial_t + \partial^3_{xxx} + \partial^3_{xyy}$ are investigated. The obtained estimates are used further in Sect. 3.4 but also have their own significance.

In Sect. 3.4 the following results on continuity of derivatives of considered solutions to the Zakharov–Kuznetsov equation are established.

Theorem 4 *Let* $u_0 \in L_2^\alpha$, $f \in L_\infty(0,T;L_2^\alpha)$ *for a certain* $\alpha > 3/4$ *and* $D^\nu f \in L_\infty(0,T;L_2^{\alpha-|\nu|/2})$ *for a certain natural* $m < 2\alpha - 1/2$ *and all* $1 \le |\nu| \le m$. *Then there exists a solution* $u(t,x,y)$ *to problem* (1), (2) *from the space* $L_\infty(0,T;L_2^\alpha)$ *satisfying* (4)–(7), *continuous in* Π_T (*possibly, after modification on a set of measure zero*) *and possessing in* Π_T *continuous derivatives* $D^\nu u$ *for* $|\nu| \le m-1$. *Moreover,* *for all* $\delta \in (0,T)$ *and* $x_0 \in \mathbb{R}$

$$\sup_{t\in[\delta,T],x\geq x_0} \left|D^\nu u(t,x,y)\right| < \infty, \quad 0 \leq |\nu| \leq m-1. \tag{12}$$

Theorem 5 *Let* $u_0 \in H^{1,\alpha}$, $f \in L_\infty(0,T;H^{1,\alpha})$ *for a certain* $\alpha > 3/4$ *and* $D^\nu f \in L_\infty(0,T;L_2^{\alpha-|\nu|/2+1/2})$ *for a certain natural* $2 \leq m < 2\alpha + 1/2$ *and all* $2 \leq |\nu| \leq m$. *Then the solution* $u(t,x,y)$ *to problem* (1), (2) *from the space* $K_1(0,T)$ *is continuous in* Π_T *(possibly, after modification on a set of measure zero) and possesses in* Π_T *continuous derivatives* $D^\nu u$ *for* $|\nu| \leq m-1$. *Moreover, for all* $\delta \in (0,T)$ *and* $x_0 \in \mathbb{R}$ *inequality* (12) *holds.*

Further we use the following auxiliary functions. Let $\eta(x)$ be a certain "cut-off" function, namely, η is an infinitely smooth non-decreasing on $\mathbb{R}$ function such that $\eta(x) \equiv 0$ for $x \leq 0$, $\eta(x) \equiv 1$ for $x \geq 1$, $\eta(x) + \eta(1-x) \equiv 1$.

Define a weight function $\rho_{\alpha,\beta}(x)$, $\alpha \geq 0$, $\beta > 0$, in the following way: $\rho_{\alpha,\beta} \in C^\infty(\mathbb{R})$ is an increasing function such that $\rho_{\alpha,\beta}(x) \equiv e^{\beta x}$ for $x \leq -1$, $\rho_{\alpha,\beta}(x) \equiv (1+x)^\alpha$ if $\alpha > 0$ and $\rho_{0,\beta}(x) \equiv 2 - (1+x)^{-1/2}$ for $x \geq 0$, $\rho'_{\alpha,\beta}(x) > 0$ for $-1 < x < 0$. Note that $\rho'_{\alpha,\beta}(x) \leq c(\alpha,\beta)\rho_{\alpha,\beta}(x)$, $|\rho^{(k)}_{\alpha,\beta}(x)| \leq c(k,\alpha,\beta)\rho'_{\alpha,\beta}(x)$ for all $x \in \mathbb{R}$ and natural $k \geq 2$.

Further we use the following interpolation inequality succeeding from [3]. Let $\psi_0(x,y)$, $\psi_1(x,y)$ be two positive infinitely smooth on $\mathbb{R}^2$ functions such that $\psi_0 \leq c\psi_1$, $|D^\nu \psi_j| \leq c(\nu)\psi_j$ for all multi-indexes ν, $j = 0$ or 1, and w be a function such that $w_x\psi_0^{1/2}, w_y\psi_0^{1/2}, w\psi_1^{1/2} \in L_2(\mathbb{R}^2)$. Then for $q \in [2,+\infty)$

$$\left\| w\psi_0^s \psi_1^{1/2-s} \right\|_{L_q} \leq c(q) \left\| \big(|w_x| + |w_y|\big)\psi_0^{1/2} \right\|_{L_2}^{2s} \left\| w\psi_1^{1/2} \right\|_{L_2}^{1-2s} + c(q)\left\| w\psi_1^{1/2}\right\|_{L_2}, \tag{13}$$

where $s = 1/2 - 1/q$.

As a rule further we omit limits of integration in the integrals over the whole plane $\mathbb{R}^2$.

3.2 Sobolev Derivatives

We obtain estimates on solutions to problem (1)–(2) under both smooth initial data and right-hand side of the equation but depending only on norms of these functions contained in the hypotheses of Theorems 1–3. In the consequent six lemmas we assume that $u_0 \in C_0^\infty(\mathbb{R}^2)$ and $f \in C_0^\infty(\Pi_T)$. Then it follows from [4] that there exists a solution $u(t,x,y)$ to the considered problem such that $\partial_t^j u \in C([0,T];H^m(\mathbb{R}^2))$ for all non-negative integers j and m. Moreover, for such a solution one can apply the results of [10] and then $\partial_t^j D^\nu u \in C([0,T];L_2^\alpha)$ for all $\alpha > 0$.

First establish estimates on the solution depending on the norms of u_0 and f in the space L_2^α.

Lemma 1 *Let $\alpha > 0$. Then for any $\beta > 0$ and $x_0 \in \mathbb{R}$*

$$\sup_{0\le t\le T} \iint_{\mathbb{R}^2} u^2 \rho_{2\alpha,\beta}(x - x_0)dxdy + \sup_{x_1\ge x_0} \int_0^T \iint_{\mathbb{R}^2} \bigl(u_x^2 + u_y^2\bigr)\rho'_{2\alpha,\beta}(x - x_1)dxdydt \le c, \tag{14}$$

where the constant c depends on T, α, β, x_0, $\|u_0\|_{L_2^\alpha}$, $\|f\|_{L_1(0,T;L_2^\alpha)}$.

Proof This estimate succeeds from [3] and it is an analog to (4). We present here the sketch of its proof for completeness.

First of all multiplying equality (1) by $2u(t, x, y)$ and integrating we obtain similarly to the first of the conservation laws (3) that

$$\sup_{0\le t\le T} \bigl\|u(t,\cdot,\cdot)\bigr\|_{L_2} \le \|u_0\|_{L_2} + \|f\|_{L_1(0,T;L_2)}. \tag{15}$$

Let $\rho(x) \equiv \rho_{2\alpha,\beta}(x - x_1)$. Multiplying (1) by $2u(t, x, y)\rho(x)$ and integrating over $\mathbb{R}^2$ we derive the equality

$$\frac{d}{dt}\iint u^2\rho dxdy + \iint \bigl(3u_x^2 + u_y^2\bigr)\rho' dxdy - \iint u^2\rho''' dxdy - \frac{2}{3}\iint u^3\rho' dxdy = 2\iint fu\rho dxdy. \tag{16}$$

Taking into account the already obtained estimate (15) and interpolation inequality (13) (where $\psi_0 = \psi_1 \equiv \rho'$) we deduce that

$$\begin{aligned}\left|\iint u^3\rho' dxdy\right| &\le \Bigl(\iint u^2 dxdy\Bigr)^{1/2}\Bigl(\iint u^4\bigl(\rho'\bigr)^2 dxdy\Bigr)^{1/2} \\ &\le c\Bigl(\iint \bigl(u_x^2 + u_y^2\bigr)\rho' dxdy\Bigr)^{1/2}\Bigl(\iint u^2\rho dxdy\Bigr)^{1/2} \\ &\quad + c\iint u^2\rho dxdy,\end{aligned} \tag{17}$$

whence the desired estimate follows. □

Lemma 2 *Let $\alpha \ge 1/2$. Then for all $\beta > 0$, $\delta \in (0, T)$, $x_0 \in \mathbb{R}$ if $|\nu| = 1$ then*

$$\sup_{\delta\le t\le T} \iint_{\mathbb{R}^2} \bigl(D^\nu u\bigr)^2 \rho_{2\alpha-|\nu|,\beta}(x - x_0)dxdy + \sup_{x_1\ge x_0} \int_\delta^T \iint_{\mathbb{R}^2} \bigl(\bigl(D^\nu u_x\bigr)^2 + \bigl(D^\nu u_y\bigr)^2\bigr)\rho'_{2\alpha-|\nu|,\beta}(x - x_1)dxdydt \le c, \tag{18}$$

where the constant on the right-hand side depends on T, α, β, δ, x_0, $\|u_0\|_{L_2^\alpha}$, $\|f\|_{L_1(0,T;L_2^\alpha)}$ and $\|D^\nu f\|_{L_1(0,T;L_2^{\alpha-|\nu|/2})}$ for $|\nu| = 1$.

Proof Let $\varphi(t) \equiv \eta(2t/\delta - 1)$, $\rho(x) \equiv \rho_{2\alpha-1,\beta}(x - x_1)$.

Multiply equality (1) by $-2((u_x\rho(x))_x + u_{yy}\rho(x))\varphi(t)$ and integrate over $\mathbb{R}^2$, then

$$\begin{aligned}
&\frac{d}{dt}\iint \big(u_x^2 + u_y^2\big)\rho\varphi dxdy + \iint \big(3u_{xx}^2 + 4u_{xy}^2 + u_{yy}^2\big)\rho'\varphi dxdy \\
&\quad - \iint \big(u_x^2 + u_y^2\big)\rho'''\varphi dxdy - \iint \big(u_x^2 + u_y^2\big)\rho\varphi' dxdy \\
&\quad + \iint \big(u_x\rho - u\rho'\big)\big(u_x^2 + u_y^2\big)\varphi dxdy \\
&= 2\iint (f_x u_x + f_y u_y)\rho\varphi dxdy. \qquad (19)
\end{aligned}$$

Next, multiply (1) by $-u^2(t,x,y)\rho(x)\varphi(t)$ and integrate over $\mathbb{R}^2$, then

$$\begin{aligned}
&-\frac{d}{dt}\iint \frac{u^3}{3}\rho\varphi dxdy + \frac{1}{3}\iint u^3\rho\varphi' dxdy \\
&\quad - \iint u_x\big(u_x^2 + u_y^2\big)\rho\varphi dxdy - \iint u\big(3u_x^2 + u_y^2\big)\rho'\varphi dxdy \\
&\quad + \frac{1}{3}\iint u^3\rho'''\varphi dxdy + \frac{1}{4}\iint u^4\rho'\varphi dxdy \\
&= -\iint fu^2\rho\varphi dxdy.
\end{aligned}$$

Adding this equality and (19) we find that

$$\begin{aligned}
&\frac{d}{dt}\iint \left(u_x^2 + u_y^2 - \frac{u^3}{3}\right)\rho\varphi dxdy - \iint \left(u_x^2 + u_y^2 - \frac{u^3}{3}\right)\rho\varphi' dxdy \\
&\quad + \iint \big(3u_{xx}^2 + 4u_{xy}^2 + u_{yy}^2\big)\rho'\varphi dxdy - \iint \big(u_x^2 + u_y^2\big)\rho'''\varphi dxdy \\
&\quad - \iint u\big(4u_x^2 + 2u_y^2\big)\rho'\varphi dxdy + \frac{1}{3}\iint u^3\rho'''\varphi dxdy + \frac{1}{4}\iint u^4\rho'\varphi dxdy \\
&= \iint \big(2f_x u_x + 2f_y u_y - fu^2\big)\rho\varphi dxdy. \qquad (20)
\end{aligned}$$

Here, similarly to (17),

$$\begin{aligned}
&\left|\iint u^3\rho\varphi dxdy\right| \\
&\quad \le c\left(\iint \big(u_x^2 + u_y^2\big)\rho\varphi dxdy\right)^{1/2}\left(\iint u^2\rho\varphi dxdy\right)^{1/2} + c\iint u^2\rho\varphi dxdy.
\end{aligned}$$

Also similarly to (17) for $|\nu|=1$

$$\begin{aligned}
&\left|\iint u\big(D^\nu u\big)^2\rho'\varphi\,dxdy\right| \\
&\quad\le\left(\iint u^2dxdy\right)^{1/2}\left(\iint \big(D^\nu u\big)^4\big(\rho'\big)^2\varphi^2dxdy\right)^{1/2} \\
&\quad\le c\left(\iint \big(\big(D^\nu u_x\big)^2+\big(D^\nu u_y\big)^2\big)\rho'\varphi\,dxdy\right)^{1/2}\left(\iint \big(D^\nu u\big)^2\rho\varphi\,dxdy\right)^{1/2} \\
&\qquad+c\iint \big(D^\nu u\big)^2\rho\varphi\,dxdy.
\end{aligned}$$

Finally, note that $\rho_{2\alpha-1,\beta}\sim\rho'_{2\alpha,\beta}$, therefore, by virtue of (14)

$$\int_0^T\iint\left(u_x^2+u_y^2-\frac{u^3}{3}\right)\rho\varphi'\,dxdydt\le c.$$

Then equality (20) yields the desired inequality. □

Remark 3 Evidently, equality (20) is an analog to the second conservation law (3).

Lemma 3 *Let $\alpha>1$. Then for any $\beta>0$, $\delta\in(0,T)$, $x_0\in\mathbb{R}$ inequality (18) holds for $|\nu|=2$, where the constant on the right-hand side depends on T, α, β, δ, x_0, $\|u_0\|_{L_2^\alpha}$ and $\|D^\nu f\|_{L_1(0,T;L_2^{\alpha-|\nu|/2})}$ for $|\nu|\le 2$.*

Proof Let $\varphi(t)\equiv\eta(2t/\delta-1)$, $\rho(x)\equiv\rho_{2\alpha-2,\beta}(x-x_1)$. For any multi-index ν, $|\nu|=2$, multiply equality (1) by $2D^\nu(D^\nu u\rho)\varphi$ and integrate over $\mathbb{R}^2$, then

$$\begin{aligned}
&\frac{d}{dt}\iint\big(D^\nu u\big)^2\rho\varphi\,dxdy-\iint\big(D^\nu u\big)^2\rho\varphi'\,dxdy \\
&\qquad+\iint\big(3\big(D^\nu u_x\big)^2+\big(D^\nu u_y\big)^2\big)\rho'\varphi\,dxdy-\iint\big(D^\nu u\big)^2\rho'''\varphi\,dxdy \\
&\qquad+2\iint D^\nu(uu_x)D^\nu u\rho\varphi\,dxdy \\
&\quad=2\iint D^\nu f D^\nu u\rho\varphi\,dxdy. \qquad (21)
\end{aligned}$$

Since $\rho_{2\alpha-2,\beta}\sim\rho'_{2\alpha-1,\beta}$ by virtue of (18)

$$\int_0^T\iint\big(D^\nu u\big)^2\rho\varphi'\,dxdydt\le c. \qquad (22)$$

In order to estimate the integral of the nonlinear term note that

$$2\iint uD^{\nu}u_x D^{\nu}u\rho\varphi\,dxdy = -\iint \big(u_x\rho + u\rho'\big)\big(D^{\nu}u\big)^2\varphi\,dxdy. \tag{23}$$

Therefore in fact, one must estimate an integral of $D^{\nu_1}uD^{\nu_2}uD^{\nu}u\rho\varphi$, where $|\nu_1|\le 1$, $|\nu_2| = 2$. Note that $2\alpha - 2 < (\alpha - 3/4) + (\alpha - 1) + (\alpha - 5/4)$ since $\alpha > 1$ and thus

$$\begin{aligned}
&\left|\iint D^{\nu_1}uD^{\nu_2}uD^{\nu}u\rho\varphi\,dxdy\right| \\
&\quad\le \left(\iint \big(D^{\nu_1}u\big)^4\rho^2_{2\alpha-3/2,\beta/2}\varphi\,dxdy\right)^{1/4}\left(\iint \big(D^{\nu_2}u\big)^2\rho_{2\alpha-2,\beta/2}\varphi^{1/2}dxdy\right)^{1/2} \\
&\qquad\times\left(\iint \big(D^{\nu}u\big)^4\rho'\rho\varphi^2dxdy\right)^{1/4}.
\end{aligned} \tag{24}$$

Applying the interpolation inequality (13) (where $\psi_0 \equiv \rho'_{2\alpha-j,\beta/2}$, $\psi_1 \equiv \rho_{2\alpha-j,\beta/2}$, $j = 1$ or 2) we find that

$$\begin{aligned}
&\iint \big(D^{\nu_1}u\big)^4\rho^2_{2\alpha-3/2,\beta/2}\varphi\,dxdy \\
&\quad\sim \iint \big(D^{\nu_1}u\big)^4\rho'_{2\alpha-1,\beta/2}\rho_{2\alpha-1,\beta/2}\varphi\,dxdy \\
&\quad\le c\iint \Big(\big(D^{\nu_1}u_x\big)^2 + \big(D^{\nu_1}u_y\big)^2\Big)\rho'_{2\alpha-1,\beta/2}\varphi^{1/2}dxdy \\
&\qquad\times\iint \big(D^{\nu_1}u\big)^2\rho_{2\alpha-1,\beta/2}\varphi^{1/2}dxdy \\
&\qquad + c\left(\iint \big(D^{\nu_1}u\big)^2\rho_{2\alpha-1,\beta/2}\varphi^{1/2}dxdy\right)^2, \\
&\iint \big(D^{\nu}u\big)^4\rho'\rho\varphi^2dxdy \\
&\quad\le c\iint \Big(\big(D^{\nu}u_x\big)^2 + \big(D^{\nu}u_y\big)^2\Big)\rho'\varphi\,dxdy\iint \big(D^{\nu}u\big)^2\rho\varphi\,dxdy \\
&\qquad + c\left(\iint \big(D^{\nu}u\big)^2\rho\varphi\,dxdy\right)^2.
\end{aligned} \tag{25}$$

Finally, using the already established estimates (14) and (18) we derive that

$$\begin{aligned}
&\left|\iint D^{\nu_1}uD^{\nu_2}uD^{\nu}u\rho\varphi\,dxdy\right| \\
&\quad\le \varepsilon\iint \Big[\Big(\big(D^{\nu}u_x\big)^2 + \big(D^{\nu}u_y\big)^2\Big)\rho' + \big(D^{\nu}u\big)^2\rho\Big]\varphi\,dxdy
\end{aligned}$$

$$
\begin{aligned}
&+ c(\varepsilon) \iint \big[\big(\big(D^{\nu_1} u_x\big)^2 + \big(D^{\nu_1} u_y\big)^2\big)\rho'_{2\alpha-1,\beta/2} \\
&+ \big(D^{\nu_1} u\big)^2 \rho_{2\alpha-1,\beta/2}\big]\varphi^{1/2} dxdy \\
&\times \iint \big(D^{\nu_1} u\big)^2 \rho_{2\alpha-1,\beta/2}\varphi^{1/2} dxdy \iint \big(D^{\nu} u\big)^2 \rho_{2\alpha-2,\beta}\varphi dxdy \\
&+ c(\varepsilon) \iint \big(D^{\nu_2} u\big)^2 \rho'_{2\alpha-1,\beta/2}\varphi^{1/2} dxdy \\
\le{}& \varepsilon \iint \big(\big(D^{\nu} u_x\big)^2 + \big(D^{\nu} u_y\big)^2\big)\rho'\varphi dxdy \\
&+ \gamma(t)\Big(\iint \big(D^{\nu} u\big)^2 \rho\varphi dxdy + 1\Big),
\end{aligned} \tag{26}
$$

where $\|\gamma\|_{L_1(0,T)} \le c$ and $\varepsilon > 0$ can be chosen arbitrarily small. Therefore, equality (21) yields the desired estimate. □

Lemma 4 *Let $\alpha \ge 3/2$, $3 \le m \le 2\alpha$. Then for any $\beta > 0$, $\delta \in (0,T)$, $x_0 \in \mathbb{R}$ inequality (18) holds for $3 \le |\nu| \le m$, where the constant on the right-hand side depends on T, α, β, δ, x_0, $\|u_0\|_{L_2^\alpha}$ and $\|D^\nu f\|_{L_1(0,T;L_2^{\alpha-|\nu|/2})}$ for $|\nu| \le m$.*

Proof Use the induction on $l = |\nu|$.

By virtue of the induction argument for any $\beta > 0$, $\delta \in (0,T)$ and $x_0 \in \mathbb{R}$ for $|\nu| \le l \le m$ uniformly with respect to $x_1 \ge x_0$

$$
\int_{\delta/2}^{T} \iint \big(D^{\nu} u\big)^2 \rho_{2\alpha-l,\beta}(x-x_1) dxdydt \le c, \tag{27}
$$

since $2\alpha - l + 1 > 0$ and therefore $\rho_{2\alpha-l,\beta} \sim \rho'_{2\alpha-(l-1),\beta}$.

Let $\varphi(t) \equiv \eta(2t/\delta - 1)$, $\rho(x) \equiv \rho_{2\alpha-l,\beta}(x-x_1)$. For any multi-index ν, $|\nu| = l$, multiplying (1) by $2(-1)^l D^\nu(D^\nu u\rho)\varphi$ and integrating over $\mathbb{R}^2$ we derive equality (21).

By virtue of (27)

$$
\int_0^T \iint \big(D^{\nu} u\big)^2 \rho\varphi' dxdydt \le c.
$$

In order to estimate the integral of the nonlinear term consider items of the type

$$
\iint D^{\nu_1} u D^{\nu_2} u_x D^{\nu} u \rho\varphi dxdy,
$$

where $\nu_1 + \nu_2 = \nu$. For $|\nu_1| = 0$ (then $\nu_2 = \nu$) repeat argument (23).

Similarly to the proof of the preceding lemma consider the integral of $D^{\nu_1} u \times D^{\nu_2} u D^{\nu} u \rho\varphi$, where $|\nu_1| \le 1$, $|\nu_2| = l$. Note that $2\alpha - l < (\alpha - 3/4) + (\alpha - l/2) +$

$(\alpha - l/2 - 1/4)$ since $\alpha \geq 3/2$ and as in (24)

$$\left| \iint D^{\nu_1} u D^{\nu_2} u D^{\nu} u \rho\varphi dxdy \right|$$
$$\leq \left(\iint \left(D^{\nu_1} u\right)^4 \rho^2_{2\alpha-3/2,\beta/2} \varphi dxdy \right)^{1/4}$$
$$\times \left(\iint \left(D^{\nu_2} u\right)^2 \rho_{2\alpha-l,\beta/2} \varphi^{1/2} dxdy \right)^{1/2} \left(\iint \left(D^{\nu} u\right)^4 \rho' \rho \varphi^2 dxdy \right)^{1/4}.$$

Here

$$\iint \left(D^{\nu_1} u\right)^4 \rho^2_{2\alpha-3/2,\beta/2} \varphi dxdy$$
$$\sim \iint \left(D^{\nu_1} u\right)^4 \rho_{2\alpha-2,\beta/2} \rho_{2\alpha-1,\beta/2} \varphi dxdy$$
$$\leq c \iint \left(\left(D^{\nu_1} u_x\right)^2 + \left(D^{\nu_1} u_y\right)^2\right) \rho_{2\alpha-2,\beta/2} \varphi^{1/2} dxdy$$
$$\times \iint \left(D^{\nu_1} u\right)^2 \rho_{2\alpha-1,\beta/2} \varphi^{1/2} dxdy$$
$$+ \left(\iint \left(D^{\nu_1} u\right)^2 \rho_{2\alpha-1,\beta/2} \varphi^{1/2} dxdy \right)^2.$$

Similarly to (25), (26) by virtue of the already obtained inequality (18) for $|\nu| = 2$ we find that

$$\left| \iint D^{\nu_1} u D^{\nu_2} u D^{\nu} u \rho\varphi dxdy \right| \leq \varepsilon \iint \left(\left(D^{\nu} u_x\right)^2 + \left(D^{\nu} u_y\right)^2\right) \rho' \varphi dxdy$$
$$+ c(\varepsilon) \iint \left(D^{\nu} u\right)^2 \rho\varphi dxdy + \gamma(t),$$

where $\|\gamma\|_{L_1(0,T)} \leq c$ and $\varepsilon > 0$ can be chosen arbitrarily small.

Now let $|\nu_1| \leq l - 1$, $|\nu_2| \leq l - 2$, then

$$\left| \iint D^{\nu_1} u D^{\nu_2} u_x D^{\nu} u \rho\varphi dxdy \right|$$
$$\leq \left(\iint \left(D^{\nu_1} u D^{\nu_2} u_x\right)^2 \rho\varphi dxdy \right)^{1/2} \left(\iint \left(D^{\nu} u\right)^2 \rho\varphi dxdy \right)^{1/2}. \tag{28}$$

Note that

$$\rho_{2\alpha-l,\beta}(x) \leq c\rho^2_{2\alpha-l,\beta/2}(x). \tag{29}$$

Again applying the interpolation inequality (13) we derive that

$$\Big(\iint \big(D^{\nu_1} u D^{\nu_2} u_x\big)^2 \rho\varphi\,dxdy\Big)^{1/2}$$
$$\le c \sum_{|\nu|\le l-1} \Big(\iint \big(D^{\nu} u\big)^4 \rho^2_{2\alpha-l,\beta/2}\varphi\,dxdy\Big)^{1/2}$$
$$\le c_1 \sum_{|\nu|\le l} \iint \big(D^{\nu} u\big)^2 \rho_{2\alpha-l,\beta/2}\varphi^{1/2}\,dxdy. \tag{30}$$

With the use of (27) we finish the proof. □

In [3] weak solutions are constructed as limits of solutions to "smooth" problems. Therefore, Theorem 1 succeeds from Lemmas 1–4.

Now we turn to estimates of solutions depending on norms of u_0 and f in the spaces $H^{1,\alpha}$. Note that by virtue of the results from [4] the norms of the function $u \in K_1(0,T)$ in the spaces $C([0,T]; H^1(\mathbb{R}^2))$ and $L_3(0,T; W^1_\infty(\mathbb{R}^2))$ are already estimated in an appropriate way. Then all the consequent estimates can be obtained considerably easier.

Lemma 5 *Let $\alpha > 0$. Then for any $\beta > 0$ and $x_0 \in \mathbb{R}$ for $|\nu| = 1$*

$$\sup_{0\le t\le T} \iint_{\mathbb{R}^2} \big(D^{\nu} u\big)^2 \rho_{2\alpha,\beta}(x-x_0)\,dxdy$$
$$+ \sup_{x_1\ge x_0} \int_0^T \iint_{\mathbb{R}^2} \big(\big(D^{\nu} u_x\big)^2 + \big(D^{\nu} u_y\big)^2\big)\rho'_{2\alpha,\beta}(x-x_1)\,dxdydt \le c, \tag{31}$$

where the constant c depends on T, α, β, x_0, $\|u_0\|_{H^{1,\alpha}}$, $\|f\|_{L_1(0,T;H^{1,\alpha})}$.

Proof Use equality (19) for $\varphi \equiv 1$. Since

$$\Big|\iint \big(u_x\rho - u\rho'\big)\big(u_x^2 + u_y^2\big)\,dxdy\Big|$$
$$\le c \sup_{(x,y)\in\mathbb{R}^2} \big(|u_x| + |u|\big) \iint \big(u_x^2 + u_y^2\big)\rho\,dxdy,$$

where the norm of sup in the space $L_1(0,T)$ can be estimated in an appropriate way by virtue of the estimate of the solution in the space $L_3(0,T; W^1_\infty(\mathbb{R}^2))$, equality (19) provides (31). □

Remark 4 In the proof of the preceding lemma the second conservation law (3) is formally not used, but it is used in [4] for the proof of global estimates of solutions to the initial value problem for Zakharov–Kuznetsov equation, in particular, in the space $K_1(0,T)$.

Lemma 6 *Let $\alpha \geq 1/2$, $2 \leq m \leq 2\alpha+1$. Then for any $\beta > 0$, $\delta \in (0,T)$, $x_0 \in \mathbb{R}$ if $2 \leq |\nu| \leq m$*

$$\sup_{\delta \leq t \leq T} \iint_{\mathbb{R}^2} \big(D^\nu u\big)^2 \rho_{2\alpha-|\nu|+1,\beta}(x-x_0)\,dxdy \\ + \sup_{x_1 \geq x_0} \int_\delta^T \iint_{\mathbb{R}^2} \big(\big(D^\nu u_x\big)^2 + \big(D^\nu u_y\big)^2\big)\rho'_{2\alpha-|\nu|+1,\beta}(x-x_1)\,dxdydt \leq c, \tag{32}$$

where the constant on the right-hand side of the inequality depends on T, α, β, δ, x_0, $\|u_0\|_{H^{1,\alpha}}$, $\|f\|_{L_1(0,T;H^{1,\alpha})}$ and $\|D^\nu f\|_{L_1(0,T;L_2^{\alpha-|\nu|/2+1/2})}$ for $2 \leq |\nu| \leq m$.

Proof Use the induction on $l = |\nu|$.

By virtue of the induction argument for any $\beta > 0$, $\delta \in (0,T)$ and $x_0 \in \mathbb{R}$ for $|\nu| \leq l \leq m$ uniformly with respect to $x_1 \geq x_0$

$$\int_{\delta/2}^T \iint \big(D^\nu u\big)^2 \rho_{2\alpha-l+1,\beta}(x-x_1)\,dxdydt \leq c, \tag{33}$$

since $2\alpha - l + 2 \geq 1$ and therefore $\rho_{2\alpha-l+1,\beta} \sim \rho'_{2\alpha-(l-1)+1,\beta}$.

Let $\varphi(t) \equiv \eta(2t/\delta - 1)$, $\rho(x) \equiv \rho_{2\alpha-l+1,\beta}(x-x_1)$. As in the proof of Lemma 4 consider equality (21) for $|\nu| = l$. By virtue of (33)

$$\int_0^T \iint \big(D^\nu u\big)^2 \rho\varphi'\,dxdydt \leq c.$$

In order to estimate the integral of the nonlinear term consider items of the type

$$\iint D^{\nu_1}uD^{\nu_2}u_x D^\nu u\rho\varphi\,dxdy,$$

where $\nu_1 + \nu_2 = \nu$. If $|\nu_1| = 0$ then $\nu_2 = \nu$ and we repeat argument (23). Thus, for $|\nu_1| = 0$, $|\nu_1| = 1$ and $|\nu_2| = 0$ the absolute value of the corresponding integral does not exceed

$$c \sup_{(x,y)\in\mathbb{R}^2} \big(|u_x| + |u_y| + |u|\big) \sum_{|\nu|=l} \iint \big(D^\nu u\big)^2 \rho\varphi\,dxdy.$$

Now let $|\nu_1| \leq l-1$, $|\nu_2| \leq l-2$. Write down inequality (28). Similarly to (29)

$$\rho_{2\alpha-l+1,\beta}(x) \leq c\rho^2_{2\alpha-l+1,\beta/2}(x),$$

and, similarly to (30),

$$\Big(\iint \big(D^{\nu_1}uD^{\nu_2}u_x\big)^2\rho\varphi\,dxdy\Big)^{1/2} \le c\sum_{|\nu|\le l-1}\Big(\iint \big(D^{\nu}u\big)^4\rho^2_{2\alpha-l+1,\beta/2}\varphi\,dxdy\Big)^{1/2}$$
$$\le c_1\sum_{|\nu|\le l}\iint \big(D^{\nu}u\big)^2\rho_{2\alpha-l+1,\beta/2}\varphi^{1/2}\,dxdy.$$

The end of the proof is the same as for Lemma 4. □

By virtue of well-posedness of the considered problem in the space $K_1(0,T)$ Theorems 2 and 3 follow from Lemmas 5 and 6.

3.3 Fundamental Solution to the Linearized Equation

The fundamental solutions to the operator $\partial_t + \partial^3_{xxx} + \partial^3_{xyy}$ is evidently given by the formula

$$G(t,x,y) = \theta(t)\mathcal{F}^{-1}\big[e^{it(\xi_1^3+\xi_1\xi_2^2)}\big] \equiv \frac{\theta(t)}{t^{2/3}}S\Big(\frac{x}{t^{1/3}},\frac{y}{t^{1/3}}\Big), \tag{34}$$

where

$$S(x,y) \equiv \mathcal{F}^{-1}\big[e^{i(\xi_1^3+\xi_1\xi_2^2)}\big]$$

(see, for example, pp. 200–201 in [13]) and θ is the Heaviside function. The study of the function S was begun in [4], where it was proved that this function existed and was bounded on $\mathbb{R}^2$. Besides that, in that paper the following representation was derived:

$$S(x,y) = \frac{1}{4\pi^{3/2}}\int_{\mathbb{R}} |\xi|^{-1/2} e^{i(\xi^3+\xi x - y^2/(4\xi)+(\pi/4)\operatorname{sign}\xi)}\,d\xi. \tag{35}$$

In fact, in the proof of (35) the inverse Fourier transform is applied, firstly with respect to y and then with respect to x. In the consequent lemma for an alternative representation of the function S the order is changed to the opposite one.

Lemma 7 *The function $S(x,y)$ is infinitely differentiable on $\mathbb{R}^2$, in any point it satisfies the equation*

$$3S_{xx} + S_{yy} - xS = 0, \tag{36}$$

and is represented in the form

$$S(x,y) = \mathcal{F}_y^{-1}\big[A\big(x+\xi^2\big)\big](y) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\xi y}A\big(x+\xi^2\big)\,d\xi, \tag{37}$$

where $A(x) \equiv \mathcal{F}^{-1}[e^{i\xi^3}](x)$ *is the Airy function. Moreover, for any* $x \in \mathbb{R}$ *and integer* $k \geq 0$ *the derivative* $\partial_x^k S(x, y)$ *belongs to the Schwartz space* $\mathcal{S}(\mathbb{R})$ *with respect to* y *and there exists a constant* $c_0 > 0$ *such that for any* $x_0 \in \mathbb{R}$, *integer* $m \geq 0$ *and multi-index* ν

$$\big(1 + |y|\big)^m \big|D^\nu S(x, y)\big| \leq c\big(m, |\nu|, x_0\big)e^{-c_0(x-x_0)^{3/2}} \quad \forall x \geq x_0, \forall y \in \mathbb{R}. \tag{38}$$

Proof Properties of the Airy function are well-known (see, for example, pp. 184–186 in [6]). In particular, the function $A(x)$ is infinitely differentiable and $|A^{(n)}(x)| \leq c(n)e^{-c_0 x^{3/2}}$ if $x \geq 0$ for any integer $n \geq 0$ and with certain positive constants $c(n)$ and c_0. Then for any $x_0 \in \mathbb{R}$ it holds $|A^{(n)}(x)| \leq c(n, x_0)e^{-c_0(x-x_0)^{3/2}}$ if $x \geq x_0$ and, therefore,

$$\big|A^{(n)}\big(x + \xi^2\big)\big| \leq c(n, x_0)e^{-c_0|\xi|^3} e^{-c_0(x-x_0)^{3/2}} \quad \forall x \geq x_0.$$

It is easy to see that

$$\partial_\xi^n A\big(x + \xi^2\big) = \sum_{0 \leq n_1, n_2 \leq n} c_{n_1,n_2} \xi^{n_1} A^{(n_2)}\big(x + \xi^2\big),$$

and so for any integer $k, n, m \geq 0$ if $x \geq x_0$

$$\big(1 + |\xi|\big)^m \big|\partial_x^k \partial_\xi^n A\big(x + \xi^2\big)\big| \leq c(k, n, m, x_0)e^{-c_0(x-x_0)^{3/2}}. \tag{39}$$

Therefore, the function $\partial_x^k A(x+\xi^2)$ belongs to the space $\mathcal{S}(\mathbb{R})$ with respect to ξ. The properties of the Fourier transform provide that the function $\mathcal{F}_y^{-1}[\partial_x^k A(x+\xi^2)]$ also belongs to $\mathcal{S}(\mathbb{R})$ with respect to y. In addition, since $S(x, y) = \mathcal{F}_y^{-1}[\mathcal{F}_x^{-1}[e^{i(\xi_1^3+\xi_1\xi_2^2)}]](x, y)$ applying inequality (39) we obtain formula (37) and inequality (38). Moreover, since the Airy function satisfies the equation $3A''(x) - xA(x) = 0$ equality (37) provides (36). □

Remark 5 Previously in [5] it was proved that $S \in \mathcal{S}(\overline{\mathbb{R}}_+^2)$ (restriction of the space $\mathcal{S}(\mathbb{R}^2)$ on $\overline{\mathbb{R}}_+^2$; in fact, in that paper a more general case was considered). Lemma 7 refines this result.

The next lemma refines behavior of the function S as $x \to -\infty$.

Lemma 8 *For any* $r \in [0, 2/3]$ *and integer* $n \in [0, 2]$

$$\big|\partial_x^n S(x, y)\big| \leq c(r)\big(1 + |y|\big)^{-r} \big(1 + |x|\big)^{r+n/2-1/4} \quad \forall x \leq 0, \forall y \in \mathbb{R}. \tag{40}$$

Proof By virtue of the previous lemma without loss of generality one can assume that $x \leq -1$. Let

$$\varphi(\xi) \equiv \xi x + \xi^3 - \frac{y^2}{4\xi}, \qquad \psi_n(\xi) \equiv (-i\xi)^{n-1/2},$$

where the branch of the root $\sqrt{z}$ is defined in the domain $\mathbb{C} \setminus (-\infty; 0]$ and $\sqrt{z} = r^{1/2} e^{i\varphi/2}$ for $z = re^{i\varphi}$, $-\pi < \varphi < \pi$,

$$I_n(x, y) \equiv \int_{\mathbb{R}} \psi_n(\xi) e^{i\varphi(\xi)} d\xi.$$

Equality (35) implies that $S(x, y) \equiv cI_0(x, y)$. It is easy to see that

$$\begin{aligned} &\varphi'(\xi) = x + 3\xi^2 + \frac{y^2}{4\xi^2}, \qquad \varphi''(\xi) = 6\xi - \frac{y^2}{2\xi^3}, \\ &\varphi'''(\xi) = 6 + \frac{3y^2}{2\xi^4} \geq c|y|^{3r} |\xi|^{-6r} \end{aligned} \tag{41}$$

for $r \in [0, 2/3]$.

In order to estimate I_n we divide the real line into several parts. First consider $\Omega_1 = \{\xi : \xi^2 \geq |x|/2\}$. Then

$$\varphi'(\xi) \geq \frac{x^2 + 4\xi^4 + 2y^2}{8\xi^2} \tag{42}$$

ant integrating by parts twice we derive that

$$\begin{aligned} \left| \int_{\Omega_1} \psi_n e^{i\varphi} d\xi \right| \leq &\left(\left| \frac{\psi_n}{\varphi'} \right| + \left| \frac{\psi_n'}{(\varphi')^2} \right| + \left| \frac{\psi_n \varphi''}{(\varphi')^3} \right| \right) \Big|_{|\xi| = |x/2|^{1/2}} \\ &+ \int_{\Omega_1} \left(\left| \frac{\psi_n''}{(\varphi')^2} \right| + 3 \left| \frac{\psi_n' \varphi''}{(\varphi')^3} \right| + \left| \frac{\psi_n \varphi'''}{(\varphi')^3} \right| + 3 \left| \frac{\psi_n (\varphi'')^2}{(\varphi')^4} \right| \right) d\xi. \end{aligned} \tag{43}$$

Since $n < 5/2$ and

$$\left|\varphi''(\xi)\right| \leq c \frac{\xi^4 + y^2}{|\xi|^3}, \qquad \left|\varphi'''(\xi)\right| \leq c \frac{\xi^4 + y^2}{\xi^4}, \tag{44}$$

inequalities (42)–(44) yield that

$$\begin{aligned} \left| \int_{\Omega_1} \psi_n e^{i\varphi} d\xi \right| &\leq c \frac{|x|^{n/2+3/4}}{x^2 + y^2} + c \int_{\mathbb{R}} \frac{|\xi|^{n+3/2}}{(\xi^4 + x^2 + y^2)^2} d\xi \\ &\leq c \frac{|x|^{n/2+3/4}}{x^2 + y^2} + c_1 \left(x^2 + y^2\right)^{n/4 - 11/8} \\ &\leq c_2 \left(1 + |y|\right)^{-r} \left(1 + |x|\right)^{r + n/2 - 5/4}. \end{aligned} \tag{45}$$

In particular, note that the performed argument provides that the integrals I_n converge uniformly in the neighborhood of any considered point (x, y) and so $\partial_x^n S(x, y) = (-1)^n c I_n(x, y)$.

In order to estimate the remaining part of the integral I_n we consider separately two cases.

(1) First, let $y^2 \ge x^2/4$. In this case estimate (40) for all y is sufficient to prove only for $r=0$ while for $r>0$ assume that $|y| \ge 1$.

Let $\Omega_2 = \{\xi : |x|/32 \le \xi^2 \le |x|/2\}$, $\Omega_3 = \{\xi : \xi^2 \le |x|/32\}$.

Further we use the classical van der Corput lemma (see pp. 309–312 in [12]).

Lemma 9 *Let I be a certain interval on $\mathbb{R}$, $k \ge 2$, $\varphi \in C^k(\overline{I})$ be a real-valued function. Assume that there exists $\lambda > 0$ such that $|\varphi^{(k)}(x)| \ge \lambda$ for any $x \in I$. Let a function ψ be such that $\psi \in L_\infty(I)$, $\psi' \in L_1(I)$. Then if the integral $\int_I e^{i\varphi(x)}\psi(x)dx$ converges there exists a constant $c(k) > 0$ independent of I such that*

$$\left|\int_I e^{i\varphi(x)}\psi(x)dx\right| \le c(k)\lambda^{-1/k}\big(\|\psi\|_{L_\infty(I)} + \|\psi'\|_{L_1(I)}\big).$$

Apply van der Corput lemma on the set Ω_2 for $k=3$. Then (41) implies that $|\varphi'''(\xi)| \ge c|y|^{3r}|x|^{-3r}$ and, therefore,

$$\begin{aligned}\left|\int_{\Omega_2} \psi_n e^{i\varphi} d\xi\right| &\le c|y|^{-r}|x|^r \sup_{\xi\in\Omega_2} |\xi|^{n-1/2} \\ &\le c_1\big(1+|y|\big)^{-r}\big(1+|x|\big)^{r+|n|/2-1/4}. \qquad (46)\end{aligned}$$

Next, it is easy to see that on the set Ω_3

$$\varphi'(\xi) \ge 3\xi^2 + \frac{4x\xi^2 + x^2/8}{4\xi^2} + \frac{y^2}{8\xi^2} \ge 3\xi^2 + \frac{y^2}{8\xi^2} \ge \frac{192\xi^4 + 4y^2 + x^2}{64\xi^2},$$

that is, the analog of inequality (42) holds. Therefore, similarly to (43)–(45) one can obtain for Ω_3 the same estimate as for Ω_1.

(2) Now, let $y^2 \le x^2/4$. Obviously, in this case it is sufficient to prove estimate (40) for $r=0$. Let $y^2 = px^2$, $0 \le p \le 1/4$. Define

$$\Omega_4 = \left\{\xi : \frac{|x|}{6} \le \xi^2 \le \frac{|x|}{2}\right\}, \qquad \Omega_5 = \left\{\xi : \frac{p}{2}|x| \le \xi^2 \le \frac{|x|}{6}\right\},$$

$$\Omega_6 = \left\{\xi : \xi^2 \le \frac{p}{2}|x|\right\}.$$

Then the estimate for Ω_4 is the same as (46) for $r=0$.

Consider Ω_5. We show that $\varphi'(\xi) < 0$ and $|\varphi'(\xi)| \ge |x|/8$ for $\xi \in \Omega_5$. In fact, $\varphi'(\pm\sqrt{p|x|/2}) = \varphi'(\pm\sqrt{|x|/6}) = |x|(3p-1)/2 \le -|x|/8$. Moreover, $\varphi''(\xi) = 0$ if $\xi = \pm\sqrt[4]{p/12}|x|^{1/2}$ and $\varphi'(\pm\sqrt[4]{p/12}|x|^{1/2}) = |x|(\sqrt{3p}-1) \le -|x|/8$.

Since

$$\big|\varphi''(\xi)\big| \le 6|\xi| + \frac{px^2}{2|\xi|^3} \le 6|\xi| + \frac{|x|}{|\xi|},$$

integrating by parts we derive that

$$\begin{aligned}\left|\int_{\Omega_5}\psi_n e^{i\varphi}d\xi\right| &\le \int_{|\xi|\le|x|^{-1/2}}|\xi|^{n-1/2}d\xi + \left|\int_{\Omega_5\setminus\{|\xi|\le|x|^{-1/2}\}}\psi_n e^{i\varphi}d\xi\right|\\
&\le c|x|^{-n/2-1/4} + \sup_{\xi\in\Omega_5\setminus\{|\xi|\le|x|^{-1/2}\}}\left|\frac{\psi_n}{\varphi'}\right|\\
&\quad + \int_{\Omega_5\setminus\{|\xi|\le|x|^{-1/2}\}}\left(\left|\frac{\psi_n'}{\varphi'}\right| + \left|\frac{\psi_n\varphi''}{(\varphi')^2}\right|\right)d\xi\\
&\le c|x|^{-n/2-1/4} + \frac{c}{|x|}\sup_{|x|^{-1/2}\le|\xi|\le\sqrt{|x|/6}}|\xi|^{n-1/2}\\
&\quad + \frac{c}{|x|}\int_{|x|^{-1/2}}^{\sqrt{|x|/6}}|\xi|^{n-3/2}d\xi + \frac{c}{x^2}\int_0^{\sqrt{|x|/6}}|\xi|^{n+1/2}d\xi\\
&\le c_1|x|^{n/2-1/4}.\end{aligned}$$

Finally, consider Ω_6. Note that the function $\varphi''(\xi)$ increases on $\mathbb{R}_+$, $\varphi''(\sqrt{p|x|/2}) = \sqrt{2/p}(3p-1)|x|^{1/2} < 0$, therefore,

$$\inf_{\xi\in\Omega_6,\xi>0}|\varphi''(\xi)| = \left|\varphi''\left(\sqrt{p|x|/2}\right)\right| = \sqrt{\frac{2}{p}}(1-3p)|x|^{1/2} \ge |x|^{1/2}/\sqrt{2}.$$

The case $\xi < 0$ is similar. Applying van der Corput lemma for $k = 2$ we find that

$$\begin{aligned}\left|\int_{\Omega_6\cap\{|\xi|\ge1\}}\psi_n e^{i\varphi}d\xi\right| &\le c\left(\inf_{\xi\in\Omega_6}|\varphi''(\xi)|\right)^{-1/2}\sup_{1\le|\xi|\le\sqrt{|x|/8}}|\xi|^{n-1/2}\\
&\le c_1|x|^{n/2-1/4}.\end{aligned}$$

Now let $|\xi| \le 1$. Then if $p|x| \ge 2$ since $|x| \ge 2/p \ge 8$

$$\inf_{\xi\in\Omega_6,|\xi|\le1}|\varphi''(\xi)| = |\varphi''(1)| = \frac{px^2}{2} - 6 \ge \frac{|x|}{4},$$

while if $p|x| \le 2$ then $2/p \ge |x|$ and also

$$\inf_{\xi\in\Omega_6,|\xi|\le1}|\varphi''(\xi)| = \left|\varphi''\left(\sqrt{p|x|/2}\right)\right| = \sqrt{\frac{2}{p}}(1-3p)|x|^{1/2} \ge \frac{|x|}{4}.$$

Thus, application of van der Corput lemma for $k = 2$ yields that

$$\begin{aligned}\left|\int_{\Omega_6}\psi_n e^{i\varphi}d\xi\right| &\le \int_{|\xi|\le|x|^{-1/2}}|\xi|^{n-1/2}d\xi + \left|\int_{\Omega_6\cap\{|x|^{-1/2}\le|\xi|\le1\}}\psi_n e^{i\varphi}d\xi\right|\\
&\le c|x|^{-1/4}.\end{aligned}$$

□

Remark 6 The fundamental solution to the operator $\partial_t + \partial^3_{xxx}$ is the function $\theta(t)t^{-1/3}A(x/t^{1/3})$, where A is the aforementioned Airy function. The asymptotic behavior at infinity of this function is well-known: $A(x) \sim c_-|x|^{-1/4}\sin(2|x|^{3/2}/3^{3/2} + \phi)$ as $x \to -\infty$, $A(x) \sim c_+ x^{-1/4}\exp(-2x^{3/2}/3^{3/2})$ as $x \to +\infty$ (see, for example, pp. 184–186 in [6]). Therefore, we can conclude that the estimates (38) and (40) are sharp.

Corollary 1 *For any* $r \in [0, 2/3]$ *and natural* $n \le 2$

$$\left|\partial^n_y S(x, y)\right| \le c(r)\big(1 + |y|\big)^{-r}\big(1 + |x|\big)^{r+n/2-1/4} \quad \forall x \le 0, \forall y \in \mathbb{R}. \tag{47}$$

Proof For $n = 2$ this estimate follows evidently from (40) and equality (36). For $n = 1$ apply the interpolation inequality

$$\max_{|y|\ge y_0} S^2_y(x, y) \le 8 \max_{|y|\ge y_0}\left|S(x, y)\right| \cdot \max_{|y|\ge y_0}\left|S_{yy}(x, y)\right|$$

(see, for example, p. 237 in [1]), which together with the non-increasing behavior of the function $(1 + |y|)^{-r}\operatorname{sign} y$ yields the desired result. □

Lemma 10 *The function* $S(x, y)$ *satisfies in any point of the plane* $\mathbb{R}^2$ *the equation*

$$2S_{xy} - yS = 0. \tag{48}$$

Proof As in the previous lemma we use representation (35). Then for all points (x, y) if $n \in [0, 2]$

$$\partial^n_x S(x, y) = \frac{1}{4\pi^{3/2}} \int_{\mathbb{R}} (-i\xi)^{-1/2}(i\xi)^n e^{i(\xi^3+\xi x-y^2/(4\xi))} d\xi, \tag{49}$$

where the integrals on the right-hand side converge uniformly in a neighborhood of any point. In the proof of Lemma 8 it was obtained for $x \le -1$. In the case $x > -1$ note that $\varphi'(\xi) \ge \xi^2 + 1$ for $\xi^2 \ge 1$ (we use the same notation φ and ψ_n as in the proof of Lemma 8) and thus after integration by parts we find that

$$\left|\int_{|\xi|\ge N} \psi_n e^{i\varphi} d\xi\right| \le \left|\frac{\psi_n}{\varphi'}\Big|_{|\xi|=N}\right| + \int_{|\xi|\ge N}\left(\left|\frac{\psi_n'}{\varphi'}\right| + \left|\frac{\psi_n\varphi''}{(\varphi')^2}\right|\right)d\xi,$$

whence the desired uniform convergence succeeds. Therefore,

$$\begin{aligned} S_{xy}(x, y) &= \frac{1}{4\pi^{3/2}} \int_{\mathbb{R}} (-i\xi)^{-1/2}(i\xi)e^{i(\xi^3+\xi x-y^2/(4\xi))}\left(-\frac{2iy}{4\xi}\right)d\xi \\ &= \frac{y}{2}S(x, y). \end{aligned}$$

□

Remark 7 Using both already established estimates (40), (47) and equalities (36), (48) one can easily estimate derivatives of the function S of any order for $x \le 0$. For example, (48) yields that for $r \in [0, 2/3]$

$$|S_{xy}| \le c(r)\big(1+|y|\big)^{1-r}\big(1+|x|\big)^{r-1/4},$$

$$|S_{xxy}|, |S_{xyy}| \le c(r)\big(1+|y|\big)^{1-r}\big(1+|x|\big)^{r+1/4},$$

and then with the use of (36) we find that

$$|S_{xxx}|, |S_{yyy}| \le c(r)\big(1+|y|\big)^{1-r}\big(1+|x|\big)^{r+1/4} + c(r)\big(1+|y|\big)^{-r}\big(1+|x|\big)^{r+5/4}.$$

The defect of these estimates is their unboundedness as $y \to \infty$.

Remark 8 In [11] on the basis of the method from [4] it was established that the fractional derivative with respect to x of the function S was bounded on $\mathbb{R}^2$, namely, it was proved that if

$$D_x^{\varepsilon+i\beta} S(x,y) \equiv \mathcal{F}^{-1}\big[|\xi_1|^{\varepsilon+i\beta} e^{i(\xi_1^3+\xi_1\xi_2^2)}\big],$$

then for $0 \le \varepsilon < 1/2$, $\beta \in \mathbb{R}$

$$\big|D_x^{\varepsilon+i\beta} S(x,y)\big| \le c.$$

Using the argument of the present paper one can easily establish this property for $\varepsilon = 1/2$ also. In fact, similarly to (35)

$$D_x^{1/2+i\beta} S(x,y) = \frac{1}{4\pi^{3/2}} \int_{\mathbb{R}} |\xi|^{i\beta} e^{i(\xi^3+\xi x - y^2/(4\xi) + (\pi/4)\operatorname{sign}\xi)}\, d\xi.$$

Obviously, in order to estimate this integral it is sufficient to assume that $|\xi| \ge 1$. Then for $x \le -1$ the argument repeats the one from Lemma 8 (for $r = 0$), while for $x > -1$ it is sufficient to note that $\varphi'(\xi) \ge \xi^2 + (4\xi^4 + y^2)/(4\xi^2)$ and argue similarly to (43)–(45).

3.4 Continuous Derivatives

Theorems 4 and 5 succeed from the following lemma.

Lemma 11 *Let $D^{\nu_0} f \in L_\infty(0,T; L_2^\alpha)$ for a certain $\alpha > 3/4$ and multi-index ν_0. Assume that a function $u(t,x,y) \in L_\infty(0,T; L_2^\alpha)$ satisfies (1) in the layer Π_T (possibly, in the weak sense) and possesses in Π_T Sobolev derivatives $D^\nu u$ for $|\nu| \le |\nu_0|$ and $D^{\nu_0} u_x$, where for any $\delta \in (0,T)$, $x_0 \in \mathbb{R}$*

$$(x - x_0 + 1)^\alpha D^{\nu_0} u \in L_\infty\big(\delta, T; L_2\big((x_0, +\infty) \times \mathbb{R}\big)\big),$$

$$D^{\nu_0} u_x, D^\nu u \in L_\infty\big(\delta, T; L_2\big((x_0, +\infty) \times \mathbb{R}\big)\big), \quad |\nu| \le |\nu_0|.$$

Then the derivative $D^{\nu_0} u$ is continuous in Π_T and bounded on any set of the type $[\delta, T] \times [x_0, +\infty) \times \mathbb{R}$.

Proof Let $\varphi(t) \equiv \eta(2t/\delta - 1)$, $\psi(x) \equiv \eta(x - x_0 + 2)$. Then $w(t,x,y) \equiv D^{\nu_0}u(t,x,y)\varphi(t)\psi(x)$ is a solution (generally speaking, in the sense of distributions) in the layer Π_T to the linear initial value problem

$$\begin{aligned} w_t + w_{xxx} + w_{xyy} &= D^{\nu_0} f\varphi\psi - D^{\nu_0}(uu_x)\varphi\psi + D^{\nu_0}u\varphi'\psi \\ &\quad + 3D^{\nu_0}u_{xx}\varphi\psi' + 3D^{\nu_0}u_x\varphi\psi'' + D^{\nu_0}u\varphi\psi''' + D^{\nu_0}u_{yy}\varphi\psi' \\ &\equiv F(t,x,y), \qquad (50) \\ w|_{t=0} &= 0. \qquad (51) \end{aligned}$$

With the use of the fundamental solution to the operator $\partial_t + \partial^3_{xxx} + \partial^3_{xyy}$ (see (34)) we write the function w in the form

$$w(t,x,y) = \int_0^t \iint G(t-\tau, x-\xi, y-\zeta)F(\tau,\xi,\zeta)\,d\zeta\,d\xi\,d\tau. \qquad (52)$$

Let $\delta \le t \le T$, $x \ge x_0$. Then $w(t,x,y) = D^{\nu_0}u(t,x,y)$. Let us estimate the right-hand side of equality (52). First consider the integral of $D^{\nu_0} f\varphi\psi$. By virtue of (38) and (40)

$$\begin{aligned} &\int_{\delta/2}^t \int_{x_0-2}^{+\infty} \int_{\mathbb{R}} \big|G(t-\tau,x-\xi,y-\zeta)D^{\nu_0} f(\tau,\xi,\zeta)\big|\,d\zeta\,d\xi\,d\tau \\ &\le c\int_{\delta/2}^t \frac{1}{(t-\tau)^{2/3}}\bigg(\int_{x_0-2}^x \int_{\mathbb{R}} \frac{e^{-c_0(x-\xi)^{3/2}T^{-1/2}}|D^{\nu_0} f(\tau,\xi,\zeta)|}{(1+|\zeta-y|)}\,d\zeta\,d\xi \\ &\quad + \frac{1}{(t-\tau)^{(1+\varepsilon)/12}} \int_x^{+\infty} \int_{\mathbb{R}} \frac{(1+\xi-x)^{3/4+\varepsilon/2}|D^{\nu_0} f(\tau,\xi,\zeta)|}{(1+|\zeta-y|)^{(2+\varepsilon)/4}(1+\xi-x)^{(2+\varepsilon)/4}}\,d\zeta\,d\xi\bigg)d\tau \\ &\le c_1 \operatorname{ess\,sup}_{\tau\in(\delta/2,T)}\bigg(\int_{x_0-2}^{+\infty}\int_{\mathbb{R}} (3+\xi-x_0)^{3/2+\varepsilon}\big(D^{\nu_0} f(\tau,\xi,\zeta)\big)^2\,d\zeta\,d\xi\bigg)^{1/2} \\ &< \infty. \end{aligned}$$

The integral of $D^{\nu_0}u\varphi'\psi$ is estimated in a similar way. Next, $D^{\nu_0}(uu_x)\varphi\psi \in L_\infty(0,T;L_1(\mathbb{R}^2))$ and by virtue of the boundedness of the function S on the whole plane $\mathbb{R}^2$

$$\begin{aligned} &\int_{\delta/2}^t \int_{x_0-2}^{+\infty} \int_{\mathbb{R}} \big|G(t-\tau,x-\xi,y-\zeta)D^{\nu_0}(uu_\xi)\big|\,d\zeta\,d\xi\,d\tau \\ &\quad \le c\int_{\delta/2}^t \frac{1}{(t-\tau)^{2/3}} \int_{x_0-2}^{+\infty} \int_{\mathbb{R}} \big|D^{\nu_0}(uu_\xi)\big|\,d\zeta\,d\xi\,d\tau < \infty. \end{aligned}$$

Finally, since $\operatorname{supp}\psi' \subset [x_0-2, x_0-1]$ by virtue of (38)

$$\Big| \int_{\delta/2}^{t} \int_{x_0-2}^{+\infty} \int_{\mathbb{R}} G(t-\tau, x-\xi, y-\zeta)$$
$$\times \big(3D^{\nu_0} u_{\xi\xi} \psi' + 3D^{\nu_0} u_{\xi} \psi'' + D^{\nu_0} u \psi''' + D^{\nu_0} u_{\zeta\zeta} \psi'\big) \varphi \, d\zeta \, d\xi \, d\tau \Big|$$
$$= \Big| \int_{\delta/2}^{t} \int_{x_0-2}^{x_0-1} \int_{\mathbb{R}} \big(3(G\psi')_{\xi\xi} - 3(G\psi'')_{\xi} + G\psi''' + G_{\zeta\zeta}\psi'\big) D^{\nu_0} u \varphi \, d\zeta \, d\xi \, d\tau \Big|$$
$$\le c \int_{\delta/2}^{t} \int_{x_0-2}^{x_0-1} \int_{\mathbb{R}} \frac{e^{-c_0(t-\tau)^{-1/2}}}{(t-\tau)^{4/3}(1+|\zeta-y|)} \big|D^{\nu_0} u\big| \, d\zeta \, d\xi \, d\tau < \infty.$$

It is easy to see that by a similar argument one can prove uniform convergence of the integral on the right-hand side of (52) in a neighborhood of any point $(t, x, y) \in \Pi_T$ and, consequently, its continuity.

Finally, note that for the justification of the performed argument one can first assume that the function F is smooth (it can be obtained, for example, via the averaging procedure) and then pass to the limit (in more details for the KdV equation such an argument can be found, for example, in [2]). □

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Chapter 4
Singular Semilinear Elliptic Equations with Subquadratic Gradient Terms

Marius Ghergu

Abstract We investigate the semilinear elliptic equation $-\Delta u = a(\delta(x))g(u) + f(x,u) + \lambda|\nabla u|^q$ in a smooth and bounded domain Ω subject to an homogeneous Dirichlet boundary condition. Here g is an unbounded decreasing function, a is positive and continuous, f grows at most linearly at infinity, $\delta(x) = \text{dist}(x, \partial\Omega)$ and $0 < q \le 2$. We emphasize the effect of all these terms in the study of existence, nonexistence and asymptotic behavior of positive solutions.

Mathematics Subject Classification 35J15 · 35J75

4.1 Introduction

We are concerned with semilinear elliptic problems in the form

$$\begin{cases} -\Delta u = a(\delta(x))g(u) + f(x,u) + \lambda|\nabla u|^q & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is a smooth and bounded domain in $\mathbb{R}^N$, $N \ge 2$, $\delta(x) = \text{dist}(x, \partial\Omega)$, $\lambda \in \mathbb{R}$ and $0 < q \le 2$.

We assume that $g \in C^1(0,\infty)$ is a positive decreasing function and

$(g1)$ $\lim_{t\to 0^+} g(t) = \infty$.

The function $f : \overline{\Omega} \times [0,\infty) \to [0,\infty)$ is Hölder continuous, nondecreasing with respect to the second variable and f is positive on $\overline{\Omega} \times (0,\infty)$. The analysis we develop in this paper concerns the cases where f is either linear or sublinear with respect to the second variable. This latter case means that f fulfills the hypotheses

$(f1)$ the mapping $(0,\infty) \ni t \longmapsto \frac{f(x,t)}{t}$ is nonincreasing for all $x \in \overline{\Omega}$;
$(f2)$ $\lim_{t\to 0^+} \frac{f(x,t)}{t} = \infty$ and $\lim_{t\to\infty} \frac{f(x,t)}{t} = 0$, uniformly for $x \in \overline{\Omega}$.

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Such singular boundary value problems arise in the context of chemical heterogeneous catalysts and chemical catalyst kinetics, in the theory of heat conduction in electrically conducting materials, singular minimal surfaces, as well as in the study of non-Newtonian fluids or boundary layer phenomena for viscous fluids (we refer for more details to [3–5, 8, 10, 11] and the more recent papers [6, 13, 18–20, 22, 24, 25, 28]). We also point out that, due to the meaning of the unknowns (concentrations, populations, etc.), only the positive solutions are relevant in most cases.

The main features of this paper are the presence of the convection term $|\nabla u|^q$ combined with the singular weight $a : (0,\infty) \to (0,\infty)$ which is assumed to be nonincreasing and Hölder continuous.

Many papers have been devoted to the case $a \equiv 1$ and $\lambda = 0$ (see [7, 9, 13, 23, 24, 27, 29] and the references therein). One of the first works in the literature dealing with singular weights in connection with singular nonlinearities is due to Taliaferro [26]. In [26] the following problem has been considered

$$\begin{cases} -y'' = \varphi(x) y^{-p} & \text{in } (0,1), \\ y(0) = y(1) = 0, \end{cases} \tag{2}$$

where $p > 0$ and $\varphi(x)$ is positive and continuous on $(0,1)$. It was proved that problem (2) has solutions if and only if $\int_0^1 t(1-t)\varphi(t)dt < \infty$. Later, Agarwal and O'Regan (Sect. 2 in [1]) studied the more general problem

$$\begin{cases} H''(t) = -a(t)g(H(t)) & \text{in } (0,1), \\ H > 0 & \text{in } (0,1), \\ H(0) = H(1) = 0, \end{cases} \tag{3}$$

where g satisfies $(g1)$ and p is positive and continuous on $(0,1)$. It is shown in [1] that if

$$\int_0^1 t(1-t)a(t)dt < \infty, \tag{4}$$

then (3) has at least one classical solution. In our framework, p is continuous at $t = 1$ so condition (4) reads as

$$\int_0^1 ta(t)dt < \infty. \tag{5}$$

In this paper we prove that the assumption (5) is also necessary for (1) to have solutions.

4.2 Main Results

We start this section by a nonexistence result in which we prove the necessity of condition (5).

Theorem 1 (Nonexistence) *Assume* $\int_0^1 ta(t)dt = \infty$. *Then* (1) *has no solutions.*

We next assume that (5) holds.

Theorem 2 (Sublinear case) *Assume* (5) *and conditions* $(f1)$, $(f2)$, $(g1)$ *hold.*

(i) *If* $0 < a < 1$, *then problem* (1) *has at least one solution, for all* $\lambda \in \mathbb{R}$;
(ii) *If* $1 < a \leq 2$, *then there exists* $\lambda^* > 0$ *such that* (1) *has at least one classical solution for all* $-\infty < \lambda < \lambda^*$ *and no solution exists if* $\lambda > \lambda^*$.

We shall next focus on the case $a = 1$. This case was left as an open question in [14]. We are able here to give a complete answer in the case where Ω is a ball centered at the origin.

Theorem 3 (Case $a = 1$) *Assume* $(f1)$, $(f2)$, (5), $a = 1$ *and* $\Omega = B_R(0)$ *for some* $R > 0$. *Then the problem* (1) *has at least one solution for all* $\lambda \in \mathbb{R}$.

The existence of a solution to (1) is achieved by the sub and super-solution method. In particular, the super-solution of (1) is expressed in terms of the solution H to (3). In some particular cases we are able to describe the asymptotic behavior of solutions near the boundary. This is our next task here.

Let $a(t) = t^{-\alpha}$, $g(t) = t^{-p}$, α, $p > 0$ and consider the following related problem:

$$\begin{cases} -\Delta u = \delta(x)^{-\alpha} u^{-p} + f(x,u) + \lambda |\nabla u|^q & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{6}$$

Then we have:

Theorem 4 (Asymptotic behavior) *Assume* $(g1)$, $(f1)$, $(f2)$.

(i) *If* $\alpha \geq 2$, *then the problem* (6) *has no classical solutions.*
(ii) *If* $\alpha < 2$, *then there exists* $\lambda^* \in (0, \infty]$ (*with* $\lambda^* = \infty$ *if* $0 < a < 1$) *such that problem* (6) *has at least one classical solution* u, *for all* $-\infty < \lambda < \lambda^*$. *Moreover, for all* $0 < \lambda < \lambda^*$, *there exist* $0 < \eta < 1$ *and* $C_1, C_2 > 0$ *such that* u *satisfies*
 (ii1) *If* $\alpha + p > 1$, *then*

$$C_1 \delta(x)^{(2-\alpha)/(1+p)} \leq u(x) \leq C_2 \delta(x)^{(2-\alpha)/(1+p)}, \quad \textit{for all } x \in \Omega; \tag{7}$$

 (ii2) *If* $\alpha + p = 1$, *then*

$$C_1 \delta(x) \ln^{1/(2-\alpha)} \left(\frac{1}{\delta(x)} \right) \leq u(x) \leq C_2 \delta(x) \ln^{1/(2-\alpha)} \left(\frac{1}{\delta(x)} \right), \tag{8}$$

for all $x \in \Omega$ *with* $\delta(x) < \eta$;

(ii3) *If* $\alpha + p < 1$, *then*

$$C_1\delta(x) \le u(x) \le C_2\delta(x), \quad \textit{for all } x \in \Omega. \tag{9}$$

We have seen that if $a(t) = t^{-\alpha}$ then (1) has no solutions if $\alpha \ge 2$. Motivated by the results in [12], let us now consider the extremal case $a(t) = t^{-2}\ln^{\alpha}(A/t)$ where $A > \operatorname{diam}(\Omega)$ and the corresponding boundary value problem

$$\begin{cases} -\Delta u = \delta(x)^{-2}\ln^{\alpha}(\frac{A}{\delta(x)})u^{-p} + f(x,u) + \lambda|\nabla u|^q & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{10}$$

Theorem 5 (Asymptotic behavior) *Assume* $(g1)$, $(f1)$, $(f2)$.

(i) *If* $\alpha \ge -1$, *then problem* (10) *has no classical solutions.*
(ii) *If* $\alpha < -1$, *then there exists* $\lambda^* \in (0, \infty]$ (*with* $\lambda^* = \infty$ *if* $0 < a < 1$) *such that problem* (6) *has at least one classical solution* u, *for all* $-\infty < \lambda < \lambda^*$. *Moreover, there exist* $C_1, C_2 > 0$ *such that* u *satisfies*

$$C_1 \ln^{(1-\alpha)/(1+p)}\left(\frac{A}{\delta(x)}\right) \le u(x) \le C_2 \ln^{(1-\alpha)/(1+p)}\left(\frac{A}{\delta(x)}\right), \quad \textit{for all } x \in \Omega. \tag{11}$$

In the following we study the problem (1) in which we drop out the sublinearity assumptions $(f1)$, $(f2)$ on f but we require in turn that f is linear. More precisely, we assume that $f(x,t) = \mu t$, for some $\mu > 0$. Note that the existence results established in Lemma 4 in [24] or [25] do not apply here since the mapping

$$\Psi(x,t) = a\big(\delta(x)\big)g(t) + \lambda t, \quad (x,t) \in \Omega \times (0,\infty),$$

is not defined on $\partial\Omega \times (0,\infty)$.

Theorem 6 (Linear case) *Assume* (5), $(g1)$, $f(x,u) = \mu u$ *for some* $\mu > 0$ *and* $0 < a < 1$. *Then for any* $\lambda \ge 0$ *problem* (1) *has solutions if and only if* $\mu < \lambda_1$.

4.3 Proof of Theorem 1

The proof of Theorem 1 follows from the following more general result.

Proposition 1 *Assume that $\int_0^1 ta(t)dt = \infty$. Then the inequality boundary value problem*

$$\begin{cases} -\Delta u + \lambda|\nabla u|^2 \geq a(\delta(x))g(u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{12}$$

has no classical solutions.

Proof Let (λ_1, φ_1) be the first eigenvalue/eigenfunction of $-\Delta$ in Ω subject to a homogeneous Dirichlet boundary condition. It is known that $\lambda_1 > 0$ and by normalization, one can assume $\varphi_1 > 0$ in Ω. It suffices to prove the result only for $\lambda > 0$. We argue by contradiction and assume that there exists $u \in C^2(\Omega) \cap C(\overline{\Omega})$ a solution of (12). Using $(g1)$, we can find $c_1 > 0$ such $\underline{u} := c_1\varphi_1$ verifies

$$-\Delta\underline{u} + \lambda|\nabla\underline{u}|^2 \leq a\big(\delta(x)\big)g(\underline{u}) \quad \text{in } \Omega.$$

Since g is decreasing, we easily obtain

$$u \geq \underline{u} \quad \text{in } \Omega. \tag{13}$$

We make in (12) the change of variable $v = 1 - e^{-\lambda u}$. Therefore

$$\begin{cases} -\Delta v = \lambda(1-v)(\lambda|\nabla u|^2 - \Delta u) \geq \lambda(1-v)a(\delta(x))g(-\frac{\ln(1-v)}{\lambda}) & \text{in } \Omega, \\ v > 0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{14}$$

In order to avoid the singularities in (14) let us consider the approximated problem

$$\begin{cases} -\Delta v = \lambda(1-v)a(\delta(x))g(\varepsilon - \frac{\ln(1-v)}{\lambda}) & \text{in } \Omega, \\ v > 0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{15}$$

with $0 < \varepsilon < 1$. Clearly v is a super-solution of (15). By (13) and the fact that $\lim_{t\to 0^+} \frac{1-e^{-\lambda t}}{t} = \lambda > 0$, there exists $c_2 > 0$ such that $v \geq c_2\varphi_1$ in Ω. On the other hand, there exists $0 < c < c_2$ such that $c\varphi_1$ is a sub-solution of (15) and obviously $c\varphi_1 \leq v$ in Ω. Then, by the standard sub- and super-solution method (see, e.g., [16, 21]) the problem (15) has a solution $v_\varepsilon \in C^2(\overline{\Omega})$ such that

$$c\varphi_1 \leq v_\varepsilon \leq v \quad \text{in } \Omega. \tag{16}$$

Multiplying by φ_1 in (15) and integrating we find

$$\lambda_1 \int_\Omega \varphi_1 v_\varepsilon dx = C \int_\Omega (1 - v_\varepsilon)\varphi_1 a\big(\delta(x)\big)g\bigg(\varepsilon - \frac{\ln(1-v_\varepsilon)}{\lambda}\bigg)dx.$$

Using (16) we obtain

$$M =: \lambda_1 \int_\Omega \varphi_1 v dx \geq \lambda \int_\Omega (1-v)\varphi_1 a\big(\delta(x)\big) g\left(-\frac{\ln(1-v)}{\lambda}\right) dx$$
$$\geq C_1 \int_{\Omega_\delta} \varphi_1 a\big(\delta(x)\big) dx, \tag{17}$$

where $\Omega_\delta \supset \{x \in \Omega; \delta(x) < \delta\}$, for some $\delta > 0$ sufficiently small. Since $\varphi_1(x)$ behaves like $\delta(x)$ in Ω_δ and $\int_0^1 ta(t)dt = \infty$, by (17) we find a contradiction. Hence, problem (12) has no classical solutions and the proof is now complete. □

4.4 Proof of Theorem 2

The existence part in this result relies on the sub and super-solution method. Basic to our approach is the following comparison result whose proof may be found in [15].

Lemma 1 *Let* $\Psi : \overline{\Omega} \times (0, +\infty) \to \mathbb{R}$ *be a Hölder continuous function such that the mapping* $(0, +\infty) \ni s \longmapsto \frac{\Psi(x,s)}{s}$ *is strictly decreasing for each* $x \in \Omega$. *Assume that there exist* $v, w \in C^2(\Omega) \cap C(\overline{\Omega})$ *such that*

(a) $\Delta w + \Psi(x, w) \leq 0 \leq \Delta v + \Psi(x, v)$ *in* Ω;
(b) $v, w > 0$ *in* Ω *and* $v \leq w$ *on* $\partial\Omega$;
(c) $\Delta v \in L^1(\Omega)$ *or* $\Delta w \in L^1(\Omega)$.

Then $v \leq w$ *in* Ω.

We shall divide our arguments into two cases according to the values of λ.
(i) CASE $\lambda > 0$. By Lemma 4 in [24] there exists $\zeta \in C^2(\overline{\Omega})$ such that

$$\begin{cases} -\Delta\zeta = f(x, \zeta) & \text{in } \Omega, \\ \zeta > 0 & \text{in } \Omega, \\ \zeta = 0 & \text{on } \partial\Omega. \end{cases} \tag{18}$$

Thus, ζ is a sub-solution of (1) provided $\lambda > 0$. We focus now on finding a super-solution $\overline{u}_\lambda$ of (1) such that $\zeta \leq \overline{u}_\lambda$ in Ω.

Let H be the solution of (3). Since H is concave, there exists $H'(0+) \in (0, \infty]$. Taking $0 < b < 1$ small enough, we can assume that $H' > 0$ in $(0, b]$, so H is increasing on $[0, b]$. Multiplying by H' in (3) and integrating on $[t, b]$, we find

$$\big(H'\big)^2(t) - \big(H'\big)^2(b) = 2\int_t^b a(s) g\big(H(s)\big) H'(s) ds \leq 2a(t) \int_{H(t)}^{H(b)} g(\tau) d\tau. \tag{19}$$

Using the monotonicity of g it follows that

$$\big(H'\big)^2(t) \le 2H(b)a(t)g\big(H(t)\big) + \big(H'\big)^2(b), \quad \text{for all } 0 < t \le b. \tag{20}$$

Hence, there exist $C_1, C_2 > 0$ such that

$$\big(H'\big)(t) \le C_1 a(t)g\big(H(t)\big), \quad \text{for all } 0 < t \le b \tag{21}$$

and

$$\big(H'\big)^2(t) \le C_2 a(t)g\big(H(t)\big), \quad \text{for all } 0 < t \le b. \tag{22}$$

Now we can proceed to construct a super-solution for (1). First, we fix $c > 0$ such that

$$c\varphi_1 \le \min\big\{b, \delta(x)\big\} \quad \text{in } \Omega. \tag{23}$$

By Hopf's maximum principle, there exist $\omega \subset\subset \Omega$ and $\delta > 0$ such that

$$|\nabla\varphi_1| > \delta \quad \text{in } \Omega \setminus \omega. \tag{24}$$

Moreover, since

$$\lim_{\delta(x)\to 0^+} \big\{c^2 a(c\varphi_1)g\big(H(c\varphi_1)\big)|\nabla\varphi_1|^2 - 3f\big(x, H(c\varphi_1)\big)\big\} = \infty,$$

we can assume that

$$c^2 a(c\varphi_1)g\big(H(c\varphi_1)\big)|\nabla\varphi_1|^2 \ge 3f\big(x, H(c\varphi_1)\big) \quad \text{in } \Omega \setminus \omega. \tag{25}$$

Let $M > 1$ be such that

$$Mc^2\delta^2 > 3. \tag{26}$$

Since $H'(0+) > 0$ and $0 < a < 1$, we can choose $M > 1$ such that

$$M\frac{(c\delta)^2}{C_1}H'(c\varphi_1) \ge 3\lambda\big(McH'(c\varphi_1)|\nabla\varphi_1|\big)^q \quad \text{in } \Omega \setminus \omega,$$

where C_1 is the constant appearing in (21). By (21), (24) and (26) we derive

$$Mc^2 a(c\varphi_1)g\big(H(c\varphi_1)\big)|\nabla\varphi_1|^2 \ge 3\lambda\big(McH'(c\varphi_1)|\nabla\varphi_1|\big)^q \quad \text{in } \Omega \setminus \omega. \tag{27}$$

Since g is decreasing and $H'(c\varphi_1) > 0$ in $\overline{\omega}$, there exists $M > 0$ such that

$$Mc\lambda_1\varphi_1 H'(c\varphi_1) \ge 3a\big(\delta(x)\big)g\big(H(c\varphi_1)\big) \quad \text{in } \omega. \tag{28}$$

In the same manner, using $(f2)$ and the fact that $\varphi_1 > 0$ in $\overline{\omega}$, we can choose $M > 1$ large enough such that

$$Mc\lambda_1\varphi_1 H'(c\varphi_1) \ge 3\lambda\big(McH'(c\varphi_1)|\nabla\varphi_1|\big)^q \quad \text{in } \omega, \tag{29}$$

and

$$Mc\lambda_1\varphi_1 H'(c\varphi_1) \geq 3f\big(x, MH(c\varphi_1)\big) \quad \text{in } \omega. \tag{30}$$

For M satisfying (26)–(30), we prove that

$$\overline{u}_\lambda(x) := MH\big(c\varphi_1(x)\big), \quad \text{for all } x \in \Omega, \tag{31}$$

is a super-solution of (1). We have

$$-\Delta\overline{u}_\lambda = Mc^2 a(c\varphi_1) g\big(H(c\varphi_1)\big)|\nabla\varphi_1|^2 + Mc\lambda_1\varphi_1 H'(c\varphi_1) \quad \text{in } \Omega. \tag{32}$$

We first show that

$$\begin{aligned} &Mc^2 a(c\varphi_1) g\big(H(c\varphi_1)\big)|\nabla\varphi_1|^2 \\ &\quad \geq a\big(\delta(x)\big) g(\overline{u}_\lambda) + f(x, \overline{u}_\lambda) + \lambda|\nabla\overline{u}_\lambda|^q \quad \text{in } \Omega\setminus\omega. \end{aligned} \tag{33}$$

Indeed, by (23), (24) and (26) we get

$$\begin{aligned} \frac{M}{3}c^2 a(c\varphi_1) g\big(H(c\varphi_1)\big)|\nabla\varphi_1|^2 &\geq a\big(\delta(x)\big) g\big(H(c\varphi_1)\big) \\ &\geq a\big(\delta(x)\big) g\big(MH(c\varphi_1)\big) \\ &= a\big(\delta(x)\big) g(\overline{u}_\lambda) \quad \text{in } \Omega\setminus\omega. \end{aligned} \tag{34}$$

The assumption $(f1)$ and (25) produce

$$\begin{aligned} \frac{M}{3}c^2 a(c\varphi_1) g\big(H(c\varphi_1)\big)|\nabla\varphi_1|^2 &\geq Mf\big(x, H(c\varphi_1)\big) \\ &\geq f\big(x, MH(c\varphi_1)\big) \\ &= f(x, \overline{u}_\lambda) \quad \text{in } \Omega\setminus\omega. \end{aligned} \tag{35}$$

From (27) we obtain

$$\begin{aligned} \frac{M}{3}c^2 a(c\varphi_1) g\big(H(c\varphi_1)\big)|\nabla\varphi_1|^2 &\geq \lambda\big(McH'(c\varphi_1)|\nabla\varphi_1|\big)^q \\ &= \lambda|\nabla\overline{u}_\lambda|^q \quad \text{in } \Omega\setminus\omega. \end{aligned} \tag{36}$$

Now, relation (33) follows by (34), (35) and (36).

Next we prove that

$$Mc\lambda_1\varphi_1 H'(c\varphi_1) \geq a\big(\delta(x)\big) g(\overline{u}_\lambda) + f(x, \overline{u}_\lambda) + \lambda|\nabla\overline{u}_\lambda|^q \quad \text{in } \omega. \tag{37}$$

From (28) and (29) we get

$$\begin{aligned}\frac{M}{3}c\lambda_1\varphi_1 H'(c\varphi_1) &\geq a\big(\delta(x)\big)g\big(H(c\varphi_1)\big)\\ &\geq a\big(\delta(x)\big)g\big(MH(c\varphi_1)\big)\\ &= a\big(\delta(x)\big)g(\overline{u}_\lambda) \quad \text{in } \omega \end{aligned} \tag{38}$$

and

$$\begin{aligned}\frac{M}{3}c\lambda_1\varphi_1 H'(c\varphi_1) &\geq \lambda\big(McH'(c\varphi_1)|\nabla\varphi_1|\big)^q\\ &= \lambda|\nabla\overline{u}_\lambda|^q \quad \text{in } \omega.\end{aligned} \tag{39}$$

Finally, from (30) we derive

$$\frac{M}{3}c\lambda_1\varphi_1 H'(c\varphi_1) \geq f\big(x, MH(c\varphi_1)\big) = f(x,\overline{u}_\lambda) \quad \text{in } \omega. \tag{40}$$

Clearly, relation (37) follows from (38), (39) and (40).

Combining (32) with (33) and (37) we conclude that $\overline{u}_\lambda$ is a super-solution of (1). Thus, by Lemma 1 we obtain $\zeta \leq \overline{u}_\lambda$ in Ω and by sub and super-solution method it follows that (1) has at least one classical solution for all $\lambda > 0$.

CASE $\lambda \leq 0$. We fix $\nu > 0$ and let $u_\nu \in C^2(\Omega)\cap C(\overline{\Omega})$ be a solution of (1) for $\lambda = \nu$. Then u_ν is a super-solution of (1) for all $\lambda \leq 0$. Set

$$m := \inf_{(x,t)\in\overline{\Omega}\times(0,\infty)} \Big(a\big(\delta(x)\big)g(t) + f(x,t)\Big).$$

Since $\lim_{t\to 0^+} g(t) = \infty$ and the mapping $(0,\infty)\ni t \longmapsto \min_{x\in\overline{\Omega}} f(x,t)$ is positive and nondecreasing, we deduce that m is a positive real number. Consider the problem

$$\begin{cases} -\Delta v = m + \lambda|\nabla v|^q & \text{in } \Omega,\\ v = 0 & \text{on } \partial\Omega.\end{cases} \tag{41}$$

Clearly zero is a sub-solution of (41). Since $\lambda \leq 0$, the solution w of the problem

$$\begin{cases} -\Delta w = m & \text{in } \Omega,\\ w = 0 & \text{on } \partial\Omega,\end{cases}$$

is a super-solution of (41). Hence, (41) has at least one solution $v \in C^2(\Omega)\cap C(\overline{\Omega})$. We claim that $v > 0$ in Ω. Indeed, if not, we deduce that $\min_{x\in\overline{\Omega}} v$ is achieved at some point $x_0 \in \Omega$. Then $\nabla v(x_0) = 0$ and

$$-\Delta v(x_0) = m + \lambda\big|\nabla v(x_0)\big|^q = m > 0, \quad \text{contradiction.}$$

Therefore, $v > 0$ in Ω. It is easy to see that v is sub-solution of (1) and $-\Delta v \leq m \leq -\Delta u_\nu$ in Ω, which gives $v \leq u_\nu$ in Ω. Again by the sub and super-solution method we conclude that (1) has at least one classical solution $u_\lambda \in C^2(\Omega)\cap C(\overline{\Omega})$.

(ii) The proof follows the same steps as above. The only difference is that (27) and (29) are no more valid for any $\lambda > 0$. The main difficulty when dealing with estimates like (27) is that $H'(c\varphi_1)$ may blow-up at the boundary. However, combining the assumption $1 < a \le 2$ with (22), we can choose $\lambda > 0$ small enough such that (27) and (29) hold. This implies that the problem (1) has a classical solution provided $\lambda > 0$ is sufficiently small.

Set

$$A = \{\lambda > 0;\ \text{problem (1) has at least one classical solution}\}.$$

From the above arguments, A is nonempty. Let $\lambda^* = \sup A$. We first claim that if $\lambda \in A$, then $(0, \lambda) \subseteq A$. To this aim, let $\lambda_1 \in A$ and $0 < \lambda_2 < \lambda_1$. If u_{λ_1} is a solution of (1) with $\lambda = \lambda_1$, then u_{λ_1} is a super-solution of (1) with $\lambda = \lambda_2$, while ζ defined in (18) is a sub-solution. Using Lemma 1 once more, we get $\zeta \le u_{\lambda_1}$ in Ω so (1) has at least one classical solution for $\lambda = \lambda_2$. This proves the claim. Since $\lambda_1 \in A$ was arbitrary, we conclude that $(0, \lambda^*) \subset A$.

Next, we prove that $\lambda^* < \infty$. To this aim, we use the following result which is a consequence of Theorem 2.1 in [2].

Lemma 2 *Assume that $a > 1$. Then there exists a positive number $\bar{\sigma}$ such that the problem*

$$\begin{cases} -\Delta v \ge |\nabla v|^q + \sigma & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{42}$$

has no solutions for $\sigma > \bar{\sigma}$.

Consider $\lambda \in A$ and let u_λ be a classical solution of (1). Set $v = \lambda^{1/(a-1)} u_\lambda$. Using our assumption $1 < a \le 2$, we deduce that v fulfills

$$\begin{cases} -\Delta v \ge |\nabla v|^q + m\lambda^{1/(a-1)} & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{43}$$

According to Lemma 2, we obtain $m\lambda^{1/(a-1)} \le \bar{\sigma}$, that is, $\lambda \le (\frac{\bar{\sigma}}{m})^{a-1}$. This means that $\lambda^* \le (\frac{\bar{\sigma}}{m})^{a-1}$, hence λ^* is finite. The existence of a solution in the case $\lambda \le 0$ can be achieved in the same manner as above.

This finishes the proof of Theorem 2.

4.5 Proof of Theorem 3

Let us note first that in our setting problem (1) reads

$$\begin{cases} -\Delta u = a(R - |x|)g(u) + f(x, u) + \lambda|\nabla u| & |x| < R, \\ u > 0 & |x| < R, \\ u = 0 & |x| = R. \end{cases} \tag{44}$$

The case $\lambda \le 0$ is the same as in the proof of Theorem 2(i). In what follows, we assume that $\lambda > 0$. Using Theorem 2(i) it is easy to see that there exists $\underline{u} \in C^2(\Omega) \cap C(\overline{\Omega})$ such that

$$\begin{cases} -\Delta \underline{u} = a(R-|x|)g(\underline{u}) & |x| < R, \\ \underline{u} > 0 & |x| < R, \\ \underline{u} = 0 & |x| = R. \end{cases}$$

It is obvious that $\underline{u}$ is a sub-solution of (44) for all $\lambda > 0$. In order to provide a super-solution of (44) we consider the problem

$$\begin{cases} -\Delta u = a(R-|x|)g(u) + 1 + \lambda|\nabla u| & |x| < R, \\ u > 0 & |x| < R, \\ u = 0 & |x| = R. \end{cases} \tag{45}$$

We need the following auxiliary result.

Lemma 3 *Problem* (45) *has at least one solution.*

Proof We are looking for radially symmetric solution u of (45), that is, $u = u(r)$, $0 \le r = |x| \le R$. In this case, problem (45) becomes

$$\begin{cases} -u'' - \frac{N-1}{r}u'(r) = a(R-r)g(u(r)) + 1 + \lambda|u'(r)| & 0 \le r < R, \\ u > 0 & 0 \le r < R, \\ u(R) = 0. \end{cases} \tag{46}$$

This implies

$$-\big(r^{N-1}u'(r)\big)' \ge 0 \quad \text{for all } 0 \le r < R,$$

which yields $u'(r) \le 0$ for all $0 \le r < R$. Then (46) gives

$$-\left(u'' + \frac{N-1}{r}u'(r) + \lambda u'(r)\right) = a(R-r)g\big(u(r)\big) + 1, \quad 0 \le r < R.$$

We obtain

$$-\big(e^{\lambda r}r^{N-1}u'(r)\big)' = e^{\lambda r}r^{N-1}\psi\big(r, u(r)\big), \quad 0 \le r < R, \tag{47}$$

where

$$\psi(r,t) = a(R-r)g(t) + 1, \quad (r,t) \in [0,R) \times (0,\infty).$$

From (47) we obtain

$$u(r) = u(0) - \int_0^r e^{-\lambda t}t^{-N+1}\int_0^t e^{\lambda s}s^{N-1}\psi\big(s,u(s)\big)ds\,dt, \quad 0 \le r < R. \tag{48}$$

On the other hand, in view of Theorem 2 and using the fact that g is decreasing, there exists a unique solution $w \in C^2(B_R(0)) \cap C(\overline{B}_R(0))$ of the problem

$$\begin{cases} -\Delta w = a(R-|x|)g(w)+1 & |x|<R, \\ w>0 & |x|<R, \\ w=0 & |x|=R. \end{cases} \tag{49}$$

Clearly, w is a sub-solution of (45). Due to the uniqueness and to the symmetry of the domain, w is radially symmetric, so, $w=w(r)$, $0 \le r = |x| \le R$. As above we get

$$w(r) = w(0) - \int_0^r t^{-N+1} \int_0^t s^{N-1} \psi\big(s, w(s)\big) ds dt, \quad 0 \le r < R. \tag{50}$$

We claim that there exists a solution $v \in C^2[0,R) \cap C[0,R]$ of (48) such that $v>0$ in $[0,R)$.

Let $A = w(0)$ and define the sequence $(v_k)_{k\ge 1}$ by

$$\begin{cases} v_k(r) = A - \int_0^r e^{-\lambda t} t^{-N+1} \int_0^t e^{\lambda s} s^{N-1} \psi(s, v_{k-1}(s)) ds dt, \\ \quad 0 \le r < R, k \ge 1, \\ v_0 = w. \end{cases} \tag{51}$$

Note that v_k is decreasing in $[0,R)$ for all $k \ge 0$. From (50) and (51) we easily check that $v_1 \ge v_0$ and by induction we deduce $v_k \ge v_{k-1}$ for all $k \ge 1$. Hence

$$w = v_0 \le v_1 \le \cdots \le v_k \le \cdots \le A \quad \text{in } B_R(0).$$

Thus, there exists $v(r) := \lim_{k\to\infty} v_k(r)$, for all $0 \le r < R$ and $v>0$ in $[0,R)$. We now can pass to the limit in (51) in order to get that v is a solution of (48). By classical regularity results we also obtain $v \in C^2[0,R) \cap C[0,R]$. This proves the claim.

We have obtained a super-solution v of (45) such that $v \ge w$ in $B_R(0)$. So, the problem (45) has at least one solution and the proof of our Lemma is now complete. □

Let u be a solution of the problem (45). For $M>1$ we have

$$\begin{aligned} -\Delta(Mu) &= Ma\big(R-|x|\big)g(u) + M + \lambda\big|\nabla(Mu)\big| \\ &\ge a\big(R-|x|\big)g(Mu) + M + \lambda\big|\nabla(Mu)\big|. \end{aligned} \tag{52}$$

Since f is sublinear, we can choose $M>1$ such that

$$M \ge f\big(x, M|u|_\infty\big) \quad \text{in } B_R(0).$$

Then $\overline{u}_\lambda := Mu$ satisfies

$$-\Delta \overline{u}_\lambda \ge a\big(R-|x|\big)g(\overline{u}_\lambda) + f(x,\overline{u}_\lambda) + \lambda|\nabla \overline{u}_\lambda| \quad \text{in } B_R(0).$$

It follows that $\overline{u}_\lambda$ is a super-solution of (44). Since g is decreasing we easily deduce $\underline{u} \le \overline{u}_\lambda$ in $B_R(0)$ so, problem (1) has at least one solution.

The proof of Theorem 3 is now complete.

4.6 Proof of Theorem 4 and Theorem 5

Proof of Theorem 4 The existence and nonexistence of a solution to (6) follows directly from Theorems 1 and 2. We next prove the boundary estimates (7)–(9).

Recall that if $\int_0^1 ta(t)dt < \infty$ and λ belongs to a certain range, then Theorem 2 asserts that (1) has at least one classical solution u satisfying $u \le MH(c\varphi_1)$ in Ω, for some $M, c > 0$. Here H is the solution of

$$\begin{cases} H''(t) = -t^{-\alpha}H^{-p}(t), & \text{for all } 0 < t \le b < 1, \\ H, H' > 0 & \text{in } (0, b], \\ H(0) = 0. \end{cases} \tag{53}$$

With the same idea as in the proof of Theorem 2, we can show that there exists $m > 0$ small enough such that $v := mH(c\varphi_1)$ satisfies

$$-\Delta v \le \delta(x)^{-\alpha} v^{-p} \quad \text{in } \Omega. \tag{54}$$

Indeed, we have

$$-\Delta v = m\big[c^{2-\alpha}|\nabla\varphi_1|^2\varphi_1^{-\alpha}H^{-p}(c\varphi_1) + \lambda_1 c\varphi_1 H'(c\varphi_1)\big] \quad \text{in } \Omega.$$

Thus, there exist two positive constants $c_1, c_2 > 0$ such that

$$-\Delta v \le m\big[c_1|\nabla\varphi_1|^2 + c_2\varphi_1\big]\delta(x)^{-\alpha}H^{-p}(c\varphi_1) \quad \text{in } \Omega.$$

Clearly (54) holds if we choose $m > 0$ small enough such that $m[c_1|\nabla\varphi_1|^2 + c_2\varphi_1] < 1$ in Ω. Moreover, v is a sub-solution of (6) for all $\mu > 0$ and one can easily see that $v \le u_\mu$ in Ω. Hence

$$mH(c\varphi_1) \le u \le MH(c\varphi_1) \quad \text{in } \Omega. \tag{55}$$

Now, a careful analysis of (53) together with (55) is used in order to obtain boundary estimates for the solution of (6). Our estimates complete the results in Theorem 2.1 in [17] since here the potential $a(\delta(x))$ blows-up at the boundary.

(ii1) Remark that

$$H(t) = \left(\frac{(1+p)^2}{(2-\alpha)(\alpha+p-1)}\right)^{1/(1+p)} t^{(2-\alpha)/(1+p)}, \quad t > 0,$$

is a solution of (53) provided $\alpha + p > 1$. The conclusion in this case follows now from (55).

(ii2) Note that in this case problem (53) becomes

$$\begin{cases} H''(t) = -t^{-\alpha} H^{\alpha-1}(t), & \text{for all } 0 < t \le b < 1, \\ H(0) = 0, & \\ H > 0 & \text{in } (0, b]. \end{cases} \tag{56}$$

Let $w = t \ln^{1/(2-\alpha)}(\frac{1}{t})$, $t > 0$. Then

$$-w''(t) \sim t^{-1} \ln^{(\alpha-1)/(2-\alpha)}\left(\frac{1}{t}\right) \sim t^{-\alpha} w^{-p}$$

in a neighborhood of the origin. Now if $m > 0$ is small enough it follows that w satisfies $-(mw)'' \le t^{-\alpha}(mw)^{\alpha-1}$ in $(0, b)$ and $mw(b) \le H(b)$. By the maximum principle we find $H \ge mw$ in $(0, b)$, that is

$$H(t) \ge c_1 t \ln^{1/(2-\alpha)}\left(\frac{1}{t}\right) \quad \text{in } (0, b).$$

Similarly, if $M > 1$ is large enough we have $-(Mw)'' \le t^{-\alpha}(Mw)^{\alpha-1}$ in $(0, b)$ and $Mw(b) \ge H(b)$. By the maximum principle we find $H \le Mw$ in $(0, b)$, that is

$$H(t) \le c_2 t \ln^{1/(2-\alpha)}\left(\frac{1}{t}\right) \quad \text{in } (0, b).$$

Now the desired estimate follows from (55).

(ii3) Using the fact that $H'(0+) \in (0, \infty]$ we get the existence of $c > 0$ such that

$$H(t) > ct, \quad \text{for all } 0 < t \le b.$$

This yields

$$-H''(t) \le c^{-p} t^{-(\alpha+p)}, \quad \text{for all } 0 < t \le b.$$

Since $\alpha + p < 1$, it follows that $H'(0+) < \infty$, that is, $H \in C^1[0, b]$. Thus, there exists $c_1, c_2 > 0$ such that

$$c_1 t \le H(t) \le c_2 t, \quad \text{for all } 0 < t \le b. \tag{57}$$

The conclusion in Theorem 4(iii) follows directly from (57) and (55).

This completes the proof of Theorem 4. □

Proof of Theorem 5 This follows in the same way as above. The estimate (11) follows by using the approach in Theorem 4(ii2) with $w(t) = \ln^{(1-\alpha)/(1+p)}(A/t)$. □

4.7 Proof of Theorem 6

Fix $\mu \in (0, \lambda_1)$ and $\lambda \geq 0$. By Theorem 2(i) there exists $u \in C^2(\Omega) \cap C(\overline{\Omega})$ a solution of the problem

$$\begin{cases} -\Delta u = a(\delta(x))g(u) + \lambda|\nabla u|^q & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Obviously, $\underline{u}_{\lambda\mu} := u$ is a sub-solution of (1). Since $\mu < \lambda_1$, there exists $v \in C^2(\overline{\Omega})$ such that

$$\begin{cases} -\Delta v = \mu v + 2 & \text{in } \Omega, \\ v > 0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

Since $0 < a < 1$, we can choose $M > 0$ large enough such that

$$M > \mu|u|_\infty \quad \text{and} \quad M > \lambda(M|\nabla v|)^q \quad \text{in } \Omega.$$

Then $w := Mv$ satisfies

$$-\Delta w \geq \mu(u + w) + \lambda|\nabla w|^q \quad \text{in } \Omega.$$

We claim that $\overline{u}_{\lambda\mu} := u + w$ is a super-solution of (1). Indeed, we have

$$-\Delta\overline{u}_{\lambda\mu} \geq a\big(\delta(x)\big)g(u) + \lambda\overline{u}_{\lambda\mu} + \lambda|\nabla u|^q + \lambda|\nabla w|^q \quad \text{in } \Omega. \tag{58}$$

Using the assumption $0 < a < 1$ one can easily deduce

$$t_1^q + t_2^q \geq (t_1 + t_2)^q, \quad \text{for all } t_1, t_2 \geq 0.$$

Hence

$$|\nabla u|^q + |\nabla w|^q \geq \big(|\nabla u| + |\nabla w|\big)^q \geq \big|\nabla(u + w)\big|^q \quad \text{in } \Omega. \tag{59}$$

Combining (58) with (59) we obtain

$$-\Delta\overline{u}_{\lambda\mu} \geq a\big(\delta(x)\big)g(\overline{u}_{\lambda\mu}) + \mu\overline{u}_{\lambda\mu} + \lambda|\nabla\overline{u}_{\lambda\mu}|^q \quad \text{in } \Omega.$$

Hence, $(\underline{u}_{\lambda\mu}, \overline{u}_{\lambda\mu})$ is an ordered pair of sub and super-solution of (1), so there exists a classical solution $u_{\lambda\mu}$ of (1), provided $\lambda \geq 0$ and $0 < \mu < \lambda_1$. Assume by contradiction that there exist $\mu \geq \lambda_1$ and $\lambda \geq 0$ such that the problem (1) has a classical solution $u_{\lambda\mu}$. If $m = \min_{x \in \overline{\Omega}} a(\delta(x))g(u_{\lambda\mu}) > 0$ it follows that $u_{\lambda\mu}$ is a super-solution of

$$\begin{cases} -\Delta u = \mu u + m & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{60}$$

Clearly, zero is a sub-solution of (60), so there exists a classical solution u of (60) such that $u \le u_{\lambda\mu}$ in Ω. By maximum principle and elliptic regularity we get $u > 0$ in Ω and $u \in C^2(\overline{\Omega})$. To raise a contradiction, we proceed as in the proof of Theorem 2(ii).

Multiplying by φ_1 in (60) and then integrating over Ω we find

$$-\int_\Omega \varphi_1 \Delta u = \mu \int_\Omega u\varphi_1 + m\int_\Omega \varphi_1.$$

This implies $\lambda_1 \int_\Omega u\varphi_1 = \mu \int_\Omega u\varphi_1 + m\int_\Omega \varphi_1$, which is a contradiction, since $\mu \ge \lambda_1$ and $m > 0$. The proof of Theorem 6 is now complete.

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Chapter 5
On the Parabolic Regime of a Hyperbolic Equation with Weak Dissipation: The Coercive Case

Marina Ghisi and Massimo Gobbino

Abstract We consider a family of Kirchhoff equations with a small parameter ε in front of the second-order time-derivative, and a dissipation term with a coefficient which tends to 0 as $t \to +\infty$.

It is well-known that, when the decay of the coefficient is slow enough, solutions behave as solutions of the corresponding parabolic equation, and in particular they decay to 0 as $t \to +\infty$.

In this paper we consider the nondegenerate and coercive case, and we prove *optimal* decay estimates for the hyperbolic problem, and optimal decay-error estimates for the difference between solutions of the hyperbolic and the parabolic problem. These estimates show a quite surprising fact: in the coercive case the analogy between parabolic equations and dissipative hyperbolic equations is weaker than in the noncoercive case.

This is actually a result for the corresponding linear equations with time-dependent coefficients. The nonlinear term comes into play only in the last step of the proof.

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5.1 Introduction

Let H be a real Hilbert space. For every x and y in H, $|x|$ denotes the norm of x, and $\langle x, y\rangle$ denotes the scalar product of x and y. Let A be a self-adjoint linear operator on H with dense domain $D(A)$. We assume that A is nonnegative, namely $\langle Ax, x\rangle \geq 0$ for every $x \in D(A)$, so that for every $\alpha \geq 0$ the power $A^{\alpha}x$ is defined provided that x lies in a suitable domain $D(A^{\alpha})$.

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We consider the Cauchy problem

$$\varepsilon u_\varepsilon''(t) + \frac{1}{(1+t)^p} u_\varepsilon'(t) + m\left(\left|A^{1/2}u_\varepsilon(t)\right|^2\right) A u_\varepsilon(t) = 0 \quad \forall t \geq 0, \tag{1.1}$$

$$u_\varepsilon(0) = u_0, \qquad u_\varepsilon'(0) = u_1, \tag{1.2}$$

where $\varepsilon > 0$ and $p \geq 0$ are real parameters, $m : [0, +\infty) \to (0, +\infty)$ is a locally Lipschitz continuous function, and $(u_0, u_1) \in D(A) \times D(A^{1/2})$.

The singular perturbation problem in its generality consists in proving the convergence of solutions of (1.1), (1.2) to solutions of the first order problem

$$\frac{1}{(1+t)^p} u'(t) + m\left(\left|A^{1/2}u(t)\right|^2\right) A u(t) = 0 \quad \forall t \geq 0, \tag{1.3}$$

$$u(0) = u_0, \tag{1.4}$$

obtained setting formally $\varepsilon = 0$ in (1.1), and omitting the second initial condition in (1.2).

Several cases have been considered in the last 30 years, depending on the nonlinearity (degenerate or nondegenerate), on the dissipative term (constant dissipation $p = 0$ or weak dissipation $p > 0$), and on the operator A (coercive or noncoercive). The main research lines concern global existence for the parabolic and the hyperbolic problem (at least when ε is small enough), decay estimates on $u(t)$, $u_\varepsilon(t)$, and $u_\varepsilon(t) - u(t)$ as $t \to +\infty$, error estimates on the difference as $\varepsilon \to 0^+$, and decay-error estimates, namely estimates describing in the same time the behavior of the difference $u_\varepsilon(t) - u(t)$ as $t \to +\infty$ and $\varepsilon \to 0^+$. The interested reader is referred to the survey [5], or to the more recent papers [2, 7, 8].

In this paper we focus on the case where the equation is *nondegenerate*, namely

$$\inf\left\{m(\sigma) : \sigma \geq 0\right\} =: \mu > 0, \tag{1.5}$$

and the operator is *coercive*, namely

$$\inf\left\{\langle Au, u\rangle : u \in D(A), |u| = 1\right\} =: \nu > 0. \tag{1.6}$$

Concerning the parabolic problem, it is well-known that it admits a global solution for every $p \geq 0$, and every $u_0 \in D(A)$ (and even for less regular data and more general nonlinearities, see [9]).

As for the hyperbolic problem, things are different depending on p. Let us begin with the linear equation in which $m(\sigma)$ is a positive constant. In this case, T. Yamazaki [14] and J. Wirth [13] proved two complementary results, which can be outlined as follows.

- When $p > 1$, the dissipative term is too weak, and solutions of (1.1), (1.2) behave as solutions of the same equation without the dissipative term. In particular, solutions do not decay to 0. This is the *hyperbolic regime*.

- When $p < 1$, inertia is negligible, and solutions of (1.1), (1.2) behave as solutions of (1.3), (1.4). In particular, they decay to 0. This is the *parabolic regime*, with the so-called effective dissipation.
- When $p = 1$, the dissipation is still effective (namely the integral of the coefficient diverges), but according to [13] "the parabolic asymptotics changes to a wave type asymptotics". In any case, solutions keep on going to 0, at least when ε is small enough, and for this reason the case $p = 1$ eventually falls in the parabolic regime.

These results have been extended to Kirchhoff equation by H. Hashimoto and T. Yamazaki [10], T. Yamazaki [15, 16] and the authors [6], in the following sense.

- When $p \in [0, 1]$, problem (1.1), (1.2) has a unique global solution provided that ε is small enough, and this solution decays to 0 as $t \to +\infty$. This is the parabolic regime.
- When $p > 1$, existence of global solutions to (1.1), (1.2) is known only for special initial data or special operators, the same ones for which global existence is known in the nondissipative case. Global existence for every $(u_0, u_1) \in D(A) \times D(A^{1/2})$, even for ε small enough, is still an open problem, exactly as in the nondissipative case. In any case, nontrivial global solutions, if they exist, can *not* decay to 0 as $t \to +\infty$. This is the hyperbolic regime.

All the results quoted above do not depend on the coerciveness of A, namely they are true also when $\nu = 0$.

Several estimates on solutions have been proved in the literature, once again without assumption (1.6). The prototype of *decay estimates* is that

$$\left|A^{1/2}u(t)\right|^2 \leq \frac{C}{(1+t)^{1+p}} \quad \text{and} \quad \left|A^{1/2}u_\varepsilon(t)\right|^2 \leq \frac{C}{(1+t)^{1+p}}$$

for every $t \geq 0$, where the constant C is independent of ε and of course also of t. As a consequence, we have also that

$$\left|A^{1/2}\big(u_\varepsilon(t) - u(t)\big)\right|^2 \leq \frac{C}{(1+t)^{1+p}} \quad \forall t \geq 0. \tag{1.7}$$

The prototype of *error estimates* is that for initial data $(u_0, u_1) \in D(A^{3/2}) \times D(A^{1/2})$ one has that

$$\left|A^{1/2}\big(u_\varepsilon(t) - u(t)\big)\right|^2 \leq C\varepsilon^2 \quad \forall t \geq 0, \tag{1.8}$$

where the constant C is once again independent of ε and t (global-in-time error estimates). It is well-known that ε^2 is the best possible convergence rate (even when looking for local-in-time error estimates), and that $D(A^{3/2}) \times D(A^{1/2})$ is the minimal requirement on initial data which guarantees this rate (even in the case of linear equations). We refer to [1, 3, 4] for these aspects.

The prototype of *decay-error estimates* is that for initial data $(u_0, u_1) \in D(A^{3/2}) \times D(A^{1/2})$ one has that

$$\left|A^{1/2}\big(u_\varepsilon(t) - u(t)\big)\right|^2 \leq C\frac{\varepsilon^2}{(1+t)^{1+p}} \quad \forall t \geq 0. \tag{1.9}$$

We point out in particular that, according to these estimates, solutions of the hyperbolic problem decay with the same rate of solutions of the parabolic problem. Moreover, in the decay-error estimates (1.9) we have the same convergence rate of the error estimates (1.8), and the same decay rate of the decay estimates (1.7). Finally, all these results hold true without coerciveness assumptions on A, and for these general operators it turns out that decay rates are optimal.

When the operator A is coercive, better decay rates are expected. For example, it is easy to see that solutions of the parabolic problem satisfy

$$\left|A^{1/2}u(t)\right|^2 \leq Ce^{-\alpha(1+t)^{1+p}} \quad \forall t \geq 0 \tag{1.10}$$

for a suitable $\alpha > 0$, depending on μ, ν, and p (see Theorem 2.1).

Therefore, the analogy with the noncoercive case could lead to guess that also solutions of the hyperbolic problem should decay with the same exponential rate, and the same rate should also appear in the decay-error estimates.

In this paper we show that this is *not* the case, because solutions of the hyperbolic problem decay to 0 with a different, slower rate. Indeed we prove (see Theorem 2.2) that

$$\left|A^{1/2}u_\varepsilon(t)\right|^2 \leq Ce^{-\alpha(1+t)^{1-p}} \quad \forall t \geq 0 \tag{1.11}$$

if $p \in [0, 1)$, and

$$\left|A^{1/2}u_\varepsilon(t)\right|^2 \leq \frac{C}{(1+t)^{\alpha}} \quad \forall t \geq 0$$

if $p = 1$, where $\alpha < 2\mu\nu$ if $p = 0$, and α is any (positive) real number if $p \in (0, 1]$ (now the constant C depends also on α). These rates are optimal, in the sense that every nonzero solution does not satisfy an estimate such as (1.11) with an exponent larger than $(1 - p)$ (see Theorem 2.4). The same slower rates appear also in the decay-error estimates (see Theorem 2.3), and of course they are optimal also in this case.

We have thus shown an essential difference between the coercive and the noncoercive case. In the noncoercive case, solutions of the hyperbolic problem mimic the behavior of solutions of the parabolic problem for every $p \in [0, 1]$. In the coercive case, this is true only for $p = 0$, when the exponent $(1 + p)$ in (1.10) and the exponent $(1 - p)$ in (1.11) coincide. On the contrary, for every $p \in (0, 1]$ there is a spread between exponents in the decay rates of $u(t)$ and $u_\varepsilon(t)$, and this spread becomes larger and larger as p approaches 1. As a consequence, from the point of view of decay rates, (1.3) is a good approximation of (1.1) for ε small in the noncoercive case, but not in the coercive case (see also Sect. 5.2.3).

In both cases (coercive and noncoercive), the parabolic problem and the hyperbolic problem take different paths when $p > 1$: solutions of the parabolic problem keep on decaying according to (1.10), hence faster and faster as p grows, while solutions of the hyperbolic problem do not decay to 0 any more (provided that they globally exist).

All our proofs are based on linear arguments. To this end, we first linearize (1.1) and (1.3). We obtain the following equations

$$\varepsilon u_\varepsilon''(t) + \frac{1}{(1+t)^p} u_\varepsilon'(t) + c_\varepsilon(t) A u_\varepsilon(t) = 0 \quad \forall t \geq 0, \tag{1.12}$$

$$\frac{1}{(1+t)^p} u'(t) + c(t) A u(t) = 0 \quad \forall t \geq 0, \tag{1.13}$$

with time-dependent coefficients $c_\varepsilon : [0, +\infty) \to (0, +\infty)$ and $c : [0, +\infty) \to (0, +\infty)$.

Then we prove decay and decay-error estimates for solutions of these linear equations, under suitable assumptions on the coefficients. This is the core of the paper.

Finally, we just observe that the coefficients $c_\varepsilon(t)$ and $c(t)$ coming from the nonlinear terms in (1.1) and (1.3) satisfy the assumptions required by the linear theory. Fortunately, these assumptions are quite weak, and follow easily from previous literature on the noncoercive case.

This paper is organized as follows. In Sect. 5.2.1 we recall the previous results and estimates needed throughout this paper. In Sect. 5.2.2 we state our main results for Kirchhoff equations. In Sect. 5.2.3 we present a heuristic argument leading to our decay rates. In Sect. 5.2.4 we state our results for linear equations with time-dependent coefficients. In Sect. 5.3 we collect all proofs.

5.2 Statements

5.2.1 Previous Works

The theory of nondegenerate Kirchhoff equations with weak dissipation has been developed in [6, 15, 16]. In the following statement we collect the existence results, and some decay and error estimates. We limit ourselves to the results which are needed in the sequel, and for this reason Theorem A below does not represent the full state of the art, especially for decay-error estimates. The interested reader is referred to Sect. 5 of [5] for further (and more refined) estimates and references.

Theorem A *Let H be a Hilbert space, let A be a self-adjoint nonnegative operator on H with dense domain $D(A)$ (no coercivity assumption on A), let $m : [0, +\infty) \to (0, +\infty)$ be a locally Lipschitz continuous function satisfying the nondegeneracy condition (1.5), and let $(u_0, u_1) \in D(A) \times D(A^{1/2})$.*

Then we have the following conclusions.

(1) (Parabolic problem) *For every* $p \geq 0$, *problem* (1.3), (1.4) *has a unique global solution*

$$u \in C^1\big([0,+\infty); H\big) \cap C^0\big([0,+\infty); D(A)\big). \tag{2.1}$$

Moreover $u \in C^1((0,+\infty); D(A^\alpha))$ *for every* $\alpha \geq 0$ (*and more generally* u *is of class* C^{k+1} *when* $m(\sigma)$ *is of class* C^k), *and there exists a constant* C *such that*

$$\begin{aligned}(1+t)^2\big|u'(t)\big|^2 + (1+t)^{1+p}\big|A^{1/2}u(t)\big|^2& \\ + (1+t)^{2(1+p)}\big|Au(t)\big|^2 \leq C \quad \forall t \geq 0.&\end{aligned} \tag{2.2}$$

(2) (Hyperbolic problem) *For every* $p \in [0,1]$, *there exists* $\varepsilon_0 > 0$ *such that*, *for every* $\varepsilon \in (0,\varepsilon_0)$, *problem* (1.1), (1.2) *has a unique global solution*

$$u_\varepsilon \in C^2\big([0,+\infty); H\big) \cap C^1\big([0,+\infty); D\big(A^{1/2}\big)\big) \cap C^0\big([0,+\infty); D(A)\big). \tag{2.3}$$

Moreover, there exists a constant C *such that for every* $\varepsilon \in (0,\varepsilon_0)$ *we have that*

$$\begin{aligned}(1+t)^2\big|u_\varepsilon'(t)\big|^2 + (1+t)^{1+p}\big|A^{1/2}u_\varepsilon(t)\big|^2& \\ + (1+t)^{2(1+p)}\big|Au_\varepsilon(t)\big|^2 \leq C \quad \forall t \geq 0.&\end{aligned} \tag{2.4}$$

(3) (Singular perturbation) *If* $p \in [0,1]$, *and* $(u_0,u_1) \in D(A^{3/2}) \times D(A^{1/2})$, *then there exist* $\varepsilon_1 \in (0,\varepsilon_0)$ *and* C *such that, for every* $\varepsilon \in (0,\varepsilon_1)$ *we have that*

$$\big|A^{1/2}\big(u_\varepsilon(t) - u(t)\big)\big|^2 \leq C\varepsilon^2 \quad \forall t \geq 0. \tag{2.5}$$

5.2.2 Main Results

In this section we state the main results of this paper. The first one concerns decay estimates for solutions of the parabolic problem.

Theorem 2.1 (Parabolic equation) *Let* H *be a Hilbert space, and let* A *be a self-adjoint operator on* H *with dense domain* $D(A)$. *Let* $u_0 \in D(A)$, *let* $p \geq 0$, *and let* $m : [0,+\infty) \to (0,+\infty)$ *be a locally Lipschitz continuous function.*

Let us assume that the nondegeneracy and coerciveness assumptions (1.5) *and* (1.6) *are satisfied.*

Then problem (1.3), (1.4) *has a unique global solution* $u(t)$ *with the regularity prescribed in statement* (1) *of Theorem A*, *and there exists a constant* C *such that*

$$\left|u(t)\right|^2+\left|A^{1/2}u(t)\right|^2+\left|Au(t)\right|^2+\frac{|u'(t)|^2}{(1+t)^{2p}}\le C\exp\left(-\frac{2\mu\nu}{1+p}(1+t)^{1+p}\right) \tag{2.6}$$

for every $t\ge 0$.

The second result concerns decay estimates for solutions of the hyperbolic problem.

Theorem 2.2 (Hyperbolic equation) *Let* H *be a Hilbert space*, *and let* A *be a self-adjoint operator on* H *with dense domain* $D(A)$. *Let* $(u_0,u_1)\in D(A)\times D(A^{1/2})$, *let* $p\in[0,1]$, *and let* $m:[0,+\infty)\to(0,+\infty)$ *be a locally Lipschitz continuous function*.

Let us assume that the nondegeneracy and coerciveness assumptions (1.5) *and* (1.6) *are satisfied*.

Then there exists $\varepsilon_0>0$ *such that*, *for every* $\varepsilon\in(0,\varepsilon_0)$, *problem* (1.1), (1.2) *has a unique global solution* $u_\varepsilon(t)$ *with the regularity prescribed by* (2.3).

Moreover the function

$$\varGamma_\varepsilon(t):=\left|u_\varepsilon(t)\right|^2+\left|A^{1/2}u_\varepsilon(t)\right|^2+\left|Au_\varepsilon(t)\right|^2+\left|u'_\varepsilon(t)\right|^2+\varepsilon\left|A^{1/2}u'_\varepsilon(t)\right|^2 \tag{2.7}$$

satisfies the following decay estimates.

- *Case* $p=0$. *For every* $\beta<2\mu\nu$, *there exist* $\varepsilon_1\in(0,\varepsilon_0]$ *and* C *such that*

$$\varGamma_\varepsilon(t)\le Ce^{-\beta t}\quad\forall t\ge 0,\forall\varepsilon\in(0,\varepsilon_1). \tag{2.8}$$

- *Case* $p\in(0,1)$. *For every* $\beta>0$, *there exist* $\varepsilon_1\in(0,\varepsilon_0]$ *and* C *such that*

$$\varGamma_\varepsilon(t)\le Ce^{-\beta(1+t)^{1-p}}\quad\forall t\ge 0,\forall\varepsilon\in(0,\varepsilon_1). \tag{2.9}$$

- *Case* $p=1$. *For every* $\beta>0$, *there exist* $\varepsilon_1\in(0,\varepsilon_0]$ *and* C *such that*

$$\varGamma_\varepsilon(t)\le\frac{C}{(1+t)^\beta}\quad\forall t\ge 0,\forall\varepsilon\in(0,\varepsilon_1). \tag{2.10}$$

Of course the constants C and ε_1 in (2.8) through (2.10) depend also on β.

The third step concerns the singular perturbation problem. Following the approach introduced in [11] in the linear case, we define the corrector $\theta_\varepsilon(t)$ as the solution of the second order *linear* ordinary differential equation

$$\varepsilon\theta''_\varepsilon(t)+\frac{1}{(1+t)^p}\theta'_\varepsilon(t)=0\quad\forall t\ge 0, \tag{2.11}$$

with initial data

$$\theta_\varepsilon(0) = 0, \qquad \theta'_\varepsilon(0) = u_1 + m\left(\left|A^{1/2}u_0\right|^2\right)Au_0 =: \theta_0.$$

Since $\theta_0 = u'_\varepsilon(0) - u'(0)$, this corrector keeps into account the boundary layer due to the loss of one initial condition.

We can now define $r_\varepsilon(t)$ and $\rho_\varepsilon(t)$ in such a way that

$$u_\varepsilon(t) = u(t) + \theta_\varepsilon(t) + r_\varepsilon(t) = u(t) + \rho_\varepsilon(t) \quad \forall t \geq 0.$$

With these notations, the singular perturbation problem consists in proving that $r_\varepsilon(t) \to 0$ or $\rho_\varepsilon(t) \to 0$ in some sense as $\varepsilon \to 0^+$. We recall that the two remainders play different roles. In particular, $r_\varepsilon(t)$ is well suited for estimating derivatives, while $\rho_\varepsilon(t)$ is used in estimates without derivatives. This distinction is essential. Indeed it is not possible to prove decay-error estimates on $A^\alpha r_\varepsilon(t)$ because it does not decay to 0 as $t \to +\infty$ (indeed $u_\varepsilon(t)$ and $u(t)$ tend to 0, while the corrector $\theta_\varepsilon(t)$ does not), and it is not possible to prove decay-error estimates on $A^\alpha \rho'_\varepsilon(t)$ because in general for $t = 0$ it does not tend to 0 as $\varepsilon \to 0^+$ (due to the loss of one initial condition).

We are now ready to state our decay-error estimates.

Theorem 2.3 (Singular perturbation) *Let H be a Hilbert space, and let A be a self-adjoint operator on H with dense domain $D(A)$. Let $(u_0, u_1) \in D(A) \times D(A^{1/2})$, let $p \in [0, 1]$, and let $m : [0, +\infty) \to (0, +\infty)$ be a locally Lipschitz continuous function.*

Let us assume that the nondegeneracy and coerciveness assumptions (1.5) and (1.6) are satisfied, and let $u(t)$, ε_0, $u_\varepsilon(t)$, $r_\varepsilon(t)$, $\rho_\varepsilon(t)$ be as above.

Let us consider the functions

$$\Gamma_{r,\varepsilon}(t) := \left|\rho_\varepsilon(t)\right|^2 + \left|A^{1/2}\rho_\varepsilon(t)\right|^2 + \varepsilon\left|r'_\varepsilon(t)\right|^2,$$

$$\Gamma_{c,\varepsilon}(t) := \left|\rho_\varepsilon(t)\right|^2 + \left|A^{1/2}\rho_\varepsilon(t)\right|^2 + \left|A\rho_\varepsilon(t)\right|^2 + \left|r'_\varepsilon(t)\right|^2 + \varepsilon\left|A^{1/2}r'_\varepsilon(t)\right|^2,$$

where indices c and r stay for "complete", and "reduced", respectively.

(1) *If in addition $(u_0, u_1) \in D(A^{3/2}) \times D(A^{1/2})$, then we have the following decay-error estimates.*

- *Case $p = 0$. For every $\beta < 2\mu\nu$, there exist $\varepsilon_1 \in (0, \varepsilon_0]$ and C such that*

$$\Gamma_{r,\varepsilon}(t) \leq C\varepsilon^2 e^{-\beta t} \quad \forall t \geq 0, \forall \varepsilon \in (0, \varepsilon_1). \tag{2.12}$$

- *Case $p \in (0, 1)$. For every $\beta > 0$, there exist $\varepsilon_1 \in (0, \varepsilon_0]$ and C such that*

$$\Gamma_{r,\varepsilon}(t) \leq C\varepsilon^2 e^{-\beta(1+t)^{1-p}} \quad \forall t \geq 0, \forall \varepsilon \in (0, \varepsilon_1). \tag{2.13}$$

- *Case $p = 1$. For every $\beta > 0$, there exist $\varepsilon_1 \in (0, \varepsilon_0]$ and C such that*

$$\Gamma_{r,\varepsilon}(t) \leq \frac{C\varepsilon^2}{(1+t)^\beta} \quad \forall t \geq 0, \forall \varepsilon \in (0, \varepsilon_1). \tag{2.14}$$

(2) *If in addition* $(u_0, u_1) \in D(A^2) \times D(A)$, *then we have the same decay-error estimates with* $\Gamma_{c,\varepsilon}(t)$ *instead of* $\Gamma_{r,\varepsilon}(t)$.

As in Theorem 2.2 above, the constants C and ε_1 in (2.12) through (2.14) depend also on β. We point out that in these estimates we have the same convergence rate as in (2.5), and the same decay rates as in (2.8) through (2.10).

The last result we state, together with Remarks 2.5 and 2.6 below, clarifies the optimality of the decay rates of Theorem 2.2, hence also of Theorem 2.3.

Theorem 2.4 (Optimality of decay rates) *Let* $H, A, p \in [0,1]$, $m : [0,+\infty) \to (0,+\infty)$, *and* $(u_0, u_1) \in D(A) \times D(A^{1/2})$ *be as in Theorem* 2.2. *Let* $\varepsilon > 0$, *and let* $u_\varepsilon(t)$ *be the solution to problem* (1.1), (1.2).

Let $\Phi : [0,+\infty) \to (0,+\infty)$ *be a function of class* C^1 *such that*

$$\lim_{t\to+\infty} (1+t)^p \frac{\Phi'(t)}{\Phi(t)} = -\infty. \tag{2.15}$$

If $(u_0, u_1) \neq (0,0)$, *then*

$$\lim_{t\to+\infty} \left(\varepsilon \left|u'_\varepsilon(t)\right|^2 + \left|A^{1/2}u_\varepsilon(t)\right|^2\right) \frac{1}{\Phi(t)} = +\infty. \tag{2.16}$$

Remark 2.5 When $p > 0$, Theorem 2.4 is exactly the counterpart of Theorem 2.2. Indeed let us consider any $\Phi : [0,+\infty) \to [0,+\infty)$, and let $\Gamma_\varepsilon(t)$ be defined as in (2.7). If (2.15) is satisfied, then we can not expect that $\Gamma_\varepsilon(t) \leq C\Phi(t)$ because of (2.16). On the contrary, if

$$(1+t)^p \frac{\Phi'(t)}{\Phi(t)} \geq -\beta > -\infty,$$

then $\Phi(t) \geq Ce^{-\beta(1-p)^{-1}(1+t)^{1-p}}$ if $p \in (0,1)$, and $\Phi(t) \geq C(1+t)^{-\beta}$ if $p = 1$, and in both cases Theorem 2.2 guarantees that $\Gamma_\varepsilon(t) \leq C\Phi(t)$.

Note in particular that the function $\Phi(t) := e^{-\beta(1+t)^\delta}$ satisfies (2.15) if and only if $\delta > 1-p$, which means that $(1-p)$ is the larger exponent for which (2.9) holds true.

Remark 2.6 When $p = 0$, estimate (2.8) can not be true when $\beta > 2\mu\nu$. This can be easily seen by considering the explicit solutions of the ordinary differential equation

$$\varepsilon y''(t) + y'(t) + \mu\nu y(t) = 0, \tag{2.17}$$

which is just the particular case of (1.1) where $H = \mathbb{R}$, A is ν times the identity, and $m(\sigma) \equiv \mu$ is a constant.

On the other hand, solutions of (2.17) satisfy (2.8) also with $\beta = 2\mu\nu$. We suspect that this could be true in general, but for the time being we have no proof.

Open Problem 2.7 Is (2.8) true also in the case $\beta = 2\mu\nu$?

5.2.3 Heuristics

According to Theorem 2.1, solutions of the parabolic problem decay as solutions of the ordinary differential equation

$$\frac{1}{(1+t)^p}y'(t)+\mu\nu y(t)=0. \tag{2.18}$$

This is hardly surprising, since (2.18) is just the special case of (1.3) corresponding to $H=\mathbb{R}$, A equal to ν times the identity, and $m(\sigma)\equiv\mu$.

Analogously, it is reasonable to expect solutions of the hyperbolic problem to decay as solutions of the ordinary differential equation

$$\varepsilon y_\varepsilon''(t)+\frac{1}{(1+t)^p}y_\varepsilon'(t)+\mu\nu y_\varepsilon(t)=0. \tag{2.19}$$

A reasonable ansatz for these solutions is that asymptotically they are the product of an oscillatory term $v_\varepsilon(t)$, and a decaying term $\lambda_\varepsilon(t)$. Plugging $y_\varepsilon(t)=\lambda_\varepsilon(t)\cdot v_\varepsilon(t)$ into (2.19), we obtain that

$$\left(\varepsilon v_\varepsilon''+\mu\nu v_\varepsilon\right)\lambda_\varepsilon+\left(2\varepsilon\lambda_\varepsilon'+\frac{\lambda_\varepsilon}{(1+t)^p}\right)v_\varepsilon'+\left(\varepsilon\lambda_\varepsilon''+\frac{\lambda_\varepsilon'}{(1+t)^p}\right)v_\varepsilon=0.$$

A reasonable guess is now that the coefficient of $\lambda_\varepsilon(t)$ in the first term is almost 0, as well as the coefficient of $v_\varepsilon'(t)$ in the second term.

The first condition is that $\varepsilon v_\varepsilon''(t)+\mu\nu v_\varepsilon(t)\sim 0$, namely

$$v_\varepsilon(t)\sim\sin\left(\sqrt{\frac{\mu\nu}{\varepsilon}}t\right),$$

which yields the same oscillations of the undamped equation.

The second condition is that

$$2\varepsilon\lambda_\varepsilon'(t)+\frac{\lambda_\varepsilon(t)}{(1+t)^p}\sim 0, \tag{2.20}$$

and for every $p\in(0,1]$ this yields a decay rate which is compatible both with Theorem 2.2 and with Theorem 2.4.

We do not know if similar asymptotics have been rigorously justified in the literature (see [13] for the case $p=1$). Nevertheless, this non-rigorous argument suggests that actually there is no sharp break between parabolic and hyperbolic regimes. For $p\leq 1$, the hyperbolic nature survives in the oscillatory behavior of $v_\varepsilon(t)$, but it is hidden by the damping imposed by (2.20). When $p>1$, solutions of (2.20) tend to a positive constant, and the hyperbolic nature emerges undisputed.

We conclude by pointing out once again that this analysis applies to the nondegenerate coercive case. Things are quite different both in the nondegenerate noncoercive case (see [6, 12–15]), and in the degenerate coercive case (see [7, 8]).

5.2.4 Linearization

Proofs of our main results are based on the analysis of the linear equations (1.12) and (1.13). We assume that the coefficient $c : [0, +\infty) \to (0, +\infty)$ is of class C^1, and satisfies the following estimates

$$c(t) \geq \mu > 0 \quad \forall t \geq 0, \tag{2.21}$$

$$c(t) \leq M_1 \quad \forall t \geq 0, \tag{2.22}$$

$$\left|c'(t)\right| \leq M_2 \quad \forall t \geq 0. \tag{2.23}$$

Similarly, we assume that $c_\varepsilon : [0, +\infty) \to (0, +\infty)$, with $\varepsilon \in (0, \varepsilon_0)$, is a family of coefficients of class C^1 satisfying the following estimates

$$c_\varepsilon(t) \geq \mu > 0 \quad \forall t \geq 0, \forall \varepsilon \in (0, \varepsilon_0), \tag{2.24}$$

$$c_\varepsilon(t) \leq M_3 \quad \forall t \geq 0, \forall \varepsilon \in (0, \varepsilon_0), \tag{2.25}$$

$$\left|c_\varepsilon'(t)\right| \leq \frac{M_4}{(1+t)^p} \quad \forall t \geq 0, \forall \varepsilon \in (0, \varepsilon_0). \tag{2.26}$$

When considering the singular perturbation, we also assume that

$$\left|c_\varepsilon(t) - c(t)\right| \leq M_5 \varepsilon \quad \forall t \geq 0, \forall \varepsilon \in (0, \varepsilon_0), \tag{2.27}$$

and we define the corrector $\theta_\varepsilon(t)$ as the solution of (2.11) with initial data

$$\theta_\varepsilon(0) = 0, \qquad \theta_\varepsilon'(0) = u_1 + c(0)Au_0 =: \theta_0. \tag{2.28}$$

The following results are the linear counterparts of Theorems 2.1 through 2.4. All of them can be extended to Lipschitz continuous coefficients through a straightforward approximation argument.

Theorem 2.8 (Linear parabolic equation) *Let H, A, $p \geq 0$, and $u_0 \in D(A)$ be as in Theorem 2.1. Let $c : [0, +\infty) \to (0, +\infty)$ be a continuous function satisfying (2.21) and (2.22).*

Then problem (1.13), (1.4) has a unique global solution $u(t)$ with the regularity prescribed by (2.1), and this solution satisfies (2.6).

Theorem 2.9 (Linear hyperbolic equation) *Let H, A, $p \in [0, 1]$, and $(u_0, u_1) \in D(A) \times D(A^{1/2})$ be as in Theorem 2.2, and let $\varepsilon_0 > 0$. Let $c_\varepsilon : [0, +\infty) \to (0, +\infty)$, with $\varepsilon \in (0, \varepsilon_0)$, be a family of coefficients of class C^1 satisfying (2.24) through (2.26).*

Then, for every $\varepsilon \in (0, \varepsilon_0)$, problem (1.12), (1.2) has a unique global solution $u_\varepsilon(t)$ with the regularity prescribed by (2.3), and this solution satisfies the same decay estimates stated in Theorem 2.2, depending on the values of p.

Theorem 2.10 (Linear singular perturbation) *Let H, A, $p \in [0, 1]$, (u_0, u_1), ε_0, $c(t)$, $u(t)$, $c_\varepsilon(t)$, $u_\varepsilon(t)$ be as in Theorems 2.8 and 2.9.*

Let us assume that also (2.23) and (2.27) hold true, and let $r_\varepsilon(t)$ and $\rho_\varepsilon(t)$ be defined as usual (keeping in mind that the corrector now satisfies (2.11) and (2.28)).

Then $r_\varepsilon(t)$ and $\rho_\varepsilon(t)$ satisfy the decay-error estimates of statements (1) *and* (2) *of Theorem 2.3, depending on the further regularity of (u_0, u_1), and on the values of p.*

Theorem 2.11 (Linear hyperbolic equation: optimality) *Let H, A, $p \in [0, 1]$, and $(u_0, u_1) \in D(A) \times D(A^{1/2})$ be as in Theorem 2.2. Let $\varepsilon > 0$, and let $u_\varepsilon(t)$ be the solution to problem (1.12), (1.2) with a coefficient $c_\varepsilon : [0, +\infty) \to (0, +\infty)$ of class C^1 satisfying (2.24) through (2.26).*

If $(u_0, u_1) \neq (0, 0)$, then (2.16) holds true for every function $\Phi : [0, +\infty) \to (0, +\infty)$ of class C^1 satisfying (2.15).

5.3 Proofs

5.3.1 Proof of Theorem 2.8

We prove a more general result, with some further estimates needed when dealing with the singular perturbation problem. These estimates easily imply (2.6). Indeed, estimate (3.2) with $k = 0, 1, 2$ allows to control the first three terms in the left-hand side of (2.6). Thanks to (1.13), the estimate for $|u'(t)|$ follows from the boundedness of $c(t)$ and the estimate on $|Au(t)|$.

Proposition 3.1 *Let H, A, and $c(t)$ be as in Theorem 2.8. Let us set*

$$\gamma := \frac{2\mu\nu}{1+p}, \qquad \Psi_{\alpha,p}(t) := \exp\left(-\alpha\left[(1+t)^{1+p} - 1\right]\right). \tag{3.1}$$

Then we have the following estimates.

(1) *If $u_0 \in D(A^{k/2})$ for some $k \in \mathbb{N}$, then*

$$\left|A^{k/2}u(t)\right|^2 \leq \left|A^{k/2}u_0\right|^2 \Psi_{\gamma,p}(t) \quad \forall t \geq 0. \tag{3.2}$$

Moreover, for every $\alpha < \gamma$ we have that

$$\int_0^{+\infty} \frac{|A^{(k+1)/2}u(t)|^2}{\Psi_{\alpha,p}(t)} dt \leq \left(2\mu - \frac{\alpha(1+p)}{\nu}\right)^{-1} \left|A^{k/2}u_0\right|^2. \tag{3.3}$$

(2) *If $u_0 \in D(A^{3/2})$, and $c(t)$ is of class C^1 and satisfies (2.23), then for every $\alpha < \gamma$ there exists a constant C (depending also on α) such that*

$$\int_0^{+\infty} \frac{|u''(t)|^2}{\Psi_{\alpha,p}(t)} dt \leq C. \tag{3.4}$$

(3) *If* $u_0 \in D(A^2)$, *and* $c(t)$ *is of class* C^1 *and satisfies* (2.23), *then there exists a constant* C *such that*

$$\left|u''(t)\right|^2 \leq C(1+t)^{4p}\Psi_{\gamma,p}(t) \quad \forall t \geq 0. \tag{3.5}$$

Moreover, for every $\alpha < \gamma$, *there exists a constant* C (*depending also on* α) *such that*

$$\int_0^{+\infty} \frac{|A^{1/2}u''(t)|^2}{\Psi_{\alpha,p}(t)} dt \leq C. \tag{3.6}$$

Proof Let us set $E_k(t) := |A^{k/2}u(t)|^2$. From (1.13), (1.6), and (2.21), we have that

$$\begin{aligned} E_k'(t) &= 2\langle A^{(k+1)/2}u(t), A^{(k-1)/2}u'(t)\rangle = -2c(t)(1+t)^p\left|A^{(k+1)/2}u(t)\right|^2 \\ &\leq -2c(t)(1+t)^p \cdot \nu\left|A^{k/2}u(t)\right|^2 \leq -2\mu\nu(1+t)^p E_k(t). \end{aligned}$$

Integrating this differential inequality, we obtain (3.2).

Moreover we have that

$$\begin{aligned} \frac{d}{dt}\left[\frac{E_k(t)}{\Psi_{\alpha,p}(t)}\right] &= \frac{E_k'(t)}{\Psi_{\alpha,p}(t)} + \alpha(1+p)(1+t)^p \frac{|A^{k/2}u(t)|^2}{\Psi_{\alpha,p}(t)} \\ &\leq -2\mu(1+t)^p \frac{|A^{(k+1)/2}u(t)|^2}{\Psi_{\alpha,p}(t)} \\ &\quad + \frac{\alpha(1+p)}{\nu}(1+t)^p \frac{|A^{(k+1)/2}u(t)|^2}{\Psi_{\alpha,p}(t)}, \end{aligned}$$

hence

$$\left(2\mu - \frac{\alpha(1+p)}{\nu}\right)\int_0^t (1+s)^p \frac{|A^{(k+1)/2}u(s)|^2}{\Psi_{\alpha,p}(s)} ds + \frac{E_k(t)}{\Psi_{\alpha,p}(t)} \leq E_k(0) \quad \forall t \geq 0,$$

which easily implies (3.3).

Let us prove the estimates on the second derivative. From (1.13) we obtain that

$$u''(t) = -p(1+t)^{p-1}c(t)Au(t) - (1+t)^p c'(t)Au(t) + (1+t)^{2p}c^2(t)A^2u(t).$$

Therefore, from (2.22) and (2.23), it follows that

$$\left|u''(t)\right|^2 \leq k_1(1+t)^{2p}\left|Au(t)\right|^2 + k_2(1+t)^{4p}\left|A^2u(t)\right|^2. \tag{3.7}$$

If $u_0 \in D(A^2)$, then (3.5) follows from (3.2) with $k = 2$ and $k = 4$.

In order to prove the integral estimates on $u''(t)$, let us choose η such that $\alpha < \alpha + \eta < \gamma$. Since $\Psi_{\alpha+\eta,p}(t) = \Psi_{\alpha,p}(t) \cdot \Psi_{\eta,p}(t)$, and since

$$\sup_{t\geq 0}\left\{\Psi_{\eta,p}(t)(1+t)^{4p}\right\} < +\infty,$$

from (3.7) it follows that

$$\frac{|u''(t)|^2}{\Psi_{\alpha,p}(t)} \leq (1+t)^{4p}\Psi_{\eta,p}(t)\cdot\frac{k_1|Au(t)|^2+k_2|A^2u(t)|^2}{\Psi_{\alpha,p}(t)\cdot\Psi_{\eta,p}(t)} \leq k_3\frac{|Au(t)|^2+|A^2u(t)|^2}{\Psi_{\alpha+\eta,p}(t)}.$$

From (3.3) with $k=1$ and $k=3$ we conclude that

$$\int_0^{+\infty}\frac{|u''(t)|^2}{\Psi_{\alpha,p}(t)}dt \leq k_3\int_0^{+\infty}\frac{|Au(t)|^2+|A^2u(t)|^2}{\Psi_{\alpha+\eta,p}(t)}dt \leq k_4$$

for a suitable k_4 depending also on η. This proves (3.4).

The proof of (3.6) is completely analogous. □

5.3.2 Comparison Results for ODEs

In this subsection we prove estimates for solutions of three ordinary differential equations needed in the sequel. To begin with, for every $\beta > 0$ and every $p \geq 0$ we define $\Phi_{\beta,p} : [0,+\infty) \to (0,+\infty)$ as the solution of the Cauchy problem

$$\Phi'_{\beta,p}(t) = -\frac{\beta}{(1+t)^p}\Phi_{\beta,p}(t) \quad \forall t \geq 0, \tag{3.8}$$

$$\Phi_{\beta,p}(0) = 1. \tag{3.9}$$

We point out that solutions of this problem decay as the right-hand sides of (2.8) through (2.10), depending on the values of p. This is the reason why we are going to exploit $\Phi_{\beta,p}(t)$ several times in the proofs of our decay and decay-error estimates.

Lemma 3.2 *Let $\beta > 0$ and $p \geq 0$ be real numbers, and let $\Phi_{\beta,p}(t)$ be the solution of the Cauchy problem* (3.8), (3.9).

Let ε and K be positive constants, with $2\varepsilon\beta \leq 1$, and let $G : [0,+\infty) \to [0,+\infty)$ be a function of class C^1 such that

$$G'(t) \leq -\frac{1}{\varepsilon}\frac{1}{(1+t)^p}G(t) + \frac{K}{\varepsilon}(1+t)^p\Phi_{\beta,p}(t) \quad \forall t \geq 0. \tag{3.10}$$

Then we have that

$$G(t) \leq \big(2K + G(0)\big)(1+t)^{2p}\Phi_{\beta,p}(t) \quad \forall t \geq 0. \tag{3.11}$$

Proof Let us consider the differential equation

$$y'(t) = -\frac{1}{\varepsilon}\frac{1}{(1+t)^p}y(t) + \frac{K}{\varepsilon}(1+t)^p\Phi_{\beta,p}(t) \quad \forall t \geq 0. \tag{3.12}$$

Assumption (3.10) says that $G(t)$ is a subsolution of (3.12). Let $z(t)$ denote the right-hand side of (3.11). We claim that $z(t)$ is a supersolution of (3.12). Indeed a simple computation shows that

$$\begin{aligned} z'(t) &= 2p\big(2K + G(0)\big)(1+t)^{2p-1}\Phi_{\beta,p}(t) + \big(2K + G(0)\big)(1+t)^{2p}\Phi'_{\beta,p}(t) \\ &\geq -\beta\big(2K + G(0)\big)(1+t)^{p}\Phi_{\beta,p}(t) \\ &\geq -\frac{1}{\varepsilon}\big(K + G(0)\big)(1+t)^{p}\Phi_{\beta,p}(t) \\ &= -\frac{1}{\varepsilon}\frac{1}{(1+t)^p}z(t) + \frac{K}{\varepsilon}(1+t)^{p}\Phi_{\beta,p}(t), \end{aligned}$$

where in the second inequality we exploited that $2\varepsilon\beta \leq 1$, and $2G(0) \geq G(0)$.

Since $G(0) \leq z(0)$, estimate (3.11) follows from the standard comparison principle between subsolutions and supersolutions. □

Lemma 3.3 *Let* $\psi_1 : [0, +\infty) \to [0, +\infty)$ *and* $\psi_2 : [0, +\infty) \to [0, +\infty)$ *be two continuous functions such that*

$$K_1 := \int_0^{+\infty} \psi_1(t)dt < +\infty, \qquad K_2 := \int_0^{+\infty} \psi_2(t)dt < +\infty.$$

Let $E : [0, +\infty) \to [0, +\infty)$ *be a function of class* C^1 *such that* $E(0) = 0$, *and*

$$E'(t) \leq \psi_1(t)\sqrt{E(t)} + \psi_2(t) \quad \forall t \geq 0.$$

Then we have that

$$E(t) \leq K_1^2 + 2K_2 \quad \forall t \geq 0. \tag{3.13}$$

Proof Let us fix any $T > 0$. For every $t \in [0, T]$ we have that

$$E'(t) \leq \psi_1(t) \cdot \Big(\sup_{s\in[0,T]} E(s)\Big)^{1/2} + \psi_2(t).$$

Since $E(0) = 0$, an easy integration gives that

$$E(t) \leq \Big(\sup_{s\in[0,T]} E(s)\Big)^{1/2} \int_0^t \psi_1(s)ds + \int_0^t \psi_2(s)ds \leq K_1\Big(\sup_{s\in[0,T]} E(s)\Big)^{1/2} + K_2$$

for every $t \in [0, T]$. Taking the supremum of the left-hand side as $t \in [0, T]$, we obtain that

$$\sup_{s\in[0,T]} E(s) \leq K_1\Big(\sup_{s\in[0,T]} E(s)\Big)^{1/2} + K_2 \leq \frac{1}{2}K_1^2 + \frac{1}{2}\Big(\sup_{s\in[0,T]} E(s)\Big) + K_2,$$

hence

$$\sup_{s\in[0,T]} E(s) \leq K_1^2 + 2K_2,$$

and in particular $E(T) \leq K_1^2 + 2K_2$. Since T is arbitrary, (3.13) is proved. □

Lemma 3.4 *Let $\beta > 0$ and $p \geq 0$ be real numbers, and let $\Phi_{\beta,p}(t)$ be the solution of the Cauchy problem* (3.8), (3.9).

Let $\psi : [0, +\infty) \to [0, +\infty)$ be a continuous function such that

$$\int_0^{+\infty} \frac{\psi(s)}{\Phi_{\beta,p}(s)} ds < +\infty.$$

Let $T > 0$, and let $F : [T, +\infty) \to [0, +\infty)$ be a function of class C^1 such that

$$F'(t) \leq -\frac{\beta}{(1+t)^p} F(t) + \psi(t) \quad \forall t \geq T. \tag{3.14}$$

Then we have that

$$F(t) \leq \left(\frac{F(T)}{\Phi_{\beta,p}(T)} + \int_0^{+\infty} \frac{\psi(s)}{\Phi_{\beta,p}(s)} ds \right) \cdot \Phi_{\beta,p}(t) \quad \forall t \geq T. \tag{3.15}$$

Proof Let us consider the differential equation

$$y'(t) = -\frac{\beta}{(1+t)^p} y(t) + \psi(t) \quad \forall t \geq 0. \tag{3.16}$$

Assumption (3.14) says that $F(t)$ is a subsolution of (3.16) for $t \geq T$. On the other hand, it is easy to see that

$$z(t) := \left(\frac{F(T)}{\Phi_{\beta,p}(T)} + \int_T^t \frac{\psi(s)}{\Phi_{\beta,p}(s)} ds \right) \cdot \Phi_{\beta,p}(t)$$

is a solution of (3.16) for $t \geq T$. Since $F(T) = z(T)$, the standard comparison principle between subsolutions and supersolutions implies that $F(t) \leq z(t)$ for every $t \geq T$, which in turn implies (3.15). □

5.3.3 Proof of Theorem 2.9

Let us describe the strategy of the proof before entering into details. Let us take any admissible value β, which means any $\beta \in (0, 2\mu\nu)$ if $p = 0$, and any $\beta > 0$ if $p > 0$. Let $\Phi_{\beta,p}(t)$ be the solution of the Cauchy problem (3.8), (3.9).

Estimates (2.8) through (2.10) are equivalent to showing that

$$\Gamma_\varepsilon(t) \leq k_1 \Phi_{\beta,p}(t) \quad \forall t \geq 0 \tag{3.17}$$

for the admissible values of β.

Let μ be the constant in (2.24), and let us choose δ and T in such a way that

$$\delta := \frac{2(\beta+1)\nu}{2\mu\nu-\beta}, \qquad T := 0 \tag{3.18}$$

if $p = 0$ (note that $\delta > 0$), and

$$\delta := \frac{\beta+2}{\mu}, \qquad (1+T)^{2p} \geq \frac{\delta\beta}{2\nu} \tag{3.19}$$

if $p > 0$. For every $\varepsilon \in (0, \varepsilon_0)$, we consider the energies

$$E_\varepsilon(t) := \frac{\varepsilon|u'_\varepsilon(t)|^2}{c_\varepsilon(t)} + \left|A^{1/2}u_\varepsilon(t)\right|^2, \tag{3.20}$$

$$\begin{aligned} F_\varepsilon(t) := {} & \frac{\varepsilon|u'_\varepsilon(t)|^2}{c_\varepsilon(t)} + \left|A^{1/2}u_\varepsilon(t)\right|^2 + \frac{\varepsilon\delta}{(1+t)^p}\left\langle u'_\varepsilon(t), u_\varepsilon(t)\right\rangle \\ & + \frac{\delta}{2}\frac{1}{(1+t)^{2p}}\left|u_\varepsilon(t)\right|^2. \end{aligned} \tag{3.21}$$

We claim that there exist $\varepsilon_2 \in (0, \varepsilon_0)$, and positive constants $k_2, \dots, k_5$, such that

$$k_2\left(\varepsilon\left|u'_\varepsilon(t)\right|^2 + \left|A^{1/2}u_\varepsilon(t)\right|^2\right) \leq E_\varepsilon(t) \leq k_3\left(\varepsilon\left|u'_\varepsilon(t)\right|^2 + \left|A^{1/2}u_\varepsilon(t)\right|^2\right), \tag{3.22}$$

$$k_4\left(\varepsilon\left|u'_\varepsilon(t)\right|^2 + \left|A^{1/2}u_\varepsilon(t)\right|^2\right) \leq F_\varepsilon(t) \leq k_5\left(\varepsilon\left|u'_\varepsilon(t)\right|^2 + \left|A^{1/2}u_\varepsilon(t)\right|^2\right) \tag{3.23}$$

for every $t \geq 0$ and every $\varepsilon \in (0, \varepsilon_2)$. Moreover we claim that

$$E'_\varepsilon(t) \leq 0 \quad \forall t \geq 0, \forall \varepsilon \in (0, \varepsilon_2), \tag{3.24}$$

$$F'_\varepsilon(t) \leq -\frac{\beta}{(1+t)^p}F_\varepsilon(t) \quad \forall t \geq T, \forall \varepsilon \in (0, \varepsilon_2). \tag{3.25}$$

Let us assume that we have proved these claims. Thanks to (3.24), and to the estimate from below in (3.22), we have that

$$\varepsilon\left|u'_\varepsilon(t)\right|^2 + \left|A^{1/2}u_\varepsilon(t)\right|^2 \leq \frac{1}{k_2}E_\varepsilon(t) \leq \frac{1}{k_2}E_\varepsilon(0) \leq k_6$$

for every $t \geq 0$. Since $\Phi_{\beta,p}(t)$ is decreasing, this implies that

$$\varepsilon\left|u'_\varepsilon(t)\right|^2 + \left|A^{1/2}u_\varepsilon(t)\right|^2 \leq \frac{k_6}{\Phi_{\beta,p}(T)} \cdot \Phi_{\beta,p}(t) = k_7\Phi_{\beta,p}(t) \quad \forall t \in [0, T]. \tag{3.26}$$

For $t \geq T$, we exploit (3.25). First of all, from (3.26) with $t = T$, and the estimate from above in (3.23), we have that

$$F_\varepsilon(T) \leq k_5\left(\varepsilon\left|u'_\varepsilon(T)\right|^2 + \left|A^{1/2}u_\varepsilon(T)\right|^2\right) \leq k_8\Phi_{\beta,p}(T).$$

Therefore, from Lemma 3.4 applied with $\psi(t) \equiv 0$, we deduce that $F_\varepsilon(t) \leq k_8 \Phi_{\beta,p}(t)$ for every $t \geq T$. Exploiting this inequality, the estimate from below in (3.23), and (3.26), we conclude that

$$\varepsilon\left|u_\varepsilon'(t)\right|^2 + \left|A^{1/2}u_\varepsilon(t)\right|^2 \leq k_9 \Phi_{\beta,p}(t) \quad \forall t \geq 0, \forall \varepsilon \in (0, \varepsilon_2). \tag{3.27}$$

Since the operator is coercive, this estimate on $|A^{1/2}u_\varepsilon(t)|^2$ yields an analogous estimate on $|u_\varepsilon(t)|^2$.

Up to now, we only assumed that $(u_0, u_1) \in D(A^{1/2}) \times H$. Let us assume now that $(u_0, u_1) \in D(A) \times D(A^{1/2})$. Since (1.12) is linear, estimate (3.27) can be applied to $A^{1/2}u_\varepsilon(t)$, which is once again a solution to (1.12). We thus obtain that

$$\varepsilon\left|A^{1/2}u_\varepsilon'(t)\right|^2 + \left|Au_\varepsilon(t)\right|^2 \leq k_{10} \Phi_{\beta,p}(t) \quad \forall t \geq 0, \forall \varepsilon \in (0, \varepsilon_2). \tag{3.28}$$

It remains to prove the ε-independent estimate on $|u_\varepsilon'(t)|^2$. To this end, we set

$$G_\varepsilon(t) := \left|u_\varepsilon'(t)\right|^2, \tag{3.29}$$

and we claim that

$$G_\varepsilon'(t) \leq -\frac{1}{\varepsilon}\frac{1}{(1+t)^p}G_\varepsilon(t) + \frac{k_{11}}{\varepsilon}(1+t)^p \Phi_{\beta,p}(t) \quad \forall t \geq 0, \forall \varepsilon \in (0, \varepsilon_2). \tag{3.30}$$

If we prove the claim, then from Lemma 3.2 if follows that

$$\left|u_\varepsilon'(t)\right|^2 = G_\varepsilon(t) \leq k_{12}(1+t)^{2p} \Phi_{\beta,p}(t) \quad \forall t \geq 0, \forall \varepsilon \in (0, \varepsilon_2). \tag{3.31}$$

What we actually need is the same estimate without the factor $(1+t)^{2p}$. If $p = 0$, there is nothing to do. If $p > 0$, we take $\beta' = \beta + 2$, and from (3.31) we obtain that

$$\left|u_\varepsilon'(t)\right|^2 = G_\varepsilon(t) \leq k_{13}(1+t)^{2p} \Phi_{\beta',p}(t) \quad \forall t \geq 0, \forall \varepsilon \in (0, \varepsilon_1),$$

of course with new positive constants k_{13} and $\varepsilon_1 \leq \varepsilon_2$, depending also on β'.

Finally, our choice of β' guarantees that

$$(1+t)^{2p} \Phi_{\beta',p}(t) \leq k_{14} \Phi_{\beta,p}(t) \quad \forall t \geq 0$$

for a suitable k_{14} depending on p, β, β' (this inequality can be easily proved exploiting the explicit formulae for $\Phi_{\beta,p}(t)$ and $\Phi_{\beta',p}(t)$, and the fact that $p \leq 1$). This completes the proof of (3.17) for every $\varepsilon \in (0, \varepsilon_1)$.

So we are left to proving (3.22) through (3.25), and (3.30).

Equivalence Between Energies Due to (2.24) and (2.25), estimate (3.22) holds true with

$$k_2 := \min\left\{\frac{1}{M_3}, 1\right\}, \qquad k_3 := \max\left\{\frac{1}{\mu}, 1\right\}.$$

In order to prove (3.23), let us estimate separately the four terms in (3.21). Due to (2.24) and (2.25), we have that

$$\frac{\varepsilon|u_\varepsilon'(t)|^2}{M_3} \leq \frac{\varepsilon|u_\varepsilon'(t)|^2}{c_\varepsilon(t)} \leq \frac{\varepsilon|u_\varepsilon'(t)|^2}{\mu}.$$

Due to (1.6) we have that

$$0 \leq \frac{\delta}{2}\frac{1}{(1+t)^{2p}}\left|u_\varepsilon(t)\right|^2 \leq \frac{\delta}{2}\left|u_\varepsilon(t)\right|^2 \leq \frac{\delta}{2\nu}\left|A^{1/2}u_\varepsilon(t)\right|^2.$$

Applying once again (1.6), and the inequality between arithmetic and geometric mean, we obtain that

$$\frac{\varepsilon|u_\varepsilon'(t)|^2}{2M_3} + \frac{1}{2}\left|A^{1/2}u_\varepsilon(t)\right|^2 \geq \frac{\varepsilon|u_\varepsilon'(t)|^2}{2M_3} + \frac{\nu}{2}\left|u_\varepsilon(t)\right|^2 \geq \sqrt{\frac{\varepsilon\nu}{M_3}}\cdot\left|u_\varepsilon'(t)\right|\cdot\left|u_\varepsilon(t)\right|.$$

If ε is small enough, this implies that

$$\frac{\varepsilon|u_\varepsilon'(t)|^2}{2M_3} + \frac{1}{2}\left|A^{1/2}u_\varepsilon(t)\right|^2 \geq \frac{\varepsilon\delta}{(1+t)^{2p}}\left|\langle u_\varepsilon'(t), u_\varepsilon(t)\rangle\right|.$$

From all these estimates, we easily obtain that

$$F_\varepsilon(t) \geq \frac{\varepsilon|u_\varepsilon'(t)|^2}{2M_3} + \frac{1}{2}\left|A^{1/2}u_\varepsilon(t)\right|^2,$$

and

$$F_\varepsilon(t) \leq \frac{\varepsilon|u_\varepsilon'(t)|^2}{\mu} + \left|A^{1/2}u_\varepsilon(t)\right|^2 + \frac{\delta}{2\nu}\left|A^{1/2}u_\varepsilon(t)\right|^2 + \frac{\varepsilon|u_\varepsilon'(t)|^2}{2M_3} + \frac{1}{2}\left|A^{1/2}u_\varepsilon(t)\right|^2,$$

from which (3.23) follows with

$$k_4 := \min\left\{\frac{1}{2M_3}, \frac{1}{2}\right\}, \qquad k_5 := \max\left\{\frac{1}{\mu} + \frac{1}{2M_3}, \frac{3}{2} + \frac{\delta}{2\nu}\right\}.$$

Differential Inequality for E_ε The time-derivative of (3.20) is

$$E_\varepsilon'(t) = -\frac{1}{(1+t)^p}\frac{|u_\varepsilon'(t)|^2}{c_\varepsilon(t)}\left(2 + \varepsilon\frac{c_\varepsilon'(t)(1+t)^p}{c_\varepsilon(t)}\right).$$

From (2.24) and (2.26) we have that

$$\varepsilon\frac{|c_\varepsilon'(t)|(1+t)^p}{c_\varepsilon(t)} \leq \frac{M_4}{\mu}\varepsilon,$$

so that $E_\varepsilon'(t) \leq 0$ for every $t \geq 0$, provided that ε is small enough. This proves (3.24).

Differential Inequality for F_ε The time-derivative of (3.21) is

$$F_\varepsilon'(t) = -\frac{1}{(1+t)^p}\frac{|u_\varepsilon'(t)|^2}{c_\varepsilon(t)}\left(2+\varepsilon\frac{c_\varepsilon'(t)(1+t)^p}{c_\varepsilon(t)} - \varepsilon\delta c_\varepsilon(t)\right)$$
$$-\frac{\delta c_\varepsilon(t)}{(1+t)^p}\left|A^{1/2}u_\varepsilon(t)\right|^2 - \delta p\frac{|u_\varepsilon(t)|^2}{(1+t)^{2p+1}} - \frac{\varepsilon\delta p}{(1+t)^{1+p}}\langle u_\varepsilon'(t), u_\varepsilon(t)\rangle$$

Therefore (3.25) holds true if and only if

$$\begin{aligned}
&\frac{|u_\varepsilon'(t)|^2}{c_\varepsilon(t)}\left(2+\varepsilon\frac{c_\varepsilon'(t)(1+t)^p}{c_\varepsilon(t)} - \varepsilon\delta c_\varepsilon(t) - \varepsilon\beta\right)\\
&+\left(\delta c_\varepsilon(t)-\beta\right)\left|A^{1/2}u_\varepsilon(t)\right|^2\\
&+\left(\frac{\delta p}{(1+t)^{1+p}} - \frac{\delta\beta}{2}\frac{1}{(1+t)^{2p}}\right)\left|u_\varepsilon(t)\right|^2\\
&+\left(\frac{\varepsilon\delta p}{1+t} - \frac{\varepsilon\delta\beta}{(1+t)^p}\right)\langle u_\varepsilon'(t), u_\varepsilon(t)\rangle \geq 0
\end{aligned}\tag{3.32}$$

holds true for every $t \geq T$, and every ε small enough.

Let $S_1, \ldots, S_4$ denote the four terms in (3.32). Due to (2.24) through (2.26), for every small enough ε we have that

$$\varepsilon\frac{|c_\varepsilon'(t)|(1+t)^p}{c_\varepsilon(t)} \leq \frac{M_4}{\mu}\varepsilon \leq \frac{1}{3}, \qquad \varepsilon\delta c_\varepsilon(t) \leq \varepsilon\delta M_3 \leq \frac{1}{3}, \qquad \varepsilon\beta \leq \frac{1}{3},$$

hence

$$S_1 \geq \frac{|u_\varepsilon'(t)|^2}{c_\varepsilon(t)} \geq \frac{1}{M_3}\left|u_\varepsilon'(t)\right|^2. \tag{3.33}$$

Since $\delta\mu \geq \beta$, from (1.6) we have that

$$\begin{aligned}
S_2 + S_3 &\geq (\delta\mu - \beta)\left|A^{1/2}u_\varepsilon(t)\right|^2 - \frac{\delta\beta}{2}\frac{1}{(1+t)^{2p}}\left|u_\varepsilon(t)\right|^2\\
&\geq \left[(\delta\mu-\beta)\nu - \frac{\delta\beta}{2}\frac{1}{(1+T)^{2p}}\right]\left|u_\varepsilon(t)\right|^2
\end{aligned}$$

for every $t \geq T$. Due to the choices (3.18) and (3.19), in both cases the term in brackets is greater than or equal to ν, hence $S_2 + S_3 \geq \nu|u_\varepsilon(t)|^2$ for every $t \geq T$. Adding this inequality to (3.33), and applying the inequality between arithmetic and geometric mean, we deduce that

$$S_1 + S_2 + S_3 \geq \frac{1}{M_3}\left|u_\varepsilon'(t)\right|^2 + \nu\left|u_\varepsilon(t)\right|^2 \geq 2\sqrt{\frac{\nu}{M_3}}\cdot\left|u_\varepsilon'(t)\right|\cdot\left|u_\varepsilon(t)\right|.$$

As a consequence, if ε is small enough and $t \geq T$, we have that

$$S_1 + S_2 + S_3 \geq \varepsilon\delta(1+\beta)\left|u_\varepsilon'(t)\right| \cdot \left|u_\varepsilon(t)\right| \geq \left(\frac{\varepsilon\delta p}{1+t} + \frac{\varepsilon\delta\beta}{(1+t)^p}\right)\left|u_\varepsilon'(t)\right| \cdot \left|u_\varepsilon(t)\right| \geq |S_4|,$$

which proves (3.32), hence also (3.25).

Differential Inequality for G_ε The time-derivative of (3.29) is

$$G_\varepsilon'(t) = -\frac{2}{\varepsilon}\frac{1}{(1+t)^p}\left|u_\varepsilon'(t)\right|^2 - \frac{2}{\varepsilon}c_\varepsilon(t)\left\langle Au_\varepsilon(t), u_\varepsilon'(t)\right\rangle.$$

From (2.25) we have that

$$-2c_\varepsilon(t)\left\langle Au_\varepsilon(t), u_\varepsilon'(t)\right\rangle \leq 2M_3\left|u_\varepsilon'(t)\right| \cdot \left|Au_\varepsilon(t)\right| \leq \frac{|u_\varepsilon'(t)|^2}{(1+t)^p} + M_3^2(1+t)^p\left|Au_\varepsilon(t)\right|^2,$$

hence

$$G_\varepsilon'(t) \leq -\frac{1}{\varepsilon}\frac{1}{(1+t)^p}G_\varepsilon(t) + \frac{M_3^2}{\varepsilon}(1+t)^p\left|Au_\varepsilon(t)\right|^2.$$

At this point (3.30) follows from (3.28).

The proof of Theorem 2.9 is thus complete.

5.3.4 Singular Perturbation: Preliminary Estimates

In this subsection we begin the analysis of the singular perturbation problem in the linear setting. If we set

$$g_\varepsilon(t) := -\big(c_\varepsilon(t) - c(t)\big)Au(t) - \varepsilon u''(t), \tag{3.34}$$

we have that $r_\varepsilon(t)$ and $\rho_\varepsilon(t)$ satisfy

$$\varepsilon r_\varepsilon''(t) + \frac{1}{(1+t)^p}r_\varepsilon'(t) + c_\varepsilon(t)A\rho_\varepsilon(t) = g_\varepsilon(t), \tag{3.35}$$

and

$$\rho_\varepsilon(0) = 0, \qquad r_\varepsilon'(0) = 0.$$

In the next two results we prove estimates on $g_\varepsilon(t)$ and on the corrector $\theta_\varepsilon(t)$.

Lemma 3.5 *Let us consider the same assumptions of Theorem 2.10. Let $g_\varepsilon(t)$ be defined according to (3.34). Let $\Phi_{\beta,p}(t)$ be the solution of the Cauchy problem (3.8), (3.9), with $\beta > 0$ if $p > 0$, and $0 < \beta < 2\mu\nu$ if $p = 0$.*

Then we have the following estimates.

(1) *If $u_0 \in D(A^{3/2})$, then there exists a constant C such that*

$$\int_0^{+\infty} \frac{(1+t)^p}{\Phi_{\beta,p}(t)} \cdot \left|g_\varepsilon(t)\right|^2 dt \le C\varepsilon^2 \quad \forall \varepsilon \in (0,\varepsilon_0). \tag{3.36}$$

(2) *If in addition we have that $u_0 \in D(A^2)$, then there exists a constant C such that*

$$\int_0^{+\infty} \frac{(1+t)^p}{\Phi_{\beta,p}(t)} \cdot \left|A^{1/2} g_\varepsilon(t)\right|^2 dt \le C\varepsilon^2 \quad \forall \varepsilon \in (0,\varepsilon_0), \tag{3.37}$$

$$\left|g_\varepsilon(t)\right|^2 \le C\varepsilon^2 \Phi_{\beta,p}(t) \quad \forall t \ge 0, \forall \varepsilon \in (0,\varepsilon_0). \tag{3.38}$$

Proof From (3.34) and (2.27) we have that

$$\left|g_\varepsilon(t)\right|^2 \le k_1\varepsilon^2\left|Au(t)\right|^2 + 2\varepsilon^2\left|u''(t)\right|^2.$$

We can estimate $|Au(t)|^2$ and $|u''(t)|^2$, or their integrals, by means of Proposition 3.1. To this end, let us consider the function $\Psi_{\alpha,p}(t)$ defined in (3.1). We claim that, for every admissible value of p and β, there exists $\alpha > 0$ for which Proposition 3.1 applies, and such that

$$\frac{(1+t)^p}{\Phi_{\beta,p}(t)} \le \frac{k_2}{\Psi_{\alpha,p}(t)} \quad \forall t \ge 0. \tag{3.39}$$

Indeed it is enough to take $\alpha = \beta$ if $p = 0$ (in which case there is basically nothing to prove), and any $\alpha \in (0,\gamma)$ if $p > 0$ (because in this case $\Psi_{\alpha,p}(t)$ has an exponential decay rate which is faster than the decay rate of $\Phi_{\beta,p}(t)$). Thus we have that

$$\int_0^{+\infty} \frac{(1+t)^p}{\Phi_{\beta,p}(t)} \cdot \left|g_\varepsilon(t)\right|^2 dt \le k_3\varepsilon^2\left(\int_0^{+\infty} \frac{|Au(t)|^2}{\Psi_{\alpha,p}(t)}dt + \int_0^{+\infty} \frac{|u''(t)|^2}{\Psi_{\alpha,p}(t)}dt\right),$$

so that (3.36) follows from (3.3) with $k = 1$, and (3.4).

The proof of (3.37) is analogous: we just exploit (3.3) with $k = 2$, and (3.6) instead of (3.4).

It remains to prove (3.38). Let γ be the constant defined in (3.1). Then, in analogy with (3.39), we have that

$$\frac{(1+t)^{4p}}{\Phi_{\beta,p}(t)} \le \frac{k_4}{\Psi_{\gamma,p}(t)} \quad \forall t \ge 0,$$

hence

$$\begin{aligned}\frac{|g_\varepsilon(t)|^2}{\Phi_{\beta,p}(t)} &= \frac{(1+t)^{4p}}{\Phi_{\beta,p}(t)} \cdot \left|g_\varepsilon(t)\right|^2 \cdot \frac{1}{(1+t)^{4p}} \le k_4\frac{|g_\varepsilon(t)|^2}{\Psi_{\gamma,p}(t)} \cdot \frac{1}{(1+t)^{4p}} \\ &\le k_5\varepsilon^2\frac{|Au(t)|^2}{\Psi_{\gamma,p}(t)} + k_6\varepsilon^2\frac{|u''(t)|^2}{\Psi_{\gamma,p}(t)} \cdot \frac{1}{(1+t)^{4p}}.\end{aligned}$$

At this point (3.38) follows from (3.2) with $k = 2$, and (3.5). □

Lemma 3.6 *Let us consider the same assumptions of Theorem 2.10. Let $\theta_\varepsilon(t)$ be the solution of the Cauchy problem (2.11), (2.28). Let $\Phi_{\beta,p}(t)$ be the solution of the Cauchy problem (3.8), (3.9).*

Let us assume that $4\varepsilon_0 \leq 1$, $2\varepsilon_0\beta \leq 1$, and that $\theta_0 \in D(A^{(k+1)/2})$ for some $k \in \mathbb{N}$. Then there exists a constant C such that for every $\varepsilon \in (0, \varepsilon_0)$ we have that

$$\int_0^{+\infty} \frac{1}{\Phi_{\beta,p}(t)} \cdot \left(\left|A^{k/2}\theta'_\varepsilon(t)\right| + \left|A^{k/2}\theta'_\varepsilon(t)\right|^2 + \left|A^{(k+1)/2}\theta'_\varepsilon(t)\right|\right) dt \leq C\varepsilon. \quad (3.40)$$

Proof Let $z_\varepsilon(t)$ be the solution of equation

$$\varepsilon z'_\varepsilon(t) + \frac{1}{(1+t)^p} z_\varepsilon(t) = 0 \quad \forall t \geq 0, \quad (3.41)$$

with initial condition $z_\varepsilon(0) = 1$. It is easy to see that $\theta'_\varepsilon(t) = \theta_0 z_\varepsilon(t)$.

Since $0 \leq z_\varepsilon(t) \leq 1$ for every $t \geq 0$, we have also that $z_\varepsilon^2(t) \leq z_\varepsilon(t)$. Therefore, (3.40) is proved if we show that

$$\int_0^{+\infty} \frac{z_\varepsilon(t)}{\Phi_{\beta,p}(t)} dt \leq 4\varepsilon. \quad (3.42)$$

Let us set $w_\varepsilon(t) := z_\varepsilon(t) \cdot [\Phi_{\beta,p}(t)]^{-1}$. From (3.41) and (3.8), it turns out that $w_\varepsilon(t)$ is the solution of the ordinary differential equation

$$w'_\varepsilon(t) = -\left(\frac{1}{\varepsilon} - \beta\right) \frac{1}{(1+t)^p} w_\varepsilon(t) \quad \forall t \geq 0, \quad (3.43)$$

with initial datum $w_\varepsilon(0) = 1$. On the other hand, when $2\varepsilon\beta \leq 1$, it is easy to show that $y_\varepsilon(t) := (1+t)^{-1/(2\varepsilon)}$ is a supersolution of (3.43). Indeed we have that

$$y'_\varepsilon(t) = -\frac{1}{2\varepsilon}\frac{y_\varepsilon(t)}{1+t} \geq -\frac{1}{2\varepsilon}\frac{y_\varepsilon(t)}{(1+t)^p} \geq -\left(\frac{1}{\varepsilon} - \beta\right)\frac{y_\varepsilon(t)}{(1+t)^p}.$$

Since $y_\varepsilon(0) = w_\varepsilon(0)$, the standard comparison principle gives that $w_\varepsilon(t) \leq y_\varepsilon(t)$ for every $t \geq 0$. Since $4\varepsilon \leq 1$, it follows that

$$\int_0^{+\infty} w_\varepsilon(t) dt \leq \int_0^{+\infty} \frac{1}{(1+t)^{1/(2\varepsilon)}} dt = \frac{2\varepsilon}{1 - 2\varepsilon} \leq 4\varepsilon.$$

This completes the proof of (3.42), hence also the proof of (3.40). □

5.3.5 Proof of Theorem 2.10

Let us describe the strategy of the proof, which is similar to Theorem 2.9. Let us take any admissible value β, which means any $\beta \in (0, 2\mu\nu)$ if $p = 0$, and any $\beta > 0$ if $p > 0$. Let $\Phi_{\beta,p}(t)$ be the solution of the Cauchy problem (3.8), (3.9).

The conclusions of statement (1) of Theorem 2.10 are equivalent to showing that

$$\Gamma_{r,\varepsilon}(t) \leq k_1 \Phi_{\beta,p}(t) \quad \forall t \geq 0 \tag{3.44}$$

for the admissible values of β.

Let μ be the constant in (2.24), and let us choose δ, σ, T in such a way that

$$\delta := \frac{4(\beta+1)\nu}{2\mu\nu - \beta}, \qquad \sigma := \mu\nu - \frac{\beta}{2}, \qquad T := 0 \tag{3.45}$$

if $p = 0$ (note that $\delta > 0$), and

$$\delta := \frac{\beta+2}{\mu}, \qquad \sigma := 1, \qquad (1+T)^{2p} \geq \frac{\delta}{2\nu}(\beta+\sigma) \tag{3.46}$$

if $p > 0$.

For every $\varepsilon \in (0, \varepsilon_0)$, we consider the energies

$$\mathcal{E}_\varepsilon(t) := \frac{\varepsilon |r'_\varepsilon(t)|^2}{c_\varepsilon(t)} + \left|A^{1/2}\rho_\varepsilon(t)\right|^2, \tag{3.47}$$

$$\mathcal{F}_\varepsilon(t) := \frac{\varepsilon |r'_\varepsilon(t)|^2}{c_\varepsilon(t)} + \left|A^{1/2}\rho_\varepsilon(t)\right|^2 + \frac{\varepsilon\delta}{(1+t)^p}\langle r'_\varepsilon(t), \rho_\varepsilon(t)\rangle + \frac{\delta}{2}\frac{1}{(1+t)^{2p}}\left|\rho_\varepsilon(t)\right|^2. \tag{3.48}$$

The arguments used in the proof of (3.22) and (3.23) can be adapted word-by-word to the energies $\mathcal{E}_\varepsilon(t)$ and $\mathcal{F}_\varepsilon(t)$. We obtain that there exist positive constants $k_2, \ldots, k_5$ such that

$$k_2\left(\varepsilon\left|r'_\varepsilon(t)\right|^2 + \left|A^{1/2}\rho_\varepsilon(t)\right|^2\right) \leq \mathcal{E}_\varepsilon(t) \leq k_3\left(\varepsilon\left|r'_\varepsilon(t)\right|^2 + \left|A^{1/2}\rho_\varepsilon(t)\right|^2\right), \tag{3.49}$$

$$k_4\left(\varepsilon\left|r'_\varepsilon(t)\right|^2 + \left|A^{1/2}\rho_\varepsilon(t)\right|^2\right) \leq \mathcal{F}_\varepsilon(t) \leq k_5\left(\varepsilon\left|r'_\varepsilon(t)\right|^2 + \left|A^{1/2}\rho_\varepsilon(t)\right|^2\right) \tag{3.50}$$

for every $t \geq 0$, provided that ε is small enough.

Moreover, we claim that there exists $\varepsilon_2 \in (0, \varepsilon_0)$ such that

$$\mathcal{E}'_\varepsilon(t) \leq \psi_{1,\varepsilon}(t)\sqrt{\mathcal{E}_\varepsilon(t)} + \psi_{2,\varepsilon}(t) \quad \forall t \geq 0, \forall \varepsilon \in (0, \varepsilon_2), \tag{3.51}$$

$$\mathcal{F}'_\varepsilon(t) \leq -\frac{\beta}{(1+t)^p}\mathcal{F}_\varepsilon(t) + \psi_{3,\varepsilon}(t) \quad \forall t \geq T, \forall \varepsilon \in (0, \varepsilon_2), \tag{3.52}$$

where the functions $\psi_{i,\varepsilon}(t)$ (with $i = 1, 2, 3$) are nonnegative continuous functions depending on $A^{1/2}\theta'_\varepsilon(t)$ and $g_\varepsilon(t)$, and such that

$$\int_0^{+\infty} \psi_{1,\varepsilon}(t)dt \leq k_6\varepsilon, \qquad \int_0^{+\infty} \psi_{2,\varepsilon}(t)dt \leq k_7\varepsilon^2, \tag{3.53}$$

$$\int_0^{+\infty} \frac{\psi_{3,\varepsilon}(t)}{\Phi_{\beta,p}(t)}dt \leq k_8\varepsilon^2. \tag{3.54}$$

Let us assume that we have proved these claims. Thanks to (3.51) and (3.53), we can apply Lemma 3.3 to the function $\mathcal{E}_\varepsilon(t)$ (note that now $\mathcal{E}_\varepsilon(0) = 0$). We obtain that

$$\mathcal{E}_\varepsilon(t) \leq k_9\varepsilon^2 \quad \forall t \geq 0. \tag{3.55}$$

Due to the estimate from below in (3.49), this implies that

$$\varepsilon\left|r'_\varepsilon(t)\right|^2 + \left|A^{1/2}\rho_\varepsilon(t)\right|^2 \leq \frac{1}{k_2}\mathcal{E}_\varepsilon(t) \leq k_{10}\varepsilon^2$$

for every $t \geq 0$. Since $\Phi_{\beta,p}(t)$ is decreasing, we can conclude that

$$\varepsilon\left|r'_\varepsilon(t)\right|^2 + \left|A^{1/2}\rho_\varepsilon(t)\right|^2 \leq \frac{k_{10}\varepsilon^2}{\Phi_{\beta,p}(T)} \cdot \Phi_{\beta,p}(t) = k_{11}\varepsilon^2\Phi_{\beta,p}(t) \quad \forall t \in [0, T]. \tag{3.56}$$

For $t \geq T$, we exploit (3.52). First of all, from (3.56) with $t = T$, and the estimate from above in (3.50), we have that

$$\mathcal{F}_\varepsilon(T) \leq k_5\left(\varepsilon\left|r'_\varepsilon(T)\right|^2 + \left|A^{1/2}\rho_\varepsilon(T)\right|^2\right) \leq k_{12}\varepsilon^2\Phi_{\beta,p}(T).$$

Due to (3.52) and (3.54), we can apply Lemma 3.4 to the function $\mathcal{F}_\varepsilon(t)$. We obtain that $\mathcal{F}_\varepsilon(t) \leq k_{13}\varepsilon^2\Phi_{\beta,p}(t)$ for every $t \geq T$. Exploiting this inequality, the estimate from below in (3.50), and (3.56), we conclude that

$$\varepsilon\left|r'_\varepsilon(t)\right|^2 + \left|A^{1/2}\rho_\varepsilon(t)\right|^2 \leq k_{14}\varepsilon^2\Phi_{\beta,p}(t) \quad \forall t \geq 0, \forall \varepsilon \in (0, \varepsilon_2).$$

Since the operator is coercive, this estimate on $|A^{1/2}\rho_\varepsilon(t)|^2$ yields an analogous estimate on $|\rho_\varepsilon(t)|^2$. This completes the proof of (3.44), hence of statement (1), for initial data $(u_0, u_1) \in D(A^{3/2}) \times D(A^{1/2})$, the regularity of data being required in the verification of (3.53) and (3.54).

Let us proceed now to statement (2), where it is assumed that $(u_0, u_1) \in D(A^2) \times D(A)$, and it is required to prove in addition that

$$\varepsilon\left|A^{1/2}r'_\varepsilon(t)\right|^2 + \left|A\rho_\varepsilon(t)\right|^2 + \left|r'_\varepsilon(t)\right|^2 \leq k_{15}\varepsilon^2\Phi_{\beta,p}(t) \quad \forall t \geq 0, \forall \varepsilon \in (0, \varepsilon_1)$$

for some $\varepsilon_1 \in (0, \varepsilon_2]$. Due to the linearity of (3.35), an analogous identity holds true with $A^{1/2}\rho_\varepsilon(t)$, $A^{1/2}r_\varepsilon(t)$, and $A^{1/2}g_\varepsilon(t)$ instead of $\rho_\varepsilon(t)$, $r_\varepsilon(t)$, and $g_\varepsilon(t)$, respectively. So we can repeat the arguments used so far, paying attention to verifying

(3.53) and (3.54) also for the new functions $\psi_{\varepsilon,i}(t)$, which now depend on $A\theta'_\varepsilon(t)$ and $A^{1/2}g_\varepsilon(t)$. We end up with

$$\varepsilon\left|A^{1/2}r'_\varepsilon(t)\right|^2+\left|A\rho_\varepsilon(t)\right|^2\leq k_{16}\varepsilon^2\Phi_{\beta,p}(t)\quad\forall t\geq 0,\forall\varepsilon\in(0,\varepsilon_2). \tag{3.57}$$

It remains to prove the ε-independent estimate on $r'_\varepsilon(t)$. To this end, we set

$$\mathcal{G}_\varepsilon(t):=\left|r'_\varepsilon(t)\right|^2, \tag{3.58}$$

and we claim that

$$\mathcal{G}'_\varepsilon(t)\leq-\frac{1}{\varepsilon}\frac{1}{(1+t)^p}\mathcal{G}_\varepsilon(t)+\frac{1}{\varepsilon}(1+t)^p\cdot k_{17}\varepsilon^2\Phi_{\beta,p}(t)\quad\forall t\geq 0,\forall\varepsilon\in(0,\varepsilon_2). \tag{3.59}$$

If we prove the claim, then from Lemma 3.2 it follows that (note that now $\mathcal{G}_\varepsilon(0)=0$)

$$\left|r'_\varepsilon(t)\right|^2=\mathcal{G}_\varepsilon(t)\leq k_{18}\varepsilon^2(1+t)^{2p}\Phi_{\beta,p}(t)\quad\forall t\geq 0,\forall\varepsilon\in(0,\varepsilon_2).$$

Finally, when $p>0$, we can get free of the factor $(1+t)^{2p}$ exactly as in the proof of Theorem 2.9, possibly changing ε_2 with some smaller ε_1.

So we are left to proving (3.51) through (3.54), both in the case of initial data $(u_0,u_1)\in D(A^{3/2})\times D(A^{1/2})$, and in the case $(u_0,u_1)\in D(A^2)\times D(A)$, and (3.59) in the second case.

Differential Inequality for $\mathcal{E}_\varepsilon$ The time-derivative of (3.47) is

$$\mathcal{E}'_\varepsilon(t)=-\frac{1}{(1+t)^p}\frac{|r'_\varepsilon(t)|^2}{c_\varepsilon(t)}\left(2+\varepsilon\frac{c'_\varepsilon(t)(1+t)^p}{c_\varepsilon(t)}\right)+\frac{2}{c_\varepsilon(t)}\left\langle r'_\varepsilon(t),g_\varepsilon(t)\right\rangle+2\left\langle A\rho_\varepsilon(t),\theta'_\varepsilon(t)\right\rangle. \tag{3.60}$$

By standard inequalities we have that

$$2\left\langle A\rho_\varepsilon(t),\theta'_\varepsilon(t)\right\rangle\leq 2\left|A^{1/2}\theta'_\varepsilon(t)\right|\cdot\left|A^{1/2}\rho_\varepsilon(t)\right|\leq 2\left|A^{1/2}\theta'_\varepsilon(t)\right|\sqrt{\mathcal{E}_\varepsilon(t)},$$

$$\frac{2}{c_\varepsilon(t)}\left\langle r'_\varepsilon(t),g_\varepsilon(t)\right\rangle\leq\frac{1}{(1+t)^p}\frac{|r'_\varepsilon(t)|^2}{c_\varepsilon(t)}+\frac{1}{c_\varepsilon(t)}(1+t)^p\left|g_\varepsilon(t)\right|^2.$$

Plugging these estimates into (3.60), when ε is small enough we obtain that

$$\mathcal{E}'_\varepsilon(t)\leq 2\left|A^{1/2}\theta'_\varepsilon(t)\right|\sqrt{\mathcal{E}_\varepsilon(t)}+\frac{1}{\mu}(1+t)^p\left|g_\varepsilon(t)\right|^2,$$

which is exactly (3.51) with

$$\psi_{1,\varepsilon}(t):=2\left|A^{1/2}\theta'_\varepsilon(t)\right|,\qquad\psi_{2,\varepsilon}(t):=\frac{1}{\mu}(1+t)^p\left|g_\varepsilon(t)\right|^2.$$

When $(u_0, u_1) \in D(A^{3/2}) \times D(A^{1/2})$, we have that $\theta_0 \in D(A^{1/2})$, hence (3.53) follows from (3.40) with $k = 0$, and (3.36).

When $(u_0, u_1) \in D(A^2) \times D(A)$, we have that $\theta_0 \in D(A)$, and we need (3.53) with $\psi_{1,\varepsilon}(t) := 2|A\theta_\varepsilon'(t)|$, and $\psi_{2,\varepsilon}(t) := \mu^{-1}(1+t)^p|A^{1/2}g_\varepsilon(t)|^2$. Due to the regularity of θ_0, estimate (3.53) follows in this case from (3.40) with $k = 1$, and (3.37).

Differential Inequality for $\mathcal{F}_\varepsilon$ The time-derivative of (3.48) is

$$\begin{aligned}\mathcal{F}_\varepsilon'(t) = &-\frac{1}{(1+t)^p}\frac{|r_\varepsilon'(t)|^2}{c_\varepsilon(t)}\left(2+\varepsilon\frac{c_\varepsilon'(t)(1+t)^p}{c_\varepsilon(t)}-\varepsilon\delta c_\varepsilon(t)\right)\\ &-\frac{\delta c_\varepsilon(t)}{(1+t)^p}\left|A^{1/2}\rho_\varepsilon(t)\right|^2-\delta p\frac{|\rho_\varepsilon(t)|^2}{(1+t)^{2p+1}}-\frac{\varepsilon\delta p}{(1+t)^{1+p}}\left\langle r_\varepsilon'(t),\rho_\varepsilon(t)\right\rangle\\ &+\frac{\varepsilon\delta}{(1+t)^p}\left\langle r_\varepsilon'(t),\theta_\varepsilon'(t)\right\rangle+2\left\langle A^{1/2}\rho_\varepsilon(t),A^{1/2}\theta_\varepsilon'(t)\right\rangle+\delta\frac{\langle\rho_\varepsilon(t),\theta_\varepsilon'(t)\rangle}{(1+t)^{2p}}\\ &+\frac{2}{c_\varepsilon(t)}\left\langle r_\varepsilon'(t),g_\varepsilon(t)\right\rangle+\frac{\delta}{(1+t)^p}\left\langle\rho_\varepsilon(t),g_\varepsilon(t)\right\rangle\\ = &\, I_1+\ldots+I_9.\end{aligned} \tag{3.61}$$

Let us estimate some of the terms. Clearly we have that $I_3 \le 0$. From (3.55) we have that

$$I_6 \le 2\left|A^{1/2}\rho_\varepsilon(t)\right|\cdot\left|A^{1/2}\theta_\varepsilon'(t)\right| \le k_{19}\varepsilon\left|A^{1/2}\theta_\varepsilon'(t)\right|,$$

$$I_7 \le \frac{\delta}{(1+t)^{2p}}\left|\rho_\varepsilon(t)\right|\cdot\left|\theta_\varepsilon'(t)\right| \le \frac{\delta}{(1+t)^{2p}}\frac{1}{\sqrt{\nu}}\left|A^{1/2}\rho_\varepsilon(t)\right|\cdot\left|\theta_\varepsilon'(t)\right| \le k_{20}\varepsilon\left|\theta_\varepsilon'(t)\right|.$$

From standard inequalities we have that

$$I_5 \le \frac{\varepsilon\delta}{(1+t)^p}\left|r_\varepsilon'(t)\right|\cdot\left|\theta_\varepsilon'(t)\right| \le \frac{\varepsilon\delta}{2}\frac{1}{(1+t)^p}\frac{|r_\varepsilon'(t)|^2}{c_\varepsilon(t)}+\frac{\varepsilon\delta}{2}\frac{c_\varepsilon(t)}{(1+t)^p}\left|\theta_\varepsilon'(t)\right|^2,$$

$$I_8 \le \frac{2}{c_\varepsilon(t)}\left|r_\varepsilon'(t)\right|\cdot\left|g_\varepsilon(t)\right| \le \frac{1}{2}\frac{1}{(1+t)^p}\frac{|r_\varepsilon'(t)|^2}{c_\varepsilon(t)}+\frac{2}{c_\varepsilon(t)}(1+t)^p\left|g_\varepsilon(t)\right|^2,$$

$$I_9 \le \frac{\delta}{(1+t)^p}\left|\rho_\varepsilon(t)\right|\cdot\left|g_\varepsilon(t)\right| \le \frac{\delta\sigma}{2}\frac{1}{(1+t)^{3p}}\left|\rho_\varepsilon(t)\right|^2+\frac{\delta}{2\sigma}(1+t)^p\left|g_\varepsilon(t)\right|^2.$$

Plugging all these estimates into (3.61), and recalling once more assumptions (2.24) through (2.26), we obtain that

$$\begin{aligned}\mathcal{F}_\varepsilon'(t) \le &-\frac{1}{(1+t)^p}\frac{|r_\varepsilon'(t)|^2}{c_\varepsilon(t)}\left(\frac{3}{2}+\varepsilon\frac{c_\varepsilon'(t)(1+t)^p}{c_\varepsilon(t)}-\varepsilon\delta c_\varepsilon(t)-\frac{\varepsilon\delta}{2}\right)\\ &-\frac{\delta c_\varepsilon(t)}{(1+t)^p}\left|A^{1/2}\rho_\varepsilon(t)\right|^2+\frac{\delta\sigma}{2}\frac{1}{(1+t)^{3p}}\left|\rho_\varepsilon(t)\right|^2\end{aligned}$$

$$- \frac{\varepsilon\delta p}{(1+t)^{1+p}}\langle r_\varepsilon'(t), \rho_\varepsilon(t)\rangle$$
$$+ k_{21}\varepsilon\left(\left|\theta_\varepsilon'(t)\right| + \left|\theta_\varepsilon'(t)\right|^2 + \left|A^{1/2}\theta_\varepsilon'(t)\right|\right) + k_{22}(1+t)^p\left|g_\varepsilon(t)\right|^2. \tag{3.62}$$

Let $\psi_{3,\varepsilon}(t)$ denote the sum of the two terms of the last line. Then (3.52) is proved if we show that the sum of the terms in the first two lines is less than or equal to $-\beta(1+t)^{-p}\mathcal{F}_\varepsilon(t)$ for every $t \geq T$. In turn, this is equivalent to showing that

$$\frac{|r_\varepsilon'(t)|^2}{c_\varepsilon(t)}\left(\frac{3}{2} + \varepsilon\frac{c_\varepsilon'(t)(1+t)^p}{c_\varepsilon(t)} - \varepsilon\delta c_\varepsilon(t) - \frac{\varepsilon\delta}{2} - \varepsilon\beta\right) + \left(\delta c_\varepsilon(t) - \beta\right)\left|A^{1/2}\rho_\varepsilon(t)\right|^2$$
$$- \frac{\delta(\sigma+\beta)}{2}\frac{|\rho_\varepsilon(t)|^2}{(1+t)^{2p}} + \left(\frac{\varepsilon\delta p}{1+t} - \frac{\varepsilon\delta\beta}{(1+t)^p}\right)\langle r_\varepsilon'(t), \rho_\varepsilon(t)\rangle \geq 0 \tag{3.63}$$

holds true for every $t \geq T$.

Let $S_1, \ldots, S_4$ denote the four terms in (3.63), which we estimate as in the proof of Theorem 2.9. From the smallness of ε we have that

$$S_1 \geq \frac{|r_\varepsilon'(t)|^2}{c_\varepsilon(t)} \geq \frac{1}{M_3}\left|r_\varepsilon'(t)\right|^2. \tag{3.64}$$

Since $\delta\mu \geq \beta$, from (1.6) we have that

$$S_2 + S_3 \geq (\delta\mu - \beta)\left|A^{1/2}\rho_\varepsilon(t)\right|^2 - \frac{\delta(\sigma+\beta)}{2}\frac{1}{(1+t)^{2p}}\left|\rho_\varepsilon(t)\right|^2$$
$$\geq \left[(\delta\mu - \beta)\nu - \frac{\delta(\sigma+\beta)}{2}\frac{1}{(1+T)^{2p}}\right]\left|\rho_\varepsilon(t)\right|^2$$

for every $t \geq T$. Due to the choices (3.45) and (3.46), in both cases the term in brackets is greater than or equal to ν, hence $S_2 + S_3 \geq \nu|\rho_\varepsilon(t)|^2$ for every $t \geq T$. Now we add this inequality to (3.64), and we apply the inequality between arithmetic and geometric mean, as in the proof of Theorem 2.9. If ε is small enough we obtain that

$$S_1 + S_2 + S_3 \geq \frac{1}{M_3}\left|r_\varepsilon'(t)\right|^2 + \nu\left|\rho_\varepsilon(t)\right|^2 \geq 2\sqrt{\frac{\nu}{M_3}}\cdot\left|r_\varepsilon'(t)\right|\cdot\left|\rho_\varepsilon(t)\right|$$
$$\geq \varepsilon\delta(1+\beta)\left|r_\varepsilon'(t)\right|\cdot\left|\rho_\varepsilon(t)\right|$$
$$\geq \left(\frac{\varepsilon\delta p}{1+t} + \frac{\varepsilon\delta\beta}{(1+t)^p}\right)\left|r_\varepsilon'(t)\right|\cdot\left|\rho_\varepsilon(t)\right| \geq |S_4|,$$

which proves (3.63), hence also (3.52). It remains to prove (3.54), with $\psi_{3,\varepsilon}(t)$ equal to the sum of the two terms in the last line of (3.62).

When $(u_0, u_1) \in D(A^{3/2}) \times D(A^{1/2})$, we have that $\theta_0 \in D(A^{1/2})$, hence (3.54) follows from (3.40) with $k = 0$, and (3.36).

When $(u_0, u_1) \in D(A^2) \times D(A)$, we have that $\theta_0 \in D(A)$, and we need (3.54) with

$$\psi_{3,\varepsilon}(t) := k_{23}\varepsilon\left(\left|A^{1/2}\theta'_\varepsilon(t)\right| + \left|A^{1/2}\theta'_\varepsilon(t)\right|^2 + \left|A\theta'_\varepsilon(t)\right|\right) + k_{24}(1+t)^p\left|A^{1/2}g_\varepsilon(t)\right|^2.$$

Due to the regularity of θ_0, estimate (3.54) follows in this case from (3.40) with $k = 1$, and (3.37).

Differential Inequality for $\mathcal{G}_\varepsilon$ The time-derivative of (3.58) is

$$\mathcal{G}'_\varepsilon(t) = -\frac{2}{\varepsilon}\frac{1}{(1+t)^p}\left|r'_\varepsilon(t)\right|^2 - \frac{2}{\varepsilon}c_\varepsilon(t)\left\langle A\rho_\varepsilon(t), r'_\varepsilon(t)\right\rangle + \frac{2}{\varepsilon}\left\langle g_\varepsilon(t), r'_\varepsilon(t)\right\rangle.$$

From standard inequalities we have that

$$-\frac{2}{\varepsilon}c_\varepsilon(t)\left\langle A\rho_\varepsilon(t), r'_\varepsilon(t)\right\rangle \leq \frac{1}{2\varepsilon}\frac{1}{(1+t)^p}\left|r'_\varepsilon(t)\right|^2 + \frac{k_{25}}{\varepsilon}(1+t)^p\left|A\rho_\varepsilon(t)\right|^2,$$

$$\frac{2}{\varepsilon}\left\langle g_\varepsilon(t), r'_\varepsilon(t)\right\rangle \leq \frac{1}{2\varepsilon}\frac{1}{(1+t)^p}\left|r'_\varepsilon(t)\right|^2 + \frac{2}{\varepsilon}(1+t)^p\left|g_\varepsilon(t)\right|^2,$$

hence

$$\mathcal{G}'_\varepsilon(t) \leq -\frac{1}{\varepsilon}\frac{1}{(1+t)^p}\left|r'_\varepsilon(t)\right|^2 + \frac{k_{25}}{\varepsilon}(1+t)^p\left|A\rho_\varepsilon(t)\right|^2 + \frac{2}{\varepsilon}(1+t)^p\left|g_\varepsilon(t)\right|^2.$$

At this point (3.59) follows from (3.57) and (3.38). This completes the proof of Theorem 2.10.

5.3.6 *Proof of Theorem 2.11*

Let us set

$$H_\varepsilon(t) := \left(\varepsilon\frac{|u'_\varepsilon(t)|^2}{c_\varepsilon(t)} + \left|A^{1/2}u_\varepsilon(t)\right|^2\right)\frac{1}{\Phi(t)} \quad \forall t \geq 0.$$

Due to (2.24) and (2.25), proving (2.16) is equivalent to showing that $H_\varepsilon(t) \to +\infty$ as $t \to +\infty$. Since $(u_0, u_1) \neq (0, 0)$, the solution is nontrivial in the sense that $H_\varepsilon(t) > 0$ for every $t \geq 0$. Moreover we have that

$$H'_\varepsilon(t) = \frac{1}{(1+t)^p}\frac{1}{\Phi(t)}\frac{\varepsilon|u'_\varepsilon(t)|^2}{c_\varepsilon(t)}\left(-\frac{\Phi'(t)}{\Phi(t)}(1+t)^p - \frac{2}{\varepsilon} - \frac{c'_\varepsilon(t)(1+t)^p}{c_\varepsilon(t)}\right)$$
$$+ \frac{1}{(1+t)^p}\frac{1}{\Phi(t)}\left|A^{1/2}u_\varepsilon(t)\right|^2\left(-\frac{\Phi'(t)}{\Phi(t)}(1+t)^p\right).$$

As usual, we have that

$$\frac{|c_\varepsilon'(t)|(1+t)^p}{c_\varepsilon(t)} \leq \frac{M_4}{\mu}.$$

Therefore, assumption (2.15) implies the existence of $T > 0$ (depending on ε, but this is not important) such that

$$-\frac{\Phi'(t)}{\Phi(t)}(1+t)^p - \frac{2}{\varepsilon} - \frac{c_\varepsilon'(t)(1+t)^p}{c_\varepsilon(t)} \geq 1 \quad \text{and} \quad -\frac{\Phi'(t)}{\Phi(t)}(1+t)^p \geq 1$$

for every $t \geq T$, hence

$$H_\varepsilon'(t) \geq \frac{1}{(1+t)^p} H_\varepsilon(t) \quad \forall t \geq T.$$

Since $H_\varepsilon(T) > 0$, and $p \leq 1$, this differential inequality implies that $H_\varepsilon(t) \to +\infty$ as $t \to +\infty$.

5.3.7 Proof of Theorems 2.1, 2.2, 2.3, 2.4

The existence of solutions to (1.3), (1.4), and (1.1), (1.2) follows from Theorem A. Let us set now

$$c(t) := m\left(\left|A^{1/2}u(t)\right|^2\right), \qquad c_\varepsilon(t) := m\left(\left|A^{1/2}u_\varepsilon(t)\right|^2\right).$$

With a standard approximation procedure, we can assume that $m(\sigma)$ is of class C^1, and not just locally Lipschitz continuous. As a consequence, also $c(t)$ and $c_\varepsilon(t)$ are of class C^1. If we show that $c(t)$ and $c_\varepsilon(t)$ satisfy (2.21) through (2.27), then all conclusions of Theorems 2.1 through 2.4 follow from the corresponding conclusions of Theorems 2.8 through 2.11.

Assumptions (2.21) and (2.24) follow from (1.5).

Assumptions (2.22) and (2.25) follow from the fact that both $|A^{1/2}u(t)|^2$ and $|A^{1/2}u_\varepsilon(t)|^2$ are bounded because of (2.2) and (2.4), respectively.

Since

$$c'(t) = 2m'\left(\left|A^{1/2}u(t)\right|^2\right)\langle Au(t), u'(t)\rangle,$$

assumption (2.23) follows from the boundedness of $|u'(t)|$, $|A^{1/2}u(t)|$, and $|Au(t)|$, resulting from (2.2).

Similarly, we have that

$$c_\varepsilon'(t) = 2m'\left(\left|A^{1/2}u_\varepsilon(t)\right|^2\right)\langle Au_\varepsilon(t), u_\varepsilon'(t)\rangle,$$

and therefore estimate (2.4) implies that

$$\left|c_\varepsilon'(t)\right| \leq k_1\left|Au_\varepsilon(t)\right| \cdot \left|u_\varepsilon'(t)\right| \leq k_2 \frac{1}{(1+t)^{1+p}} \cdot \frac{1}{1+t} \leq \frac{k_2}{(1+t)^p},$$

which is exactly (2.26).

It remains to prove (2.27). To this end, we first remark that

$$\begin{aligned}\left|\left|A^{1/2}u_\varepsilon(t)\right|^2 - \left|A^{1/2}u(t)\right|^2\right| &= \left|\left\langle A^{1/2}\big(u_\varepsilon(t)+u(t)\big), A^{1/2}\big(u_\varepsilon(t)-u(t)\big)\right\rangle\right| \\ &\leq \left(\left|A^{1/2}u_\varepsilon(t)\right| + \left|A^{1/2}u(t)\right|\right)\cdot\left|A^{1/2}\rho_\varepsilon(t)\right|.\end{aligned}$$

Now $|A^{1/2}u_\varepsilon(t)|$ and $|A^{1/2}u(t)|$ are bounded because of (2.2) and (2.4), and $|A^{1/2}\rho_\varepsilon(t)|$ can be estimated by means of (2.5). Since $m(\sigma)$ is (locally) Lipschitz continuous, we obtain that

$$\left|c_\varepsilon(t) - c(t)\right| \leq k_3\left|\left|A^{1/2}u_\varepsilon(t)\right|^2 - \left|A^{1/2}u(t)\right|^2\right| \leq k_4\varepsilon,$$

which is exactly (2.27).

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Chapter 6
H^∞ Well-Posedness for Degenerate p-Evolution Models of Higher Order with Time-Dependent Coefficients

Torsten Herrmann, Michael Reissig, and Karen Yagdjian

Abstract In this paper we deal with time dependent p-evolution Cauchy problems. The differential operators have characteristics of variable multiplicity. We consider a degeneracy only in $t = 0$. We shall prove a well-posedness result in the scale of Sobolev spaces using a C^1-approach. In this way we will prove H^∞ well-posedness with an (at most) finite loss of regularity.

Mathematics Subject Classification 35J10 · 35Q41

6.1 Introduction

In this paper we are interested in well-posedness results in Sobolev spaces for p-evolution Cauchy problems. Starting point of our considerations is the monograph [11]. The author gives a well-posedness result for the Cauchy problem for 1-evolution (hyperbolic) equation

$$D_t^l u - \sum_{\substack{0 \le j+k \le l \\ j<l}} a_{j,k}(t,x) D_x^k D_t^j u = 0, \qquad (1)$$
$$D_t^m u(0,x) = u_m(x) \quad \text{for } m = 0, \dots, l-1 \text{ and } l \ge 2.$$

For analytic functions $a_{j,k}(t,x)$ the Cauchy problem is H^∞ well-posed. In other words, for data $u_m \in H^s$ with $m = 0, \dots, l-1$ there exists a unique solution

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$u \in C([0,T], H^{s-s_0}) \cap C^1([0,T], H^{s-s_0-p}) \cap \ldots \cap C^{l-1}([0,T], H^{s-s_0-(l-1)p})$ for some s, s_0 and $T > 0$. Cauchy problem (1) is a special case of the Cauchy problem for the p-evolution equation introduced by Petrowsky, see [13]. It can be written as follows:

$$\begin{aligned} &D_t^l u - \sum_{\substack{j+k/p=l \\ j<l}} a_{j,k}(t) D_x^k D_t^j u - \sum_{0 \le j+k/p<l} a_{j,k}(t,x) D_x^k D_t^j u = 0, \\ &D_t^m u(0,x) = u_m(x) \quad \text{for } m = 0,\ldots,l-1 \text{ and } l \ge 2. \end{aligned} \tag{2}$$

For this Cauchy problem there exist only a few results about well-posedness in scales of Sobolev spaces. But as stated in [11] the Cauchy problem is no longer of Cauchy-Kovalevskaya type. In [3] the authors proved H^∞ well-posedness for Cauchy problem (2) with $l = 2$ and $p = 2$ and for complex coefficients. They had to assume some conditions on the coefficients $a_{j,k}(t,x)$. At the moment it is important that they had to pose decay conditions on the imaginary part of $a_{j,k}(t,x)$ as x tends to infinity. Furthermore, they posed decay conditions on the derivatives of some of the real parts of $a_{j,k}(t,x)$. In this paper we do not have such decay conditions for the coefficients in the spatial variables. So we want to consider Cauchy problem (2) with real coefficients. Now also from [5] we see that we have to pose decay conditions with respect to x for some t or x-derivatives of the coefficients $a_{j,k}(t,x)$ even if they are real. In this paper we are not interested to take into consideration this effect. For this reason we will restrict ourselves to the Cauchy problem

$$\begin{aligned} &D_t^l u - \sum_{\substack{0 \le j+k/p \le l \\ j<l}} a_{j,k}(t) D_x^k D_t^j u = 0, \\ &D_t^m u(0,x) = u_m(x) \quad \text{for } m = 0,\ldots,l-1 \text{ and } l \ge 2 \end{aligned} \tag{3}$$

with real-valued time dependent coefficients in the 'extended principle part', see (5). For a statement about well-posedness we need a certain regularity of the coefficients and, furthermore, separated characteristic roots. Our goal is to consider coefficients which vanish at $t = 0$. So the roots can only be expected to be separated on $(0,T]$. We will use the so-called C^1-approach and pose assumptions on the coefficients and their first derivatives to prove H^∞ well-posedness. This is an at most finite loss of derivatives in scales of Sobolev spaces. We are going to prove a statement of the following type.

"*We consider Cauchy problem (3) under assumptions on the coefficients $a_{j,k} = a_{j,k}(t)$ and their first derivatives. Furthermore, we pose assumptions on the characteristic roots of the problem. Then for initial data u_m with $m = 0,\ldots,l-1$ given in certain scales of Sobolev spaces there exists in some evolution spaces a unique solution u of (3). The solution has an (at most) finite loss of derivatives in comparison with the given regularity of the data (see Theorem 1).*"

6.2 General Notation and Main Theorem

In this section we will give the precise assumptions we need to prove our main result. Different parts of the operator given in (2) will play a different role. In order to emphasize this distinction for the special case (3) we split the coefficients into the following three groups.

The *principal part in the sense of Petrowsky* of the p-evolution operator for (3) is given by

$$D_t^l - \sum_{\substack{j+k/p=l \\ j<l}} a_{j,k}(t) D_x^k D_t^j. \tag{4}$$

The *extended principal part* for (3) is given by

$$D_t^l - \sum_{\substack{l-1<j+k/p\le l \\ j<l}} a_{j,k}(t) D_x^k D_t^j \tag{5}$$

and, finally, the *terms of lower order* for (3) are given by

$$- \sum_{0\le j+k/p\le l-1} a_{j,k}(t) D_x^k D_t^j. \tag{6}$$

Furthermore, the *terms of Levi condition* for (3) are given by

$$- \sum_{j\le l-1} a_{j,(l-1-j)p}(t) D_x^{(l-1-j)p} D_t^j. \tag{7}$$

Remark 1 Due to the Lax-Mizohata condition for H^∞ well-posedness for p-evolution equations from [12] the coefficients of the principal part in the sense of Petrowsky have to be real. If we restrict ourselves to time-dependent coefficients, then also the coefficients of the extended principle part have to be real. If we would assume complex-valued coefficients, then we need some decay behavior in x for the imaginary parts. Our assumptions for the coefficients of the extended principal part guarantee a dominance condition (see Lemma 2). The coefficients of the terms of lower order are allowed to be complex-valued.

To get a better feeling for this classification we introduce Table 6.1.

In the following we pose assumptions for the coefficients of our starting equation. We introduce the shape function $\lambda(t)$, which satisfies the assumptions

$$\begin{aligned} &\lambda(0) = 0, \qquad \lambda'(t) > 0 \quad \text{for } t > 0, \\ &d_0 \frac{\lambda(t)}{\Lambda(t)} \le \frac{\lambda'(t)}{\lambda(t)} \le d_1 \frac{\lambda(t)}{\Lambda(t)}, \quad 0 < d_0. \end{aligned} \tag{8}$$

As mentioned before we can see that our strategy is to assume only a degeneracy in $t = 0$. Let us give some examples. A shape function of finite degeneracy is given

Table 6.1 Classification of coefficients

$a_{0,lp}$	$a_{0,lp-1}\dots a_{0,(l-1)p+1}$	$a_{0,(l-1)p}$	$a_{0,(l-1)p-1}\dots a_{0,0}$
$a_{1,(l-1)p}$	$a_{1,(l-1)p-1}\dots a_{1,(l-2)p+1}$	$a_{1,(l-2)p}$	$a_{1,(l-2)p-1}\dots a_{1,0}$
$a_{2,(l-2)p}$	$\ddots$		
$\vdots$			
$a_{l-2,lp}$	$a_{l-2,2p-1}\dots a_{l-2,p+1}$	$a_{l-2,p}$	$a_{l-2,p-1}\dots a_{l-2,0}$
$a_{l-1,lp}$	$a_{l-1,p-1}\dots a_{l-1,1}$	$a_{l-1,0}$	
Petrowsky principal part		Terms of Levi size	
Extended principal part		Lower order terms	
Real coefficients		Complex coefficients	

by $\lambda(t) = t^{\beta}$ with $\beta > 0$. An example for infinite or exponential type degeneracy is given by $\lambda(t) = t^{-2}\exp(-t^{-1})$ and for super exponential type degeneracy by $\lambda(t) = \frac{\exp(-\exp^{[n]}(1)/(t))}{t^2}\prod_{k=1}^{n}\exp^{[k]}\frac{1}{t}$. For a logarithmic type degeneracy we do not have any example which satisfies (8). With these examples for the degeneracy in $t = 0$ in mind we want to formulate assumptions on the roots of the principal part in the sense of Petrowsky. The roots are defined as solutions of the characteristic equation

$$\widehat{\tau}^{l} - \sum_{\substack{j+k/p=l\\ j<l}} a_{j,k}(t)\xi^{k}\widehat{\tau}^{j} = 0. \tag{9}$$

We assume that the roots are real and, furthermore, that they satisfy the following conditions:

$$\begin{aligned} &\text{separation condition:} \quad \left|\widehat{\tau}_i(t,\xi) - \widehat{\tau}_j(t,\xi)\right| \geq C\lambda(t)|\xi|^{p} \quad \text{for } i \neq j,\\ &\text{control of oscillations:} \quad \left|D_t^m D_\xi^k \widehat{\tau}_j(t,\xi)\right| \leq C_m \lambda(t)|\xi|^{p-k}\left(\frac{\lambda(t)}{\Lambda(t)}\right)^{m}, \end{aligned} \tag{10}$$

for all $(t,\xi) \in (0,T] \times \mathbb{R}$ with $i, j = 1, 2, \dots, l$, $k \in \mathbb{N}$ and $m = 0, 1$, where $\Lambda(t) = \int_0^t \lambda(t)\mathrm{d}t$ and $\Lambda(t) < 1$. In the following statement we are only interested to describe the oscillation condition by the coefficients of the operator.

Lemma 1 *The conditions* (10) *are equivalent to the following behavior of the coefficients of the principal part in the sense of Petrowsky*:

$$\begin{aligned} &\textit{separation condition:} \quad \left|\widehat{\tau}_i(t,\xi) - \widehat{\tau}_j(t,\xi)\right| \geq C\lambda(t)|\xi|^{p} \quad \textit{for } i \neq j,\\ &\textit{control of oscillations:} \quad \left|D_t^m a_{j,p(l-j)}(t)\right| \leq C_m \lambda(t)^{l-j}\left(\frac{\lambda(t)}{\Lambda(t)}\right)^{m} \end{aligned} \tag{11}$$

for $m = 0, 1$.

Proof Using Vieta's formulas we get the following:

$$D_t^m D_\xi^\beta \sum_{i_1<\ldots<i_{l-j}} \widehat{\tau}_{i_1}(t,\xi)\ldots\widehat{\tau}_{i_{l-j}}(t,\xi) = (-1)^j D_t^m D_\xi^\beta a_{j,k}(t)\xi^k$$

for $k = p(l-j)$ and $j = 0,\ldots,l-1$. This already yields the control of oscillations of (11) if we assume (10). To prove the other direction of the statement we get the following system from Vieta's formulas:

$$\underbrace{\begin{pmatrix} 1 & \ldots & 1 \\ \sum_{j\neq 1}\widehat{\tau}_j & \ldots & \sum_{j\neq l}\widehat{\tau}_j \\ \vdots & \vdots & \vdots \\ \prod_{j\neq 1}\widehat{\tau}_j & \ldots & \prod_{j\neq l}\widehat{\tau}_j \end{pmatrix}}_{=:A} \begin{pmatrix} D_t\widehat{\tau}_1(t,\xi) \\ D_t\widehat{\tau}_2(t,\xi) \\ \vdots \\ D_t\widehat{\tau}_l(t,\xi) \end{pmatrix} = \begin{pmatrix} (-1)^{l-1}D_t a_{l-1,p}(t)\xi^p \\ (-1)^{l-2}D_t a_{l-2,2p}(t)\xi^{2p} \\ \vdots \\ D_t a_{0,lp}(t)\xi^{lp} \end{pmatrix}.$$

This can be solved for the derivatives $D_t\widehat{\tau}_k$ of the roots of the principal symbol in the sense of Petrowsky if the matrix A is invertible. The determinant of the matrix is given by

$$\det A = \prod_{k<j}(\widehat{\tau}_k - \widehat{\tau}_j).$$

Due to the separation condition the matrix is invertible and we can control the oscillations of (10) from the assumptions (11). This completes our proof. □

For all coefficients we assume

$$\left|a_{j,k}(t)\right| \leq C\lambda(t)^{l-j}\left(\frac{|\log\Lambda(t)|}{\Lambda(t)}\right)^{l-j-k/p}. \tag{12}$$

This coincides with the behavior of the coefficients of the principal part in the sense of Petrowsky coming from the assumptions on the roots. For the coefficients of the extended principal part and for the real part of the coefficients of Levi size we assume additionally

$$\left|D_t a_{j,k}(t)\right| \leq C\lambda(t)^{l-j}\left(\frac{|\log\Lambda(t)|}{\Lambda(t)}\right)^{l-j-k/p}\left(\frac{\lambda(t)}{\Lambda(t)}\right). \tag{13}$$

For some of the coefficients of the lower order terms we need additional assumptions.

- For $a_{j,0}(t)$ with $0 \leq j < l$ we assume

$$a_{j,0}(t) \in L^1(0,T). \tag{14}$$

- For $a_{j,k}(t)$ with $l-1-j-\frac{k}{p} \geq d_0(l-1-j)$ and $k \neq 0$ we assume

$$a_{j,k}(t) \in B[0,T]. \tag{15}$$

The space $B[0,T]$ is the space of all bounded functions on $[0,T]$.

- For the terms of Levi size we assume the Levi conditions

$$\big|\Im a_{l-1-k/p,k}(t)\big| \le C\lambda(t)^{k/p}\left(\frac{\lambda(t)}{\Lambda(t)}\right). \tag{16}$$

Remark 2 We want to remark that our goal is to assume $d_0 > 0$. If we would assume $d_0 > \frac{l-1}{l}$ as in [14] instead, then we can omit assumptions (14) and (15). But, as a consequence, this narrows the set of admissible shape functions.

Theorem 1 *Let us consider the Cauchy problem* (3) *under the assumptions* (8) *and* (10) *to* (16). *Then there exists non-negative constants* s_0 *and* C *such that for all initial data* $u_m \in H^{s-mp}(\mathbb{R})$, $m = 0, \ldots, l-1$ *there is a unique solution* $u \in C([0,T], H^{s-s_0}(\mathbb{R})) \cap C^1([0,T], H^{s-s_0-p}(\mathbb{R})) \cap \ldots \cap C^{l-1}([0,T], H^{s-s_0-(l-1)p}(\mathbb{R}))$. *An a priori estimate is given by*

$$\left\| D_t^m u(t,\cdot)\right\|_{H^{s-s_0-mp}} \le C\big(\|u_0\|_{H^s} + \ldots + \|u_{l-1}\|_{H^{s-(l-1)p}}\big)$$

for $m = 0, \ldots, l-1$.

Remark 3 Let us give some comments to the assumptions (12) to (16). One can only understand assumption (12) together with assumption (14) and (15). For the real parts of Levi size coefficients we can allow an additional $\log \Lambda(t)$ term in opposite to the imaginary parts. This was already observed in [14], where among other things the conditions (12), (13) and (16) are proposed for $p = 1$.

The model equation with $l = p = 2$ was studied in [1] for a finite degeneracy. Our conditions (12), (13) and (16) are in line with the assumptions which are used there apart from the fact that no $\log \Lambda(t)$ term is allowed.

Remark 4 We have an *at most* finite loss of derivatives but we can not expect optimality of the statement. The at most difference of regularity between the initial data and the solution is given by s_0. This yields H^∞ well-posedness. Using the C^1-approach implies an *at most* finite loss of derivatives but it does not explain if the loss really appears. In opposite, if we apply C^2-approach, then we are able to study the precise loss of regularity and to show its optimality [7].

6.3 Proof

We can apply partial Fourier transformation and get an ordinary differential equation with parameter ξ. We divide the extended phase space into a pseudo-differential and an evolution zone. Then, we consider in each one different micro-energies. The goal is to get a priori estimates for the micro-energies in each zone. Our techniques to get these estimates differ from the pseudo-differential to the evolution zone.

6.3.1 First Step of the Proof

At first we apply the partial Fourier transform with respect to x and obtain

$$\begin{aligned} &D_t^l v(t,\xi) - \sum_{\substack{0\le j+k/p\le l\\ j<l}} a_{j,k}(t)\xi^k D_t^j v = 0, \\ &\text{with } v = F_{x\to\xi}(u), \quad v_m = F_{x\to\xi}(u_m) \text{ for } m = 0,\dots,l-1. \end{aligned} \tag{17}$$

6.3.2 Symbol Classes and Zones

By analogy with [14] we introduce the following zones:

Definition 1 (Zones) We divide the extended phase space into two zones. We need the pseudo-differential zone $Z_{pd}(M,N)$ and the $Z_{evo}(M,N)$. They are defined as follows:

$$Z_{pd}(M,N) = \Big\{(t,\xi) \in [0,T] \times \Big\{|\xi| \ge M > 1\Big\} : \Lambda(t)|\xi|^p \le N\Big|\log \Lambda(t)\Big|\Big\},$$
$$Z_{evo}(M,N) = \Big\{(t,\xi) \in [0,T] \times \Big\{|\xi| \ge M > 1\Big\} : \Lambda(t)|\xi|^p \ge N\Big|\log \Lambda(t)\Big|\Big\}.$$

And accordingly, we define t_ξ to be the solution of $\Lambda(t)|\xi|^p = N|\log\Lambda(t)|$.

Definition 2 (Symbols in $Z_{evo}(M,N)$) By $S_n\{l_1,l_2,l_3,l_4\}$ we denote the class of all amplitudes $a = a(t,\xi) \in C(Z_{evo}(M,N))$ satisfying for all $k, j \in \mathbb{N}$ with $j \le n$ the estimates

$$\left|D_t^j D_\xi^k a(t,\xi)\right| \le C_{j,k}|\xi|^{pl_1-k}\lambda(t)^{l_2}\left(\frac{\lambda(t)}{\Lambda(t)}\right)^{l_3+j}\left(\frac{\log(1/\Lambda(t))}{\Lambda(t)}\right)^{l_4}.$$

These symbol classes satisfy the following properties:

$$\begin{aligned} &a \in S_n\{l_1,l_2,l_3,l_4\} \to D_\xi^k a \in S_n\left\{l_1 - \frac{k}{p}, l_2, l_3, l_4\right\}, \\ &a \in S_n\{l_1,l_2,l_3,l_4\} \to D_t^k a \in S_{n-k}\{l_1,l_2,l_3+k,l_4\} \quad \text{if } k \le n, \\ &a \in S_n\{l_1,l_2,l_3,l_4\}, \tilde{a} \in S_{\tilde{n}}\{\tilde{l}_1,\tilde{l}_2,\tilde{l}_3,\tilde{l}_4\} \\ &\quad \to a \cdot \tilde{a} \in S_{\min(n,\tilde{n})}\{l_1+\tilde{l}_1, l_2+\tilde{l}_2, l_3+\tilde{l}_3, l_4+\tilde{l}_4\}, \end{aligned}$$

and generate symbol hierarchies

$$\begin{aligned} S_n\{l_1,l_2,l_3,l_4\} &\subset S_{n-1}\{l_1,l_2,l_3,l_4\}, \\ S_n\{l_1,l_2,l_3+k,l_4\} &\subset S_n\{l_1,l_2+k,l_3,l_4+k\} \quad \text{for } k \ge 0, \\ S_n\{l_1,l_2,l_3,l_4\} &\subset S_n\{l_1+k,l_2,l_3,l_4-k\} \quad \text{for } k \ge 0. \end{aligned}$$

Our strategy is to have a dominance condition for the extended principal part, that is, the principal part in the sense of Petrowsky dominates the other terms of the extended principal part. By assumption (12) and the definition of zones we have the following lemma.

Lemma 2 (Dominance condition) *For all* $(t,\xi) \in Z_{evo}(M,N)$ *it holds*

$$\left|a_{j,k}(t)\right||\xi|^k \le \frac{C}{N^{l-j-k/p}}\lambda(t)^{l-j}|\xi|^{p(l-j)}. \tag{18}$$

Proof We use the first inequality of assumption (12) and the definition of the evolution zone. It holds:

$$\begin{aligned}\left|a_{j,k}(t)\right||\xi|^k &\le C\lambda(t)^{l-j}\left(\frac{|\log(1/\Lambda(t))|}{\Lambda(t)}\right)^{l-j-k/p}|\xi|^k \\ &\le C\lambda(t)^{l-j}|\xi|^{p(l-j)}\left(\frac{|\log(1/\Lambda(t))|}{\Lambda(t)}\right)^{l-j-k/p}\frac{1}{|\xi|^{-k+p(l-j)}} \\ &\le C\lambda(t)^{l-j}|\xi|^{p(l-j)}\frac{1}{N^{l-j-k/p}}. \end{aligned} \tag{19}$$

This yields the desired statement. □

Remark 5 The last line of the estimate shows that the coefficients of the extended principal part, which do not belong to the principal part in the sense of Petrowsky are always small in comparison to the used estimate of the coefficients of the principal part in the sense of Petrowsky. This holds true because the exponent of the large constant N in (19) disappears for the coefficients of the principal part in the sense of Petrowsky and this yields together with assumption (10) the dominance of those terms.

6.3.3 Treatment in the Pseudo-differential Zone

In the pseudo-differential zone we define the micro-energy

$$V(t,\xi) = \left(\rho(t,\xi)^{l-1}v, \rho(t,\xi)^{l-2}D_t v, \dots, D_t^{l-1}v\right)^T.$$

The choice of $\rho(t,\xi)$ is important for our calculus, see [14]. There are different ways to do this. Sometimes authors propose micro-energies which depend only on ξ. But we are interested to study general degeneracies (of finite or infinite order). For this reason we follow [14] and introduce

$$\rho(t,\xi) := \sqrt[l]{1 + \frac{\lambda(t)^l}{\Lambda(t)^\alpha}\left(\log\frac{1}{\Lambda(t)}\right)^\alpha |\xi|^{p(l-\alpha)}} \tag{20}$$

for a suitable positive α. This α is connected to the minimal speed of degeneracy given by d_0. We introduce the notation $\alpha_{j,k} := l\frac{l-1-j-k/p}{l-1-j}$ and with this

$$\alpha_{j^*,k^*} = \max\left\{\alpha_{j,k} \text{ with } \frac{\alpha_{j,k}}{l} < d_0\right\} \quad \text{for } j < l-1.$$

Now we define

$$\alpha := ld_0 - \varepsilon \quad \text{with } \varepsilon < \min\left\{ld_0, ld_0 - \alpha_{j^*,k^*}, \frac{1}{1+l^2}\right\}. \tag{21}$$

In (20) we use $\log \frac{1}{\Lambda(t)}$. This is always positive in the pseudo-differential zone for $|\xi|$ large. And for the proof of our regularity statement we need only to consider $|\xi|$ large (see Definition 1).

Remark 6 In the 1-evolution (hyperbolic) case with a minimal speed of finite degeneracy determined by $d_0 > \frac{l-1}{l}$, so the shape function is t^β with $\beta > l-1$, it is sufficient to choose $\alpha = (l-1)d_0$.

In the next lemma we state all the properties of $\rho(t,\xi)$ that we will use in this section.

Lemma 3 *We have the following properties for the weight $\rho(t,\xi)$ for $t \in [0, t_\xi]$:*

$$1 \le \rho(t,\xi) \le C|\xi|^p, \qquad \rho(0,\xi) = 1, \qquad \int_0^t \rho(\tau,\xi)\mathrm{d}\tau \le C\big(1 + \log|\xi|\big),$$
$$\log \rho(t_\xi,\xi) \le C \log|\xi|,$$

and for $\frac{\partial_t \rho(t,\xi)}{\rho(t,\xi)}$ it holds

$$\frac{\partial_t \rho(t,\xi)}{\rho(t,\xi)} \ge 0 \quad \textit{and} \quad \int_0^t \frac{\partial_\tau \rho(\tau,\xi)}{\rho(\tau,\xi)}\mathrm{d}\tau \le C \log|\xi|$$

provided that M and N are large.

Proof At first we need the non-negativity of $\frac{\partial_t \rho(t,\xi)}{\rho(t,\xi)}$. It holds:

$$\begin{aligned}\frac{\partial_t \rho(t,\xi)}{\rho(t,\xi)} = \frac{1}{l}\Bigg(\Bigg(l\frac{\lambda'(t)\lambda(t)^{l-1}}{\Lambda(t)^\alpha}\Big(\log\frac{1}{\Lambda(t)}\Big)^\alpha \\ - \alpha\frac{\lambda(t)^{l+1}}{\Lambda(t)^{\alpha+1}}\Big(\Big(\log\frac{1}{\Lambda(t)}\Big)^\alpha + \Big(\log\frac{1}{\Lambda(t)}\Big)^{\alpha-1}\Big)\Bigg) \\ \Big/\Big(|\xi|^{-p(l-\alpha)} + \frac{\lambda(t)^l}{\Lambda(t)^\alpha}\Big(\log\frac{1}{\Lambda(t)}\Big)^\alpha\Big)\Bigg)\end{aligned}$$

and this is non-negative if the following condition holds:

$$d_0 - \frac{\alpha}{l} - \frac{\alpha}{l}\left(\log\frac{1}{\Lambda(t)}\right)^{-1} \geq 0 \quad \to \quad d_0 \geq \frac{\alpha+\varepsilon}{l} \quad \to \quad d_0 > \frac{\alpha}{l}, \tag{22}$$

respectively. For $|\xi|$ large $\log\frac{1}{\Lambda(t)}$ is larger than $\frac{\alpha}{\varepsilon}$ for an arbitrary small $\varepsilon > 0$ and $T \leq T_0(\alpha, \varepsilon)$ in the pseudo-differential zone. So estimate (22) holds true for our choice of α. The non-negativity of $\frac{\partial_t \rho(t,\xi)}{\rho(t,\xi)}$ together with the positivity of $\rho(t,\xi)$ yields the monotonicity of $\rho(t,\xi)$. Furthermore, we get

$$\begin{aligned}
\lim_{t\to 0+}\rho(t,\xi) &= \lim_{t\to 0+}\sqrt[l]{1+\frac{\lambda(t)^l}{\Lambda(t)^\alpha}\left(\log\frac{1}{\Lambda(t)}\right)^{\alpha}|\xi|^{p(l-\alpha)}} \\
&= \lim_{t\to 0+}\sqrt[l]{1+\frac{\lambda(t)^l}{\Lambda(t)^{ld_0-\varepsilon}}\left(\log\frac{1}{\Lambda(t)}\right)^{ld_0-\varepsilon}|\xi|^{p(l-ld_0+\varepsilon)}}.
\end{aligned}$$

For the finite degenerate case $\lambda(t) = t^\beta$ we have

$$\lim_{t\to 0+}\frac{\lambda(t)^l}{\Lambda(t)^{ld_0-\varepsilon}}\left(\log\frac{1}{\Lambda(t)}\right)^{ld_0-\varepsilon} = 0$$

with $d_0 = \frac{\beta}{\beta+1}$ which brings $\lim_{t\to 0+} t^\nu = 0$ with a suitable $\nu > 0$. For the infinite degenerate case $\frac{\lambda(t)^l}{\Lambda(t)^{ld_0-\varepsilon}}$ yields a term which tends to zero of infinite order for any $d_0 < 1$. This brings $\rho(0,\xi) = 1$ for both cases.

With this we can estimate as follows:

$$1 \leq \rho(t,\xi) \leq \rho(t_\xi,\xi) \leq \sqrt[l]{1+\lambda(t_\xi)|\xi|^{pl}\left(\frac{\log(1/\Lambda(t_\xi))}{\Lambda(t_\xi)|\xi|^p}\right)^{\alpha}} \leq C_N|\xi|^p.$$

For the integrals we get

$$\int_0^t \frac{\partial_t\rho(\tau,\xi)}{\rho(\tau,\xi)}\mathrm{d}\tau \leq C\log\rho(\tau,\xi)|_0^t \leq C\log\rho(t_\xi,\xi) \leq C_N\log|\xi| \tag{23}$$

and

$$\begin{aligned}
\int_0^t \rho(\tau,\xi)\mathrm{d}\tau &\leq C\left(\int_0^t \mathrm{d}\tau + \int_0^t \frac{\lambda(t)}{\Lambda(t)^{\alpha/l}}\left(\log\frac{1}{\Lambda(t)}\right)^{\alpha/l}|\xi|^{p((l-\alpha)/l)}\right) \\
&\leq C\left(T + \Lambda(t_\xi)^{(l-\alpha)/l}\left(\log\frac{1}{\Lambda(t_\xi)}\right)^{\alpha/l}|\xi|^{p((l-\alpha)/l)}\right) \\
&\leq C\left(1+\left(\log\frac{1}{\Lambda(t_\xi)}\right)^{\alpha/l}\left(N\log\frac{1}{\Lambda(t_\xi)}\right)^{(l-\alpha)/l}\right) \\
&\leq C_N\left(1+\log\frac{1}{\Lambda(t_\xi)}\right) \leq C_N\big(1+\log|\xi|\big).
\end{aligned} \tag{24}$$

This completes the proof of Lemma 3. □

Lemma 4 *For all* $(t,\xi) \in Z_{pd}(M,N)$ *it holds*

$$\begin{cases} |v(t,\xi)| \lesssim \rho(t,\xi)^{-l+1}|\xi|^{C_{pd}}(|v_0(\xi)| + \ldots + |v_{l-1}(\xi)|), \\ |D_t v(t,\xi)| \lesssim \rho(t,\xi)^{-l+2}|\xi|^{C_{pd}}(|v_0(\xi)| + \ldots + |v_{l-1}(\xi)|), \\ \ldots \\ |D_t^{l-1} v(t,\xi)| \lesssim |\xi|^{C_{pd}}(|v_0(\xi)| + \ldots + |v_{l-1}(\xi)|). \end{cases}$$

Proof Using the micro-energy in the pseudo-differential zone for our Fourier transformed Cauchy problem (17) this leads to the system of first order $D_t V = A(t,\xi)V$ with

$$A(t,\xi) := \begin{pmatrix} (l-1)\frac{D_t\rho(t,\xi)}{\rho(t,\xi)} & \rho(t,\xi) & 0 & 0 & \ldots & 0 \\ 0 & (l-2)\frac{D_t\rho(t,\xi)}{\rho(t,\xi)} & \rho(t,\xi) & 0 & \ldots & 0 \\ \vdots & & \ddots & \ddots & & \vdots \\ 0 & \ldots & 0 & 2\frac{D_t\rho(t,\xi)}{\rho(t,\xi)} & \rho(t,\xi) & 0 \\ 0 & \ldots & 0 & 0 & \frac{D_t\rho(t,\xi)}{\rho(t,\xi)} & \rho(t,\xi) \\ \frac{\sum_{0\le k/p\le l} a_{0,k}(t)\xi^k}{\rho(t,\xi)^{l-1}} & \frac{\sum_{0\le k/p\le l-1} a_{1,k}(t)\xi^k}{\rho(t,\xi)^{l-2}} & & \ldots & & \sum_{0\le k/p\le 1} a_{l-1,k}(t)\xi^k \end{pmatrix}.$$

We are interested in the fundamental solution $E = E(t,s,\xi)$ to the system $D_t V - AV = 0$, that is, the solution of

$$D_t E - AE = 0, \quad E(s,s,\xi) = I, \quad \text{thus } V(t,\xi) = E(t,0,\xi)V(0,\xi).$$

The matrix $E(t,s,\xi)$ can be estimated by

$$\|E(t,s,\xi)\| \le \exp\Big(\int_0^t \|A(\tau,\xi)\| d\tau\Big), \quad 0 \le s \le t \le t_\xi. \tag{25}$$

Due to Lemma 3 we can estimate $\|A(t,\xi)\|$ in the following way:

$$\|A(t,\xi)\| \lesssim \frac{\partial_t\rho(t,\xi)}{\rho(t,\xi)} + \rho(t,\xi) + \sum_{\substack{0\le j+k/p\le l \\ j<l}} \frac{|a_{j,k}(t)||\xi|^k}{\rho(t,\xi)^{l-1-j}}. \tag{26}$$

The integrals of $\rho(t,\xi)$ and $\frac{\partial_t\rho(t,\xi)}{\rho(t,\xi)}$ over $[0,t]$, $t \le t_\xi$, are discussed in Lemma 3. Left is the estimate of $\int_0^t \frac{|a_{j,k}(\tau)||\xi|^k}{\rho(\tau,\xi)^{l-1-j}} d\tau$. It depends on the structure of $a_{j,k}(t)$. We begin with $a_{j,0}(t)$. Using condition (14) we can estimate

$$\int_0^t \frac{|a_{j,0}(\tau)|}{\rho(\tau,\xi)^{l-1-j}}\mathrm{d}\tau \le \int_0^t \big|a_{j,0}(\tau)\big|\mathrm{d}\tau \le C.$$

For the terms $a_{j,k}(t)$ with $l-1-j-\frac{k}{p} \ge d_0(l-1-j)$ we introduce another subzone to distinguish which part of $\rho(t,\xi)$ is dominant. Here we want to remember that only a shape function $\lambda(t)=t^\beta$ with finite degeneracy has to be considered, because for flat degeneracies, this assumption is meaningless. Let $t_{\xi,1}$ solve

$$1 = \frac{\lambda(t)^l}{\Lambda(t)^\alpha}\left(\log\frac{1}{\Lambda(t)}\right)^\alpha |\xi|^{p(l-\alpha)},$$

where α is the same as in (20). Then $0 \le t_{\xi,1} \le t_\xi$ for $|\xi|$ large. This follows from the following calculations:

$$1 = \frac{\lambda(t_{\xi,1})^l}{\Lambda(t_{\xi,1})^\alpha}\left(\log\frac{1}{\Lambda(t_{\xi,1})}\right)^\alpha |\xi|^{p(l-\alpha)}, \quad \Lambda(t_\xi)|\xi|^p = N\log\frac{1}{\Lambda(t_\xi)},$$

$$t_{\xi,1} = |\xi|^{-p/(\beta-\alpha/(l-\alpha))}\underbrace{\left(\log\frac{1}{\Lambda(t_{\xi,1})}\right)^{-\alpha/(l\beta-\alpha(\beta+1))}}_{<1},$$

$$t_\xi = |\xi|^{-p/(\beta+1)}\underbrace{N^{1/(\beta+1)}\left(\log\frac{1}{\Lambda(t_\xi)}\right)^{1/(\beta+1)}}_{>1}.$$

The definition of $t_{\xi,1}$ yields that for $0 \le t \le t_{\xi,1}$ the number 1 is dominant in the definition of $\rho(t,\xi)$ whereas for $t_{\xi,1} \le t \le t_\xi$ the second part $\frac{\lambda(t)^l}{\Lambda(t)^\alpha}(\log\frac{1}{\Lambda(t)})^\alpha|\xi|^{p(l-\alpha)}$ is dominant. With this it holds

$$\int_0^t \frac{|a_{j,k}(\tau)||\xi|^k}{\rho(\tau,\xi)^{l-1-j}}\mathrm{d}\tau = \int_0^{t_{\xi,1}} \frac{|a_{j,k}(\tau)||\xi|^k}{\rho(\tau,\xi)^{l-1-j}}\mathrm{d}\tau + \int_{t_{\xi,1}}^t \frac{|a_{j,k}(\tau)||\xi|^k}{\rho(\tau,\xi)^{l-1-j}}\mathrm{d}\tau.$$

As remarked before, we only have to consider the case of finite degeneracy. For $\lambda(t)=t^\beta$ we get $d_0=\frac{\beta}{\beta+1}$. Now we consider the first integral on the right-hand side. With assumption (15) it holds

$$\begin{aligned}\int_0^{t_{\xi,1}} \frac{|a_{j,k}(\tau)||\xi|^k}{\rho(t,\xi)^{l-1-j}}\mathrm{d}\tau &\le C\int_0^{t_{\xi,1}} |\xi|^k\mathrm{d}\tau = Ct_{\xi,1}|\xi|^k \\ &\le Ct_{\xi,1}\left(\frac{\lambda(t_{\xi,1})^l}{\Lambda(t_{\xi,1})^\alpha}\left(\log\frac{1}{\Lambda(t_{\xi,1})}\right)^\alpha\right)^{-k/(p(l-\alpha))}\end{aligned}$$

and with $\alpha = l\frac{\beta}{\beta+1} - \varepsilon$ we get

$$\int_0^{t_{\xi,1}} \frac{|a_{j,k}(\tau)||\xi|^k}{\rho(t,\xi)^{l-1-j}} \mathrm{d}\tau \le C t_{\xi,1}^{(pl-p\alpha-\beta kl+(\beta+1)k\alpha)/(p(l-\alpha))} \big(\log|\xi|\big)^{-\alpha k/(p(l-\alpha))}.$$

Now with $\varepsilon < \frac{1}{1+l^2}$, see (21), the exponent of $t_{\xi,1}$ is positive. Because of the negative exponent of $\log|\xi|$ it holds

$$\int_0^{t_{\xi,1}} \frac{|a_{j,k}(\tau)||\xi|^k}{\rho(t,\xi)^{l-1-j}} \mathrm{d}\tau \le C.$$

For the second integral we get

$$\int_{t_{\xi,1}}^{t_\xi} \frac{|a_{j,k}(\tau)||\xi|^k}{\rho(\tau,\xi)^{l-1-j}} \mathrm{d}\tau \le C \int_{t_{\xi,1}}^{t_\xi} \frac{|a_{j,k}(\tau)||\xi|^k}{((\lambda(\tau)^l/\Lambda(\tau)^\alpha)(\log(1/\Lambda(\tau)))^\alpha)^{l-1-j/(l)}} \mathrm{d}\tau$$

and for $d_0 = \frac{\beta}{\beta+1}$ it holds

$$\begin{aligned}
&= C\int_{t_{\xi,1}}^{t_\xi} \tau^{(\beta+1)(\alpha(l-1-j)/l)-\beta(l-1-j)} \left(\log\frac{1}{\tau}\right)^{-\alpha((l-1-j)/l)} |\xi|^{k-p((l-\alpha)(l-1-j)/l)} \mathrm{d}\tau \\
&\le C t_\xi^{1+(\beta+1)(\alpha(l-1-j)/l)-\beta(l-1-j)} \left(\log\frac{1}{t_{\xi,1}}\right)^{-\alpha((l-1-j)/l)} |\xi|^{k-p((l-\alpha)(l-1-j)/l)} \\
&\le C t_\xi^{1+(\beta+1)(\alpha(l-1-j)/l)-\beta(l-1-j)-k((\beta+1)/p)+(\beta+1)(l-\alpha)(l-1-j)/l} \\
&\quad \times \big(\log|\xi|\big)^{-\alpha((l-1-j)/l)+k/p-(l-\alpha)(l-1-j)/l} \\
&\le C t_\xi^{l-j-(k/p)(\beta+1)} \big(\log|\xi|\big)^{k/p-l+1+j} \\
&\le C t_\xi^{l-j-(k/p)(\beta+1)} \log|\xi|.
\end{aligned}$$

This gives an estimate for an at most finite loss of derivatives if the exponent of t_ξ is non negative. So, we have to guarantee

$$l - j - \frac{k}{p}(\beta+1) \ge 0$$

which is always satisfied for $a_{j,k}(t)$ with $d_0(l-1-j) \le l-1-j-\frac{k}{p}$. Consequently, we have shown that

$$\int_0^t \frac{|a_{j,k}(\tau)||\xi|^k}{\rho(\tau,\xi)^{l-1-j}} \mathrm{d}\tau \le C\big(1+\log|\xi|\big) \tag{27}$$

for all $0 \le t \le t_\xi$ and all coefficients $a_{j,k}(t)$ with $d_0(l-1-j) \le l-1-j-\frac{k}{p}$. This completes the explanations for the part of lower order terms satisfying assumption (15). Left is the procedure for the other part. We need to estimate

$$\int_0^t \frac{|a_{j,k}(\tau)||\xi|^k}{\rho(\tau,\xi)^{l-1-j}}\mathrm{d}\tau \le C\big(1+\log|\xi|\big) \tag{28}$$

by using assumption (12). We can estimate as follows:

$$\begin{aligned}
\frac{|a_{j,k}(t)||\xi|^k}{\rho(t,\xi)^{l-1-j}} &\le C\frac{\lambda(t)^{l-j}(\log(1/\Lambda(t))/\Lambda(t))^{l-j-k/p}|\xi|^k}{(1+(\lambda(t)^l/\Lambda(t)^\alpha)(\log(1/\Lambda(t)))^\alpha|\xi|^{p(l-\alpha)})^{(l-1-j)/l}} \\
&\le C\frac{\lambda(t)^{l-j-(l-1-j)}}{\Lambda(t)^{-\alpha(l-1-j)/l+l-j-k/p}}\left(\log\frac{1}{\Lambda(t)}\right)^{l-j-k/p-\alpha((l-1-j)/l)} \\
&\quad\times|\xi|^{k-p(l-\alpha)(l-1-j)/l} \\
&\le C\frac{\lambda(t)}{\Lambda(t)^{l-j-k/p-\alpha+\alpha/l+\alpha j/l}}\left(\log\frac{1}{\Lambda(t)}\right)^{l-j-k/p-\alpha+\alpha/l+\alpha j/l} \\
&\quad\times|\xi|^{k-pl+p+pj+\alpha p-\alpha p/l-\alpha jp/l},
\end{aligned} \tag{29}$$

which leads to

$$\begin{aligned}
\int_0^t \frac{|a_{j,k}(\tau)||\xi|^k}{\rho(\tau,\xi)^{l-1-j}}\mathrm{d}\tau &\le \Lambda(t)^{1-l+j+k/p+\alpha(1-1/l-j/l)}|\xi|^{p(1-l+j+k/p+\alpha(1-1/l-j/l))} \\
&\quad\times\left(\log\frac{1}{\Lambda(t)}\right)^{l-j-k/p-\alpha(1-1/l-j/l)} \\
&\le C_N\big(\log|\xi|\big)
\end{aligned} \tag{30}$$

for all $0\le t\le t_\xi$ by using the definition of the pseudo-differential zone. The last step only holds true for $1-l+j+\frac{k}{p}+\alpha(1-\frac{1}{l}-\frac{j}{l})\ge 0$. With our definition of α and $\varepsilon < ld_0-\alpha_{j^*,k^*}$, see (21), the condition is always satisfied. So we obtain an estimate for (25)

$$\begin{aligned}
\big\|E(t,s,\xi)\big\| &\le \exp\left(\int_0^t \big\|A(\tau,\xi)\big\|d\tau\right) \\
&\lesssim \exp\left(C\left(\int_0^t \frac{\partial_t\rho(\tau,\xi)}{\rho(\tau,\xi)}\mathrm{d}\tau+\int_0^t\rho(\tau,\xi)\mathrm{d}\tau\right.\right. \\
&\quad\left.\left.+\int_0^t\sum_{\substack{j+k/p\le l\\ j<l}}\left|\frac{a_{j,k}(\tau)\xi^k}{\rho(\tau,\xi)^{l-1-j}}\right|\mathrm{d}\tau\right)\right) \\
&\lesssim \exp\big(C\big(1+\log|\xi|\big)\big).
\end{aligned}$$

We complete the proof by using our fundamental solution E

$$\begin{aligned}
V(t,\xi) &= E(t,0,\xi)V(0,\xi), \\
\rho(t,\xi)^{l-1}\big|v(t,\xi)\big| &\lesssim \exp\big(C\big(1+\log|\xi|\big)\big)\big(\big|v_0(\xi)\big|+\big|v_1(\xi)\big|+\ldots+\big|v_{l-1}(\xi)\big|\big),
\end{aligned}$$

$$\rho(t,\xi)^{l-2}\big|D_t v(t,\xi)\big| \lesssim \exp\big(C\big(1+\log|\xi|\big)\big)\big(\big|v_0(\xi)\big|+\big|v_1(\xi)\big|+\ldots+\big|v_{l-1}(\xi)\big|\big),$$
$$\ldots$$
$$\big|D_t^{l-1} v(t,\xi)\big| \lesssim \exp\big(C\big(1+\log|\xi|\big)\big)\big(\big|v_0(\xi)\big|+\big|v_1(\xi)\big|+\ldots+\big|v_{l-1}(\xi)\big|\big).$$

Here we used $\rho(0,\xi)=1$. In this way the proof of Lemma 4 is completed. □

6.3.4 Treatment in the Evolution Zone

In the evolution zone $Z_{evo}(M,N)$ we define the micro-energy

$$V=\Big(\big(\lambda(t)|\xi|^p\big)^{l-1}v,\ \big(\lambda(t)|\xi|^p\big)^{l-2}D_t v,\ldots,D_t^{l-1}v\Big)^T.$$

Lemma 5 *For all* $(t,\xi)\in Z_{evo}(M,N)$ *it holds*

$$\begin{cases}(\lambda(t)|\xi|^p)^{l-1}|v(t,\xi)| \\ \quad\lesssim \exp(C(1+\log|\xi|))(\sum_{j=1}^{l}(\lambda(t_\xi)|\xi|^p)^{l-j}|D_t^{j-1}v(t_\xi,\xi)|), \\ (\lambda(t)|\xi|^p)^{l-2}|D_t v(t,\xi)| \\ \quad\lesssim \exp(C(1+\log|\xi|))(\sum_{j=1}^{l}(\lambda(t_\xi)|\xi|^p)^{l-j}|D_t^{j-1}v(t_\xi,\xi)|), \\ \ldots \\ |D_t^{l-1}v(t,\xi)| \lesssim \exp(C(1+\log|\xi|))(\sum_{j=1}^{l}(\lambda(t_\xi)|\xi|^p)^{l-j}|D_t^{j-1}v(t_\xi,\xi)|).\end{cases}$$

Proof First we want to consider the roots of the symbol containing the transformed extended principal part together with the real part of the terms of Levi size. They are given as the solutions to the characteristic equation

$$\tau^l-\sum_{\substack{l-1\le j+k/p\le l\\ j<l}}\Re a_{j,k}(t)\xi^k\tau^j=0. \tag{31}$$

The following proposition shows how the roots of (31) inherit the properties for the roots of (9).

Proposition 1 *We consider the roots* $\tau_1,\ldots,\tau_l$ *of* (31)*. With assumption* (10) *for the roots of the principal part in the sense of Petrowsky and with the definition of the zone we get real roots satisfying*

$$\begin{aligned}\big|\tau_i(t,\xi)-\tau_j(t,\xi)\big| &\ge C\lambda(t)|\xi|^p \quad \textit{for } i\ne j,\\ \big|D_t^m D_\xi^k\tau_j(t,\xi)\big| &\le C_m\lambda(t)|\xi|^{p-k}\left(\frac{\lambda(t)}{\Lambda(t)}\right)^m,\end{aligned}\tag{32}$$

for all $(t,\xi)\in Z_{evo}(M,N)$ *and for* $i,j=1,2,\ldots,l$, $k\in\mathbb{N}$ *and* $m=0,1$.

Proof We rewrite the assumption for the coefficients in the following way:

$$a_{j,k}(t) = \lambda(t)^{l-j}\left(\frac{\log(1/\Lambda(t))}{\Lambda(t)}\right)^{l-j-k/p}\widetilde{a}_{j,k}(t)$$

with $\widetilde{a}_{j,k}(t) \in B(0,T]$. We apply the transformation $\tau = \lambda(t)\xi^p z$. The transformation yields

$$z^l - \sum_{\substack{j+k/p=l\\ j<l}} \widetilde{a}_{j,k}(t)z^j - \sum_{\substack{l-1\le j+k/p<l\\ j<l}} \Re\widetilde{a}_{j,k}(t)\left(\frac{\log(1/\Lambda(t))}{\Lambda(t)\xi^p}\right)^{l-j-k/p} z^j = 0. \tag{33}$$

If we consider the transformation $\widehat{\tau} = \lambda(t)\xi^p\widehat{z}$ for (9) we obtain

$$\widehat{z}^l - \sum_{\substack{j+k/p=l\\ j<l}} \widetilde{a}_{j,k}(t)\widehat{z}^j = 0 \tag{34}$$

and from assumption (10) we know that equation (34) has real and distinct roots. It holds

$$\left|\widehat{z}_i(t,\xi) - \widehat{z}_j(t,\xi)\right| \ge C \quad \text{for } i \ne j,\ (t,\xi) \in [0,T] \times \big(\mathbb{R}\setminus\{0\}\big).$$

Equation (33) is a perturbed equation (34), so the roots $\tau_1, \dots, \tau_l$ are in a small neighborhood of the respective roots $\widehat{\tau}_1, \dots, \widehat{\tau}_l$ if the perturbation is sufficiently small. We know that the coefficients of the extended principal part are real. This and the distinctness of the roots $\widehat{\tau}_1, \dots, \widehat{\tau}_l$ yields that roots $z_1, \dots, z_l$ are real and distinct, because the smallness of the real perturbations is given by

$$\left|\Re\widetilde{a}_{j,k}(t)\right|\left(\frac{\log(1/\Lambda(t))}{\Lambda(t)\xi^p}\right)^{l-j-k/p} \le \frac{1}{C^*(N)} \quad \text{with } C^*(N) \to \infty \text{ for } N \to \infty.$$

And this holds true for any sufficiently large constant N in the definition of the zones. Backward transformation yields the first statement of the proposition. Furthermore, due to Vieta's formulas we have

$$\left|D_t^m D_\xi^\beta \sum_{i_1<\dots<i_{l-j}} \tau_{i_1}(t,\xi)\dots\tau_{i_{l-j}}(t,\xi)\right| = \left|D_t^m D_\xi^\beta a_{j,k}(t)\xi^k\right|$$

$$\le C_m\lambda(t)^{l-j}|\xi|^{k-\beta}\left(\frac{\lambda(t)}{\Lambda(t)}\right)^m$$

for $k = p(l-j)$ and $j = 0, \dots, l-1$.

So we know that the roots of the extended principal part satisfy Proposition 1. $\square$

Using the micro-energy in the evolution zone for our Fourier transformed Cauchy problem (17) this leads to the system of first order $D_t V = A(t,\xi)V$ with

$$A(t,\xi) := \begin{pmatrix} \frac{(l-1)}{i}\frac{\lambda'(t)}{\lambda(t)} & \lambda(t)|\xi|^p & 0 & 0 & \dots & 0 \\ 0 & \frac{(l-2)}{i}\frac{\lambda'(t)}{\lambda(t)} & \lambda(t)|\xi|^p & 0 & \dots & 0 \\ \vdots & & \ddots & \ddots & & \vdots \\ 0 & \dots & 0 & \frac{2}{i}\frac{\lambda'(t)}{\lambda(t)} & \lambda(t)|\xi|^p & 0 \\ 0 & \dots & 0 & 0 & \frac{1}{i}\frac{\lambda'(t)}{\lambda(t)} & \lambda(t)|\xi|^p \\ \frac{\sum_{0\le k/p\le l} a_{0,k}(t)\xi^k}{(\lambda(t)|\xi|^p)^{l-1}} & \frac{\sum_{0\le k/p\le l-1} a_{1,k}(t)\xi^k}{(\lambda(t)|\xi|^p)^{l-2}} & & \dots & & \sum_{0\le k/p\le 1} a_{l-1,k}(t)\xi^k \end{pmatrix}.$$

Now we split matrix $A(t,\xi)$ into several parts. We introduce

$$A_1(t,\xi) := \begin{pmatrix} 0 & \lambda(t)|\xi|^p & & & 0 \\ \vdots & & \ddots & & \\ 0 & \dots & 0 & \lambda(t)|\xi|^p & \\ 0 & \dots & 0 & 0 & \lambda(t)|\xi|^p \\ \frac{\sum_{l-1\le k/p\le l} \Re a_{0,k}(t)\xi^k}{(\lambda(t)|\xi|^p)^{l-1}} & \frac{\sum_{l-2\le k/p\le l-1} \Re a_{1,k}(t)\xi^k}{(\lambda(t)|\xi|^p)^{l-2}} & & \dots & \sum_{0\le k/p\le 1} \Re a_{l-1,k}(t)\xi^k \end{pmatrix},$$

$$A_2(t,\xi) := \begin{pmatrix} \frac{(l-1)}{i}\frac{\lambda'(t)}{\lambda(t)} & 0 & 0 & 0 & \dots & 0 \\ 0 & \frac{(l-2)}{i}\frac{\lambda'(t)}{\lambda(t)} & 0 & 0 & \dots & 0 \\ \vdots & & \ddots & \ddots & & \vdots \\ 0 & \dots & 0 & \frac{2}{i}\frac{\lambda'(t)}{\lambda(t)} & 0 & 0 \\ 0 & \dots & 0 & 0 & \frac{1}{i}\frac{\lambda'(t)}{\lambda(t)} & 0 \\ 0 & 0 & & \dots & & 0 \end{pmatrix},$$

$$A_3(t,\xi) := \begin{pmatrix} 0 & 0 & \dots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \dots & 0 \\ \frac{\Im a_{0,p(l-1)}(t)\xi^{p(l-1)}+\sum_{k/p<l-1} a_{0,k}(t)\xi^k}{(\lambda(t)|\xi|^p)^{l-1}} & \frac{\Im a_{1,p(l-2)}(t)\xi^{p(l-2)}+\sum_{k/p<l-2} a_{1,k}(t)\xi^k}{(\lambda(t)|\xi|^p)^{l-2}} & \dots & \Im a_{l-1,0}(t) \end{pmatrix}.$$

We are interested in the symbol classes for $A_2(t,\xi)$ and $A_3(t,\xi)$. It is obvious that $A_2(t,\xi) \in S_0\{0,0,1,0\}$ and for $A_3(t,\xi)$ the assumptions (12) and (16) and straight forward calculations yield $A_3(t,\xi) \in S_0\{0,0,0,0\} + S_0\{-\frac{1}{p}, 1, 0, 1+\frac{1}{p}\}$.

Remark 7 Let us come back to the assumptions (12) and (16) for the terms of Levi size. The real parts are included in the matrix A_1, this allows a $\log \Lambda(t)$ term. The imaginary parts are included in the matrix A_3. To stay in the correct symbol classes we are not able to allow a $\log \Lambda(t)$ term for the imaginary parts.

Using the system $\frac{\tau_1}{\lambda(t)|\xi|^p}, \dots, \frac{\tau_l}{\lambda(t)|\xi|^p}$ we form the Vandermonde matrix

$$M(t,\xi) := \begin{pmatrix} 1 & 1 & \dots & 1 \\ \frac{\tau_1(t,\xi)}{\lambda(t)|\xi|^p} & \frac{\tau_2(t,\xi)}{\lambda(t)|\xi|^p} & \dots & \frac{\tau_l(t,\xi)}{\lambda(t)|\xi|^p} \\ \vdots & \vdots & \vdots & \vdots \\ (\frac{\tau_1(t,\xi)}{\lambda(t)|\xi|^p})^{l-1} & (\frac{\tau_2(t,\xi)}{\lambda(t)|\xi|^p})^{l-1} & \dots & (\frac{\tau_l(t,\xi)}{\lambda(t)|\xi|^p})^{l-1} \end{pmatrix}$$

and apply the transformation $V := M(t,\xi)V_1$ to our system

$$D_t V = A_1 V + A_2 V + A_3 V. \tag{35}$$

The matrix M is chosen as a diagonalizer of A_1. The determinant of M is given by

$$\det\big(M(t,\xi)\big) = \prod_{1\le i<j\le n} \frac{\tau_j(t,\xi) - \tau_i(t,\xi)}{\lambda(t)|\xi|^p}.$$

Because of the separation condition from (32) the determinant of $M(t,\xi)$ satisfies $|\det(M(t,\xi))| \ge C > 0$ and so the inverse matrix $M^{-1}(t,\xi)$ exists for all $(t,\xi) \in Z_{evo}(M,N)$.

Lemma 6 *After the first step of diagonalization we obtain from system* (35) *the new system*

$$D_t V_1 = DV_1 + RV_1, \quad V_1(t_\xi,\xi) = V_{1,0}(\xi) := M^{-1}V(t_\xi,\xi) \tag{36}$$

with a diagonal matrix

$$D = D(t,\xi) = \begin{pmatrix} \tau_1(t,\xi) & & 0 \\ & \ddots & \\ 0 & & \tau_l(t,\xi) \end{pmatrix}$$

and a matrix

$$R = R(t,\xi) \in S_0\{0,0,0,0\} + S_0\{0,0,1,0\} + S_0\left\{-\frac{1}{p},1,0,1+\frac{1}{p}\right\}. \qquad (37)$$

Proof System (35) transforms to

$$D_t V_1 = M^{-1}A_1 M V_1 + M^{-1}A_2 M V_1 + M^{-1}A_3 M V_1 - M^{-1}(D_t M)V_1 \qquad (38)$$

with the diagonal matrix $D = M^{-1}A_1 M$. The matrix R is defined by

$$R := M^{-1}A_2 M - M^{-1}(D_t M) + M^{-1}A_3 M.$$

For the entries of M it holds

$$\left|\left(\frac{\tau_k(t,\xi)}{\lambda(t)|\xi|^p}\right)^j\right| \le C$$

for $j = 0,\dots,l-1$ and $k = 1,\dots,l$. With this $M(t,\xi)$ and its inverse $M^{-1}(t,\xi) \in S_0\{0,0,0,0\}$. So the calculus of the symbol classes yields the statement of the lemma. □

The function

$$E_2(t,r,\xi) := \begin{pmatrix} e^{i\int_r^t \tau_1(s,\xi)ds} & & 0 \\ & \ddots & \\ 0 & & e^{i\int_r^t \tau_l(s,\xi)ds} \end{pmatrix}$$

solves the Cauchy problem $(D_t - D)E(t,r,\xi) = 0$, $E(r,r,\xi) = I$. It holds for $r \ge t_\xi$

$$\left\| E_2(t,r,\xi)\right\| \le \max_{k=1,\dots,l}\left|\exp\left(i\int_r^t \sum_{k=1}^{l}\tau_k(s,\xi)\mathrm{d}s\right)\right| = 1,$$

because the roots of (31) are all real. Here we feel the dispersive character of our Cauchy problem and the dominance condition from Lemma 2. We define the matrix-valued function $H = H(t,r,\xi)$ with $t, r \ge t_\xi$:

$$H(t,r,\xi) := E_2(r,t,\xi)R(t,\xi)E_2(t,r,\xi).$$

Because $E_2(r,t,\xi)=E_2^{-1}(t,r,\xi)$, $\|E_2(r,t,\xi)\|=\|E_2^{-1}(t,r,\xi)\|=1$, and due to (37) the following estimate holds:

$$\|H(t,r,\xi)\|\le C+C\frac{\lambda(t)}{\Lambda(t)}+C\frac{\lambda(t)}{\Lambda(t)^{1+1/p}|\xi|}\left(\log\frac{1}{\Lambda(t)}\right)^{1+1/p}. \tag{39}$$

We will consider $\log\frac{1}{\Lambda(t)}$ to be positive for all $t\le T$, because we are only interested in times close to the degeneracy $t=0$. Now

$$V_1(t,\xi):=E_2(t,t_\xi,\xi)Q(t,t_\xi,\xi)V_{1,0}(\xi)$$

solves (36) if $D_tQ=H(t,r,\xi)Q$. This follows from

$$\begin{aligned}&D_t(E_2Q)-DE_2Q-RE_2Q=0,\\&\underbrace{(D_tE_2)Q-DE_2Q}_{=0}+E_2D_tQ=RE_2Q.\end{aligned}$$

Knowing that $H(t,r,\xi)$ can be estimated by (39) we are able to estimate $Q=Q(t,r,\xi)$. We see that

$$\begin{aligned}\int_{t_\xi}^{t}\|H(s,t_\xi,\xi)\|ds&\lesssim\int_{t_\xi}^{t}1+\frac{\lambda(s)}{\Lambda(s)}+\frac{\lambda(s)}{\Lambda(s)^{1+1/p}|\xi|}\left(\log\frac{1}{\Lambda(s)}\right)^{1+1/p}ds\\&\lesssim 1|_{t_\xi}^{t}+\log\frac{1}{\Lambda(s)}\bigg|_{t}^{t_\xi}-\Lambda(s)^{-1/p}\left(\log\frac{1}{\Lambda(s)}\right)^{1+1/p}|\xi|^{-1}\bigg|_{t_\xi}^{t}\\&\le C\left(1+\log\frac{1}{\Lambda(t_\xi)}\right)\le C_{evo}\log|\xi|.\end{aligned}\tag{40}$$

This leads to

$$\|Q(t,t_\xi,\xi)\|\lesssim\exp\left(C\left(1+\log\frac{1}{\Lambda(t_\xi)}\right)\right)\le C|\xi|^{C_{evo}}.$$

Now we will estimate $|V_1(t,\xi)|$ and with the backward transformation we obtain an estimate for $|V(t,\xi)|$:

$$\begin{aligned}V_1(t,\xi)&=E_2(t,t_\xi,\xi)Q(t,t_\xi,\xi)V_{1,0}(\xi),\\|V_1(t,\xi)|&\le C\exp\left(C\left(1+\log\frac{1}{\Lambda(t_\xi)}\right)\right)|V_{1,0}(\xi)|,\\|V(t,\xi)|&=|M(t,\xi)V_1(t,\xi)|\\&\le C\exp\left(C\left(1+\log\frac{1}{\Lambda(t_\xi)}\right)\right)|M^{-1}(t_\xi,\xi)V(t_\xi,\xi)|\\&\le C\exp\left(C\left(1+\log\frac{1}{\Lambda(t_\xi)}\right)\right)|V(t_\xi,\xi)|.\end{aligned}$$

Summarizing we arrive in the evolution zone at the following estimates:

$$\begin{aligned}
\big|V(t,\xi)\big| &\le C|\xi|^{C_{evo}}\big|V(t_\xi,\xi)\big|,\\
\big(\lambda(t)|\xi|^p\big)^{l-1}\big|v(t,\xi)\big| &\le C|\xi|^{C_{evo}}\left(\sum_{j=1}^{l}\big(\lambda(t_\xi)|\xi|^p\big)^{l-j}\big|D_t^{j-1}v(t_\xi,\xi)\big|\right),\\
&\cdots\\
\big|D_t^{l-1}v(t,\xi)\big| &\le C|\xi|^{C_{evo}}\left(\sum_{j=1}^{l}\big(\lambda(t_\xi)|\xi|^p\big)^{l-j}\big|D_t^{j-1}v(t_\xi,\xi)\big|\right).
\end{aligned} \tag{41}$$

With this Lemma 5 is proved. □

6.3.5 Verification

Now we want to use the estimates of both zones to get an estimate for an arbitrary $t \in [0,T]$. For $t \le t_\xi$ we get an estimate in the pseudo-differential zone. Using the initial conditions we obtain

$$\begin{aligned}
\big|D_t^m v(t,\xi)\big| \le C\rho(t,\xi)^{-l+1+m}\exp\big(C\big(1+\log|\xi|\big)\big)&\\
\times\big(\big|v_0(\xi)\big|+\ldots+\big|v_{l-1}(\xi)\big|\big)&
\end{aligned} \tag{42}$$

for $m = 0,\ldots,l-1$. In the case $t \ge t_\xi$ we use the estimates from the evolution zone

$$\begin{aligned}
&\big|D_t^m v(t,\xi)\big|\\
&\le C\big(\lambda(t)|\xi|^p\big)^{-l+m+1}\exp\big(C\big(1+\log|\xi|\big)\big)\\
&\quad\times\left(\sum_{j=1}^{l}\big(\lambda(t_\xi)|\xi|^p\big)^{l-j}\big|D_t^{j-1}v(t_\xi,\xi)\big|\right)\\
&\le C\big(\lambda(t)|\xi|^p\big)^{-l+m+1}\exp\big(C\big(1+\log|\xi|\big)\big)\\
&\quad\times\left(\sum_{j=1}^{l}\big(\lambda(t_\xi)|\xi|^p\big)^{l-j}\rho(t_\xi,\xi)^{-l+j}\big(\big|v_0(\xi)\big|+\ldots+\big|v_{l-1}(\xi)\big|\big)\right)
\end{aligned} \tag{43}$$

for $m = 0,\ldots,l-1$. Now we use that $\rho(t,\xi)$ is larger 1 and the monotonicity of $\lambda(t)$. So it holds

$$\begin{aligned}
\big|D_t^m v(t,\xi)\big| &\le C\exp\big(C\big(1+\log|\xi|\big)\big)\\
&\qquad\times\sum_{j=1}^{l}\frac{(\lambda(t_\xi)|\xi|^p)^{l-j}}{(\lambda(t_\xi)|\xi|^p)^{l-m-1}}\big(\big|v_0(\xi)\big|+\ldots+\big|v_{l-1}(\xi)\big|\big)
\end{aligned}$$

$$\begin{aligned}
&\le C|\xi|^{s_0-(l-1)p}\frac{(\lambda(t_\xi)|\xi|^p)^{l-1}}{(\lambda(t_\xi)|\xi|^p)^{l-m-1}}\big(|v_0(\xi)|+\ldots+|v_{l-1}(\xi)|\big)\\
&\le C|\xi|^{s_0-(l-1)p}\big(\lambda(t_\xi)|\xi|^p\big)^m\big(|v_0(\xi)|+\ldots+|v_{l-1}(\xi)|\big)\\
&\le C|\xi|^{s_0-(l-1)p+mp}\big(|v_0(\xi)|+\ldots+|v_{l-1}(\xi)|\big)\\
&\le C|\xi|^{s_0+mp}\big(|\xi|^{-(l-1)p}|v_0(\xi)|+\ldots+|\xi|^{-(l-1)p}|v_{l-1}(\xi)|\big)\\
&\le C|\xi|^{s_0+mp}\big(|v_0(\xi)|+\ldots+|\xi|^{-(l-1)p}|v_{l-1}(\xi)|\big)
\end{aligned}$$

for $m = 0, \ldots, l-1$ and a constant s_0 which gives an (at most) finite loss of regularity. So our solution $D_t^m u(t,\cdot)$ is in $H^{s-s_0-mp}(\mathbb{R})$ if and only if

$$\int_{\mathbb{R}} \big|D_t^m v(t,\xi)\big|^2 |\xi|^{2(s-s_0-mp)} \mathrm{d}\xi < \infty.$$

It holds

$$\begin{aligned}
&\int_{\mathbb{R}} \big|D_t^m v(t,\xi)\big|^2 |\xi|^{2(s-s_0-mp)} \mathrm{d}\xi\\
&\quad\lesssim \int_{\mathbb{R}} |\xi|^{2s}\big(|v_0(\xi)|^2+\ldots+|\xi|^{-2(l-1)p}|v_{l-1}(\xi)|^2\big)\mathrm{d}\xi < \infty
\end{aligned}$$

by taking account of the regularity of the data. The continuity of solutions and their derivatives with respect to t follows from the continuity of $V = V(t,\xi)$ with respect to t in suitable function spaces in the phase space. This completes the proof of Theorem 1.

6.4 Outlook

This last section gives an outlook about further research and open problems.

6.4.1 About Optimality—C^1-Theory

One could pose the question, whether the assumptions on the degeneracy or the assumptions on the behavior of coefficients of the extended principal part near to $t = 0$ or on their oscillating behavior are sharp. Whether a loss really appears, whether this result is optimal. But there is not much to say about optimality results in C^1-theory. There are no results to prove the sharpness of the assumptions and there are no examples that show that this loss really appears. The control of the first derivative in t allows us to diagonalize the Fourier transformed system once. This yields a diagonal part and a remainder. But this remainder belongs to a symbol class which does not allow to apply methods for proving optimality. Another approach to show optimality for the C^1-theory is the a priori knowledge of reflection points or maximum points to get some kind of classification of oscillations. This is an attempt by Prof. Hirosawa from Yamaguchi University ([8], unpublished notes). For the x-dependent case there are no results about the sharpness of the decay rates for a p-evolution Cauchy problem. In [10] sharpness for decay rates has only been shown for the Cauchy problem to Schrödinger equations with time-independent coefficients of the form

$$i\partial_t u + \partial_x^2 u - a(x)\partial_x u = 0, \quad u(0,x) = u_0(x). \tag{44}$$

An open problem that might be attackable is the sharpness of the decay rates using the ideas of the mentioned paper.

6.4.2 About Optimality—C^2-theory

The advantage of a C^2-theory would be that we can diagonalize twice so that we get a remainder which is better in some hierarchies of symbol classes. A paper about C^2-theory for the p-evolution Cauchy problem of second order in D_t is in preparation, see [6] and [7].

6.4.3 About x-Dependence—C^1-Theory

Here we want to consider the p-evolution Cauchy problem (2), where the coefficients $a_{j,k}$ may depend on space and time. The first thing we can do is to try to include x-dependence in a way that we can generalize the result for the pure time-dependent model without the need of more assumptions on the coefficients except the boundedness of the coefficients and of its derivatives with respect to the spatial variable. This is only possible for the coefficients $a_{j,k}$ of the extended principal part with the lowest order $j + \frac{k}{p} = l - 1 + \frac{1}{p}$ and for the terms of lower order. We consider the p-evolution Cauchy problem of higher order in D_t with coefficients depending on space and time as follows:

$$\begin{aligned} D_t^l u - &\sum_{\substack{l-1+1/p<j+k/p\le l \\ j<l}} a_{j,k}(t) D_x^k D_t^j u \\ &- \sum_{0\le j+k/p\le l-1+1/p} a_{j,k}(t,x) D_x^k D_t^j u = 0, \\ D_t^m u(0,x) = u_m &\quad \text{for } m = 0,\dots,l-1 \text{ and } l \ge 2. \end{aligned} \tag{45}$$

All coefficients are real and in $B^\infty(\mathbb{R})$ with respect to x.

Theorem 2 *Let us consider the Cauchy problem* (45) *under the assumptions* (8) *and* (10) *to* (16). *For initial data* $u_m \in H^{s-mp}(\mathbb{R})$, $m = 0, \dots, l-1$, *there exists a non-negative constant* s_0 *and a unique solution* $u \in C([0,T], H^{s-s_0}(\mathbb{R})) \cap C^1([0,T], H^{s-s_0-p}(\mathbb{R})) \cap \dots \cap C^{l-1}([0,T], H^{s-s_0-(l-1)p}(\mathbb{R}))$. *An a priori estimate for the solution is given by*

$$\left\| D_t^m u(t,\cdot) \right\|_{H^{s-s_0-mp}} \le C\big(\|u_0\|_{H^s} + \dots + \|u_{l-1}\|_{H^{s-(l-1)p}} \big)$$

for $m = 0, \dots, l-1$.

Remark 8 It is important to understand that the only difference in the Theorems 1 and 2 is the x-dependence of some coefficients, but this brings a complete change in the proof. We can not use the partial Fourier transformation with respect to x. We need cut-off functions techniques which help to localize the considerations to the needed zones. Moreover, we should apply methods basing on a pseudo-differential calculus.

If we include decay conditions of the coefficients with respect to x, then we can consider x-dependence for almost all coefficients. We can consider

$$\begin{aligned} &D_t^l u - \sum_{\substack{j+k/p=l \\ j<l}} a_{j,k}(t) D_x^k D_t^j u - \sum_{0 \le j+k/p \le l-1/p} a_{j,k}(t,x) D_x^k D_t^j u = 0, \\ &D_t^m u(0,x) = u_m(x) \quad \text{for } m = 0, \dots, l-1 \text{ and } l \ge 2. \end{aligned} \tag{46}$$

We propose the following decay conditions which are related to the conditions in [2]:

$$\begin{aligned} \left| D_x a_{j,(l-j)p-k}(t,x) \right| &\le C \lambda(t)^{l-j} \langle x \rangle^{-(p-k-1)/(p-1)}, \\ \left| D_x^\beta a_{j,(l-j)p-k}(t,x) \right| &\le C \lambda(t)^{l-j} \langle x \rangle^{-(p-k-[\beta/2])/(p-1)} \end{aligned} \tag{47}$$

for $2 \le \beta < 2(p-k)$, $j = 0, \dots, l-1$ and $k = 1, \dots, p-2$.

Hypothesis Let us consider the Cauchy problem (46) under the assumptions (8), (10) to (16) and (47). For initial data $u_m \in H^{s-mp}(\mathbb{R})$, $m = 0, \dots, l-1$ there exists a non-negative constant s_0 and a unique solution $u \in C([0,T], H^{s-s_0}(\mathbb{R})) \cap C^1([0,T], H^{s-s_0-p}(\mathbb{R})) \cap \dots \cap C^{l-1}([0,T], H^{s-s_0-(l-1)p}(\mathbb{R}))$. An a priori estimate for the solution is given by

$$\left\| D_t^m u(t,\cdot) \right\|_{H^{s-s_0-mp}} \le C\big(\|u_0\|_{H^s} + \dots + \|u_{l-1}\|_{H^{s-(l-1)p}} \big)$$

for $m = 0, \dots, l-1$.

Remark 9 We can also extend the calculus to Cauchy problem (46) with complex-valued coefficients depending on t and x. For the theorem to hold we need a decay for the imaginary part. We would propose the following assumptions:

$$\left|\Im a_{j,(l-j)p-k}(t,x)\right| \le C\lambda(t)^{l-j}\langle x\rangle^{-(p-k)/(p-1)} \tag{48}$$

for $j = 0, \ldots, l-1$ and $k = 2, \ldots, p-1$. Furthermore we pose an assumption for the imaginary part of $a_{j,(l-j)p-1}$ in the following way:

$$\left|\Im a_{j,(l-j)p-1}(t,x)\right| \le C\lambda(t)^{l-j} g\big(\langle x\rangle\big), \tag{49}$$

where the function $g = g(s) \in L^1(\mathbb{R}_+) \cap C[0,\infty)$ is a strictly decreasing function.

For a better understanding of the influence coming from the imaginary parts of the coefficients see [4].

6.4.4 About x-Dependence—C^2-Theory

If we merge the last results we can get a result for a Cauchy problem similar to (46). We want to propose a hypothesis for the following Cauchy problem:

$$\begin{aligned} &D_t^l u - \lambda(t)^l b(t)^l D_x^{lp} u - \sum_{0\le j+k/p\le l-1/p} a_{j,k}(t,x) D_x^k D_t^j u = 0,\\ &D_t^m u(0,x) = u_m(x) \quad \text{for } m = 0, \ldots, l-1 \text{ and } l \ge 2. \end{aligned} \tag{50}$$

We have a special choice for the principal part in the sense of Petrowsky due to the interactions in the principal part in the sense of Petrowsky shown in [9] for a strictly hyperbolic problem. The coefficients $a_{j,k}(t,x)$ are considered to be complex. We consider a shape function $\lambda(t)$ which satisfies

$$\begin{aligned} &\lambda(0) = 0, \qquad \lambda'(t) > 0 \quad \text{for } t > 0,\\ &d_0\frac{\lambda(t)}{\Lambda(t)} \le \frac{\lambda'(t)}{\lambda(t)} \le d_1\frac{\lambda(t)}{\Lambda(t)}, \quad d_0 > \frac{l-1}{l},\\ &\left|D_t^2\lambda(t)\right| \le d_2\lambda(t)\left(\frac{\lambda(t)}{\Lambda(t)}\right)^2. \end{aligned} \tag{51}$$

The function $b(t)$ describes the oscillating behavior of the coefficient and we assume

$$\begin{aligned} &c_0 := \inf_{t\in(0,T]} b(t) \le b(t) \le c_1 := \sup_{t\in(0,T]} b(t), \quad t \in (0,T], c_0, c_1 > 0,\\ &\left|D_t^m b(t)\right| \le C\left(\frac{\lambda(t)}{\Lambda(t)}\nu(t)\right)^m, \quad m = 1, 2. \end{aligned} \tag{52}$$

For the coefficients we pose the assumptions

$$\left|D_t^m a_{j,k}(t,x)\right| \le C_m \lambda(t)^{l-j} \left(\frac{\nu(t)}{\Lambda(t)}\right)^{l-j-k/p} \left(\frac{\lambda(t)}{\Lambda(t)}\nu(t)\right)^m \tag{53}$$

for $m = 0, 1, 2$. For the terms of Levi size we need the additional Levi conditions

$$\left|D_t^m \Im a_{l-1-k/p,k}(t)\right| \le C_m \lambda(t)^{k/p} \left(\frac{\lambda(t)}{\Lambda(t)}\nu(t)\right)^{m+1} \tag{54}$$

for $m = 0, 1, 2$. In some of the assumptions we used a function $\nu = \nu(t)$, which is a positive and strictly decreasing function. Furthermore, for the function $\nu(t)$ we need the assumption

$$\frac{\lambda(t)}{\Lambda(t)}\nu(t) \gg -\nu'(t). \tag{55}$$

Furthermore, we propose decay conditions

$$\left|D_t^m D_x a_{j,(l-j)p-k}(t,x)\right| \le C\lambda(t)^{l-j} \langle x\rangle^{-(p-k-1)/(p-1)}, \tag{56}$$

$$\left|D_t^m D_x^\beta a_{j,(l-j)p-k}(t,x)\right| \le C\lambda(t)^{l-j} \langle x\rangle^{-(p-k-[\beta/2])/(p-1)} \tag{57}$$

for $2 \le \beta < 2(p-k)$, $j = 0, \dots, l-1$, $k = 1, \dots, p-2$, $m = 0, 1$ and

$$\left|D_t^m \Im a_{j,(l-j)p-k}(t,x)\right| \le C\lambda(t)^{l-j} \langle x\rangle^{-(p-k)/(p-1)}, \tag{58}$$

$$\left|D_t^m \Im a_{j,(l-j)p-1}(t,x)\right| \le C\lambda(t)^{l-j} g\big(\langle x\rangle\big) \tag{59}$$

for $j = 0, \dots, l-1$, $k = 2, \dots, p-1$, $m = 0, 1$, where the function $g = g(s) \in L^1(\mathbb{R}_+) \cap C[0,\infty)$ is a strictly decreasing function.

Hypothesis Let us consider the Cauchy problem (50) under the assumptions (51) to (59). For initial data $u_0 \in H^s$ and u_m, $m = 1, \dots, l-1$ in appropriate spaces, then there exists a unique solution $u = u(t,x)$ with the properties

$$u(t,\cdot) \in \exp\left(C\nu\left(\left(\frac{\Lambda}{\nu}\right)^{(-1)}\left(\frac{N}{\langle D_x\rangle^p}\right)\right)\right) H^s(\mathbb{R}),$$

where N is a suitable positive constant. The loss of regularity of the solution is described by

$$\exp\left(C\nu\left(\left(\frac{\Lambda}{\nu}\right)^{(-1)}\left(\frac{N}{\langle D_x\rangle^p}\right)\right)\right).$$

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Chapter 7
On the Global Solvability for Semilinear Wave Equations with Smooth Time Dependent Propagation Speeds

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Abstract In this paper we consider the global existence of a solution with small data to the Cauchy problem for the semilinear wave equations with time dependent coefficient:

$$u_{tt} - a(t)^2 \Delta u = u_t^2 - a(t)^2 |\nabla u|^2.$$

We are particularly interested in the effects of the smoothness to the coefficients in the sense of C^m and the Gevrey classes, and the main theorems are natural generalization of the previous results for less smoothness of coefficients.

Mathematics Subject Classification 35L70 · 35B40

7.1 Introduction

The main purpose of this paper is to prove some results for the existence of the global solutions to the Cauchy problem for the semilinear wave equations with variable propagation speed:

$$\left(\partial_t^2 - a(t)^2 \Delta\right) u = F(t, \partial_t u, \nabla u), \tag{1}$$

where $\Delta = \sum_{j=1}^n \partial_{x_j}^2$ and $\nabla = (\partial_{x_1}, \dots, \partial_{x_n})$. There are a lot of papers which consider the global existence and non-existence of the solution to the following Cauchy

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problem of semi-linear wave equations:

$$\left(\partial_t^2 - \Delta\right)u = F(\partial_t u, \nabla u), \tag{2}$$

for instance [3, 8, 9] and references therein. The wave equation with time dependent propagation speed:

$$\left(\partial_t^2 - a(t)^2 \Delta\right)w = 0 \tag{3}$$

has been studied as a linearized model of the Kirchhoff equation, which describes the vibration of an elastic string (see [13]). On the other hand, (3) has been studied of interest in themselves for the possibility of arising a singular behavior of the solution from non-constant propagation speed. Indeed, the natural properties to the wave equations with constant propagation speed such as energy conservation, well-posedness and L^p-L^q decay estimates are not always true for time dependent propagation speed (see [1, 11] and [12]). Generally, the problems of global existence of the solution to the non-linear equations are deeply related to the properties of the solutions to the linear equations. Consequently, it is expected that not all the results for the global existence of (2) cannot be naturally generalized to the problem for (1).

The following Cauchy problem of a special semi-linear model with a time dependent coefficient was studied in [14] and [15]:

$$\begin{cases} (\partial_t^2 - a(t)^2 \Delta)u = (\partial_t u)^2 - a(t)^2 |\nabla u|^2, & (t,x) \in (0,\infty) \times \mathbb{R}^n, \\ u(0,x) = u_0(x), \qquad (\partial_t u)(0,x) = u_1(x), & x \in \mathbb{R}^n, \end{cases} \tag{4}$$

where u is real valued and $u_0, u_1 \in C_0^\infty(\mathbb{R}^n)$. Equation (4) can be reduced to the linear equation (3) by Nirenberg's transformation

$$w(t,x) = 1 - \exp\bigl(-u(t,x)\bigr),$$

and thus the influence of the time dependent coefficient on properties of the non-linear models can be understood. In this paper we restrict ourselves our consideration into the Cauchy problem (4). Here we briefly introduce the previous results for the global solvability of (4). It is studied in [15] that if $n \ge 2$ and $a(t)$ is a positive, non-constant and periodic function, then (4) has no global solution for any small initial data. On the other hand, if $a(t)$ is given by

$$a(t) = b(t) \exp\bigl(t^\alpha\bigr) \tag{5}$$

for

$$\alpha > \frac{1}{2}, \tag{6}$$

where $b(t) \in C^2(\mathbb{R})$ is a positive periodic function, then (4) has a global solution for small data. In particular, if $b(t)$ is a constant, then the global solvability of (4) is valid for any $\alpha > 0$. We see from these facts that oscillations of the coefficients might have a bad influence on the global solvability. On the other hand, the increasing property is possible to conclude the global solvability against the oscillations of the coefficient. Indeed, if we restrict ourselves that $a(t)$ is decomposed into the increasing part $\lambda(t)$ and the oscillating part $b(t)$ by

$$a(t) = \lambda(t)b(t), \tag{7}$$

where $\lambda(t)$ is a positive increasing function, and $b(t)$ satisfies

$$b_0 \le b(t) \le b_1$$

for some positive constants b_0 and b_1, then the condition (6) in (5) for the global solvability can be generalized as follows:

Theorem 1 ([2, 15]) *Let $n \ge 2$ and $a(t) \in C^2([0,\infty))$ be represented by (7). If $\lambda(t)$ and $b(t)$ satisfy*

$$\left|\lambda^{(k)}(t)\right| \le C_k \lambda(t) \left(\frac{\lambda(t)}{\Lambda(t)}\right)^k \tag{8}$$

and

$$\left|b^{(k)}(t)\right| \le C_k \left(\frac{\lambda(t)}{\Lambda(t)}\left(\log \Lambda(t)\right)^\beta\right)^k \tag{9}$$

for $t \ge 1$, $k = 1, 2$ and $\beta < 1$, where

$$\Lambda(t) = \int_0^t \lambda(\tau)d\tau,$$

and C_k are positive constants, then (4) has a global solution for small data. If $n = 1$, then the same conclusion is valid for $\beta = 0$ under the additional assumption

$$\int_0^1 \Lambda^{-1}\left(\frac{1}{r}\right)dr < \infty.$$

Remark 1 If n is large enough, then the conclusion of Theorem 1 is valid for $\beta = 1$ (see [15]).

Remark 2 We come to the condition (6) by applying Theorem 1 for a C^2 periodic function $b(t)$, $\lambda(t) = \exp(t^\alpha)$ and $\Lambda(t) \approx t^{1-\alpha}\lambda(t)$ as $t \to \infty$, where $f \approx g$ denotes that for positive functions f and g there exists a constant $C > 1$ such that $C^{-1} f \le g \le Cf$. We are interested in the asymptotic behavior as $t \to \infty$, hence all the assumptions from below are implicitly supposed that t is large enough.

Remark 3 For $n = 1$ the optimality of the assumption (9) with $\beta = 0$ for the global solvability with small data to (4) is proved in [2].

Remark 4 We shall denote universal positive constants by C and C_k with $k = 0, 1, \ldots$ without making any confusion.

If $k = 1$, then (9) is a restriction to the oscillating speed of $b(t)$ with the parameter β. In the previous research for the linear equation (3) and the non-linear equation (4) with $a(t) = \lambda(t)b(t)$, it is known that $\beta = 0$ and $\beta = 1$ are critical numbers for the behavior of the solutions; for instance the properties of the well-posedness, the energy estimates and the L^p-L^q decay estimates are changed at $\beta = 0$ or $\beta = 1$ (see [4, 11] and [12], for instance). For this reason if $\beta = 0$, $0 < \beta < 1$ or $\beta = 1$ in (9) we say that $b(t)$ has very slow oscillation, slow oscillation, or fast oscillation respectively. Thus the classification of the properties to the coefficients by (9) seems to be reasonable. However, Theorem 1 doesn't describe any effect from further smoothness of the coefficients than C^2.

The assumptions of Theorem 1 are described by essentially two properties of the coefficient: increasing and oscillating behaviors; the first, and the second ones have a good, and a bad influence for the stability of the solution. In the following we denote the increasing behavior of $a(t)$ by $\lambda(t)$, which satisfies

$$\lambda(t) \in C^1\big([0,\infty)\big), \qquad \lambda'(t) \geq 0, \qquad \lambda(0) > 0 \tag{10}$$

and

$$C_0\lambda(t) \leq a(t) \leq C_1\lambda(t). \tag{11}$$

Then we can prove the same conclusion of Theorem 1 replacing the assumptions (8) and (9) by

$$\big|a^{(k)}(t)\big| \leq C_k\lambda(t)\left(\frac{\lambda(t)}{\Lambda(t)}\big(\log\Lambda(t)\big)^{\beta}\right)^{k} \tag{12}$$

for $k = 1, 2$. Then we introduce a new condition which describes the gap between the oscillating coefficient $a(t)$ and the monotone increasing behavior $\lambda(t)$ as follows:

$$\int_0^t \big|a(\tau) - \lambda(\tau)\big|d\tau = \Theta(t) = o\big(\Lambda(t)\big) \quad (t \to \infty). \tag{13}$$

Indeed, the condition (13) is introduced in [7] to bring a benefit of further smoothness of the coefficient. Then one of our main theorems is represented as follows:

Theorem 2 *Let $m \geq 2$ and $a(t) \in C^m([0,\infty))$. If there exists $\lambda(t)$ satisfying* (10), (11), (13) *and*

$$\int_0^1 r^{n-1}\Lambda^{-1}\left(\frac{1}{r}\right)dr < \infty \tag{14}$$

such that

$$\Lambda(t)^{\varepsilon_0} = O\big(\Theta(t)\big) \quad (t \to \infty) \tag{15}$$

for a positive constant ε_0, and

$$\big|a^{(k)}(t)\big| \leq C_k\lambda(t)\left(\frac{\lambda(t)}{\Theta(t)}\left(\frac{\Theta(t)}{\Lambda(t)}\right)^{1/m}\big(\log\Lambda(t)\big)^{\beta}\right)^{k}, \quad k = 1,\ldots,m \tag{16}$$

for $\beta = 0$ *if* $n = 1$, *and* $\beta < 1$ *if* $n \geq 2$ *respectively, then* (4) *has a global solution for small data.*

Remark 5 If $m = 2$, then a corresponding result of Theorem 2 is proved in [2]. In contrast with it Theorem 2 shows that further regularity of the coefficient, that is, $m \geq 3$, contributes for the global solvability of (4) with faster oscillating coefficient.

Remark 6 The assumption (14) is trivial if $n \geq 2$. Indeed, noting $\Lambda(t) \geq \lambda(0)t$, we have

$$\int_0^1 r^{n-1} \Lambda^{-1}\left(\frac{1}{r}\right) dr \leq \frac{1}{\lambda(0)} \int_0^1 r^{n-2} dr < \infty.$$

Remark 7 The condition $\Theta(t) = O(\Lambda(t))$ is trivially satisfied. Then the condition (16) corresponds to (12) without the assumption (13). Consequently, Theorem 2 is a generalization of Theorem 1.

Remark 8 The conditions of (16) are weaker than (12) for $k = 1, 2$. This means that under the additional assumption (13) the global solvability of (4) can be valid for faster oscillating coefficient.

Here we introduce an example of $a(t)$ that the global solvability can be proved by using our theorem but not by the previous results. Let $n \geq 2$. For real numbers $0 < \alpha \leq 1$ and $\gamma > 0$ we define $a(t) = \lambda(t)b(t)$ as follows:

$$\lambda(t) = \exp\bigl(t^\alpha\bigr), \quad \alpha > 0,$$

and

$$b(t) = \begin{cases} p(t) & t \in I_j = [j^{1/\alpha}, j^{1/\alpha} + 1], \\ 1 & t \in [0, \infty) \setminus \bigcup_{j=1}^{\infty} I_j, \end{cases}$$

where $p(t) \in C^m([0, \infty))$ is a positive and 1-periodic function satisfying $p(0) = 1$ and $p^{(k)}(t) \equiv 0$ near $t = 0$ for $k = 1, \dots, m$. Then, noting $|b^{(k)}| \leq C_k$ and $|\lambda^{(k)}(t)| \leq C_k \lambda(t) t^{-k(1-\alpha)}$ we have

$$\bigl|a^{(k)}(t)\bigr| \leq C_k \lambda(t).$$

On the other hand, for $t \in [j^{1/\alpha}, (j+1)^{1/\alpha}]$ we have

$$\int_0^t \bigl|a(\tau) - \lambda(\tau)\bigr| d\tau \approx \sum_{k=1}^{j} \int_{k^{1/\alpha}}^{k^{1/\alpha}+1} \mathrm{e}^{\tau^\alpha} d\tau \approx \sum_{k=1}^{j} \mathrm{e}^k \approx \mathrm{e}^j = e^{t^\alpha} = \lambda(t).$$

Hence, we see that $\Theta(t) \approx \lambda(t)$ and $\Theta(t) = o(\Lambda(t)) = o(t^{1-\alpha}\lambda(t))$. Noting $\log \Lambda(t) \approx \log \lambda(t) = t^\alpha$, the conditions (12), and (16) with $\beta < 1$ are given by

$$\alpha > \begin{cases} \alpha_0 := \frac{1}{2} & \text{from (12)}, \\ \alpha_m := \frac{1}{m+1} & \text{from (16)}. \end{cases}$$

Then we observe that $\alpha_1 = \alpha_0$, α_m is monotone decreasing with respect to m, and $\lim_{m\to\infty}\alpha_m = 0$; thus we can conclude the global solvability of (4) for less increasing coefficients as m is larger. Generally, the conditions (16) are weaker than (12) for $k = 1, 2$ as m is larger.

By (13) and (15) we have

$$\log \Lambda(\tau) = \frac{\log(\Lambda(t)/\Theta(t))}{1 - \log\Theta(t)/\log\Lambda(t)} \le 2\log\frac{\Lambda(t)}{\Theta(t)}$$

for any large t. Therefore, for any $\tilde{\gamma} > \gamma > 0$ there exists a positive constant C such that

$$\left(\frac{\Theta(t)}{\Lambda(t)}\right)^{\tilde{\gamma}} \big(\log\Lambda(t)\big)^{\beta} \le C\left(\frac{\Theta(t)}{\Lambda(t)}\right)^{\gamma}.$$

Consequently, the following corollary immediately follows from Theorem 2:

Corollary 1 *Let $a(t) \in C^\infty([0,\infty))$. If there exist a positive constant γ and a function $\lambda(t)$ satisfying (10), (11), (13), (14), (15) and*

$$\left|a^{(k)}(t)\right| \le C_k\lambda(t)\left(\frac{\lambda(t)}{\Theta(t)}\left(\frac{\Theta(t)}{\Lambda(t)}\right)^{\beta}\right)^{k}, \quad k \in \mathbb{N} \tag{17}$$

for $\beta = 0$ if $n = 1$, and $\beta < 1$ if $n \ge 2$ respectively, then (4) has a global solution for small data.

Remark 9 The optimality of the assumption (16) for each fixed m is an open problem. Incidentally, for $a(t) \in C^\infty([0,\infty))$ the counter example in [6] implies that the global solvability of (4) is not valid if we change the assumption to β of (17) into $\beta < 0$.

Now we come to the following new critical order of $a^{(k)}(t)$ as $m \to \infty$ for the global solvability:

$$\left|a^{(k)}(t)\right| \le C_k\lambda(t)\left(\frac{\lambda(t)}{\Theta(t)}\right)^{k}, \quad k \in \mathbb{N}.$$

Actually, Corollary 1 does not give us any answer about the asymptotic as $m \to \infty$ for general C^∞ functions $a(t)$. However, if we introduce the Gevrey class as a subclass of C^∞-class, then we have the following theorem, which is a refinement of Corollary 1:

Theorem 3 *Let $a(t) \in C^\infty([0,\infty))$ and $\nu \ge 1$. If there exist a positive constant ρ_0 and a function $\lambda(t)$ satisfying (10), (11), (13), (14), (15) and*

$$\left|a^{(k)}(t)\right| \le k!^{\nu}\lambda(t)\left(\rho_0\frac{\lambda(t)}{\Theta(t)}\left(\log\frac{\Lambda(t)}{\Theta(t)}\right)^{-\nu}\big(\log\Lambda(t)\big)^{\beta}\right)^{k}, \quad k \in \mathbb{N} \tag{18}$$

for $\beta = 0$ *if* $n = 1$, *and* $\beta < 1$ *if* $n \geq 2$ *respectively, then* (4) *has a global solution for small data.*

Remark 10 If $f(t) \in C^\infty([0,\infty))$ satisfies $|f^{(k)}(t)| \leq \rho^k k!^\nu$ for any $k \in \mathbb{N}$ with real numbers $\rho > 0$ and $\nu > 0$, then we say that f belong to the Gevrey class of order ν. In particular, if the estimates hold for $\nu = 1$, then f is a real analytic function.

Let us introduce an example for a coefficient $a(t)$ that the global solvability can be proved by using Theorem 3 but not Corollary 1. Let $\kappa > 1$, $r > 1$, $\lambda(t) = \exp((\log t)^\kappa)$ for $t \geq 1$ and $b(t)$ is defined by

$$b(t) = \begin{cases} p(t) & t \in I_j = [r^j, r^j + 1], \\ 1 & t \in [0,\infty) \setminus \bigcup_{j=1}^\infty I_j, \end{cases}$$

where $p(t) \in C^\infty([0,\infty))$ is a positive and 1-periodic function satisfying $p(0) = 1$, $p^{(k)}(t) \equiv 0$ near $t = 0$ for $k \in \mathbb{N}$ and $|p^{(k)}(t)| \leq \rho^k k!^\nu$ on $[0, 1]$. Then we see that there exists a positive constant ρ_0 such that

$$\left|a^{(k)}(t)\right| \leq \rho_0^k k!^\nu \lambda(t).$$

On the other hand, for $t \in [r^j, r^{j+1}]$ we have

$$\begin{aligned}\int_0^t \left|a(\tau) - \lambda(\tau)\right| d\tau &\approx \sum_{k=1}^{j} \int_{r^j}^{r^j+1} \exp\big((\log\tau)^\kappa\big) d\tau \approx \sum_{k=1}^{j} \exp\Big(\big(\log r^k\big)^\kappa\Big) \\ &\approx j^{1-\kappa} \exp\Big(\big(\log r^j\big)^\kappa\Big) \approx (\log t)^{1-\kappa}\lambda(t),\end{aligned}$$

hence, we have $\Theta(t) \approx (\log t)^{1-\kappa}\lambda(t)$ and $\Theta(t) = o(\Lambda(t)) = o(t(\log t)^{1-\kappa}\lambda(t))$. Noting $\log \Lambda(t) \approx (\log t)^\kappa$, the conditions (18) with $\beta < 1$ is given by

$$\kappa > \frac{\nu + 1}{2}.$$

Thus we observe that the increasing order $\lambda(t)$ describing by κ is smaller for smaller ν, and the limit case is of polynomial order.

7.2 Proof of Theorem 2

7.2.1 Division of the Phase Space

Let $w(t, x)$ be the solution to the following linear Cauchy problem, which is reduced from (4) by Nirenberg's transformation:

$$\begin{cases} (\partial_t^2 - a(t)^2 \Delta) w = 0, & (t, x) \in (0,\infty) \times \mathbb{R}^n, \\ w(0, x) = w_0(x), \quad (\partial_t w)(0, x) = w_1(x), & x \in \mathbb{R}^n. \end{cases} \tag{19}$$

If the solution to the linear problem (19) satisfies

$$\sup_{(t,\xi)\in[0,\infty)\times\mathbb{R}^n}\left\{\left|w(t,x)\right|\right\} \le C\left(\left\|\langle D\rangle^s w_0(\cdot)\right\|_{L^1} + \left\|\langle D\rangle^{s-1} w_1(\cdot)\right\|_{L^1}\right) \tag{20}$$

for some $s > 1$, where $\langle\xi\rangle = \sqrt{1+|\xi|^2}$ for $\xi \in \mathbb{R}^n$ and $D = -\mathrm{i}\nabla$, then the global solvability of (4) for small data $u_0, u_1 \in C_0^\infty(\mathbb{R}^n)$ is immediately concluded.

Let us define $v(t,\xi) = \hat{w}(t,\xi)$, where $\hat{w}(t,\xi)$ denotes the partial Fourier transformation of $w(t,x)$ with respect to $x \in \mathbb{R}^n$. Then (19) is reduced to the following Cauchy problem for $v(t,\xi)$:

$$\begin{cases} (\partial_t^2 + a(t)^2|\xi|^2)v = 0, & (t,\xi) \in (0,\infty)\times\mathbb{R}^n, \\ v(0,\xi) = v_0(\xi), \quad (\partial_t v)(0,\xi) = v_1(\xi), & \xi \in \mathbb{R}^n. \end{cases} \tag{21}$$

By the assumption (15) we see that $\Theta(t) + \Lambda(t)^\varepsilon$ is strictly increasing and $\Theta(t) + \Lambda(t)^\varepsilon \approx \Theta(t)$ for any $\varepsilon \le \varepsilon_0$. Therefore, we can suppose that $\Theta(t) + \Lambda(t)^\varepsilon$ is strictly increasing to regard $\Theta(t)$ as $\Theta(t) + \Lambda(t)^\varepsilon$ from now on.

Let N and T_0 be large constants. We define the constant d by

$$d = \frac{N(\log \Lambda(T_0))^\beta}{\Theta(T_0)}. \tag{22}$$

Here we note that d can be arbitrarily small if T_0 is large enough by (15). For $|\xi| < d$ we define $\tau_0 = \tau_0(\xi)$, and $t_0 = t_0(\xi)$ implicitly by

$$\tau_0\lambda(\tau_0)|\xi| = N, \tag{23}$$

and

$$\Theta(t_0)|\xi| = N\left(\log \Lambda(t_0)\right)^\beta,$$

respectively.

Here we note that $t_0(\xi) \ge t_0(\xi)|_{|\xi|=d} = T_0$. Then we have the following lemma:

Lemma 1 *There exists a positive constant C such that*

$$1 < \frac{t_0(\xi)}{\tau_0(\xi)} \le C$$

for any $|\xi| \le d$.

Proof Since $\lambda(t)$ is monotone increasing we have

$$\frac{\Lambda(\tau_0)}{\Lambda(t_0)} \le \frac{\tau_0\lambda(\tau_0)}{\Lambda(t_0)} = \frac{\Theta(t_0)(\log\Lambda(t_0))^{-\beta}}{\Lambda(t_0)} \le \frac{\Theta(t_0)}{\Lambda(t_0)} \to 0 \quad (t_0 \to \infty),$$

it follows that $\tau_0 < t_0$. □

We shall estimate the solution of (21) independently in the following four zones of the phase space:

- the pseudo-differential zone Z_Ψ:

$$Z_\Psi = \{(t,\xi) \in [0,\infty) \times \mathbb{R}^n; 0 \le t \le \tau_0(\xi), |\xi| \le d\};$$

- the stabilized zone Z_S:

$$Z_S = \{(t,\xi) \in [0,\infty) \times \mathbb{R}^n; \tau_0(\xi) \le t \le t_0(\xi), |\xi| \le d\};$$

- the zone in finite time Z_F:

$$Z_F = \{(t,\xi) \in [0,\infty) \times \mathbb{R}^n; 0 \le t \le T_0, |\xi| \ge d\};$$

- the hyperbolic zone Z_H:

$$Z_H = \{(t,\xi) \in [0,\infty) \times \mathbb{R}^n; t \ge \max\{T_0, t_0(\xi)\}\}.$$

7.2.2 Estimates in Z_Ψ, Z_S and Z_F

Let $(t,\xi) \in Z_\Psi$. We define the energy $\mathcal{E}_0(t,\xi)$ by

$$\mathcal{E}_0(t,\xi) = \frac{1}{2}\big(\lambda(\tau_0)^2|\xi|^2|v(t,\xi)|^2 + |v_t(t,\xi)|^2\big).$$

Differentiating $\mathcal{E}_0(t,\xi)$ with respect to t and using (11) we have

$$\partial_t \mathcal{E}_0(t,\xi) = \big(\lambda(\tau_0)^2 - a(t)^2\big)|\xi|^2 \Re\{v\overline{v_t}\} \le C\lambda(\tau_0)|\xi|\mathcal{E}_0(t,\xi).$$

It follows from Gronwall's inequality and (23) that

$$\mathcal{E}_0(t,\xi) \le \exp\big(C\tau_0\lambda(\tau_0)|\xi|\big)\mathcal{E}_0(0,\xi) = \mathrm{e}^{CN}\mathcal{E}_0(0,\xi). \tag{24}$$

Let $(t,\xi) \in Z_S$. We define the energy $\mathcal{E}_1(t,\xi)$ by

$$\mathcal{E}_1(t,\xi) = \frac{1}{2}\big(\lambda(t)^2|\xi|^2|v(t,\xi)|^2 + |v_t(t,\xi)|^2\big).$$

Differentiating $\mathcal{E}_1(t,\xi)$ with respect to t and using (11) we have

$$\begin{aligned}\partial_t \mathcal{E}_1(t,\xi) &= \big(\lambda(t)^2 - a(t)^2\big)|\xi|^2 \Re\{v\overline{v_t}\} + \lambda'(t)\lambda(t)|\xi|^2|v(t,\xi)|^2 \\ &\le \left(C|\lambda(t) - a(t)||\xi| + \frac{2\lambda'(t)}{\lambda(t)}\right)\mathcal{E}_1(t,\xi).\end{aligned}$$

Let $n \geq 2$. By Gronwall's inequality we have

$$\begin{aligned}\mathcal{E}_1(t,\xi) &\leq \exp\Big(\int_{\tau_0}^{t} \frac{2\lambda'(\tau)}{\lambda(\tau)}d\tau + C|\xi|\int_{\tau_0}^{t}\big|a(\tau)-\lambda(\tau)\big|d\tau\Big)\mathcal{E}_1(\tau_0,\xi)\\ &\leq \frac{\lambda(t)^2}{\lambda(\tau_0)^2}\exp\big(C\Theta(t)|\xi|\big)\mathcal{E}_1(\tau_0,\xi)\\ &\leq \frac{\lambda(t)^2}{\lambda(\tau_0)^2}\exp\big(CN\big(\log \Lambda(t_0)\big)^{\beta}\big)\mathcal{E}_1(\tau_0,\xi)\\ &= \frac{\lambda(t)^2}{\lambda(\tau_0)^2}\Lambda(t_0)^{CN/(\log \Lambda(t_0))^{1-\beta}}\mathcal{E}_1(\tau_0,\xi).\end{aligned}$$

Therefore, for any small positive number κ we have

$$\mathcal{E}_1(t,\xi) \leq \frac{\lambda(t)^2}{\lambda(\tau_0)^2}\Lambda(t_0)^{\kappa}\mathcal{E}_0(\tau_0,\xi) \tag{25}$$

by choosing T_0 large enough. If $n = 1$, then the estimate (25) with $\kappa = 0$ is trivial since $\beta = 0$.

Let $(t,\xi) \in Z_F$. We define the energy $\mathcal{E}(t,\xi)$ by

$$\mathcal{E}(t,\xi) = \frac{1}{2}\big(a(t)^2|\xi|^2\big|v(t,\xi)\big|^2 + \big|v_t(t,\xi)\big|^2\big).$$

Differentiating $\mathcal{E}(t,\xi)$ with respect to t we have

$$\partial_t\mathcal{E}(t,\xi) = a'(t)a(t)|\xi|^2|v|^2 \leq \frac{2|a'(t)|}{a(t)}\mathcal{E}(t,\xi).$$

By Gronwall's inequality we have

$$\mathcal{E}(t,\xi) \leq \exp\Big(2T_0 \max_{0\leq t\leq T_0}\Big\{\frac{|a'(t)|}{a(t)}\Big\}\Big)\mathcal{E}(0,\xi). \tag{26}$$

Summarizing the estimates (24), (25) and (26), and using (11) we have the following lemma:

Lemma 2 *Let $n \geq 2$. For any positive small constant κ there exist positive constants N, T_0 and C such that the following estimates hold*:

$$\mathcal{E}_0(t,\xi) \leq C\mathcal{E}_0(0,\xi), \quad (t,\xi) \in Z_\Psi, \tag{27}$$

and

$$\mathcal{E}(t,\xi) \leq \begin{cases} C\frac{\lambda(t)^2}{\lambda(\tau_0)^2}\Lambda(t_0)^{\kappa}\mathcal{E}_0(0,\xi), & (t,\xi) \in Z_S,\\ C\mathcal{E}(0,\xi), & (t,\xi) \in Z_F.\end{cases} \tag{28}$$

If $n = 1$, then (27) and (28) hold with $\kappa = 0$.

7.2.3 Estimate in Z_H

Let $(t,\xi)\in Z_H$. We reduce the equation of (21) to the following first order system:

$$\partial_t V_1 = A_1 V_1, \tag{29}$$

where

$$V_1 = \begin{pmatrix} v_t + \mathrm{i}a(t)|\xi|v \\ v_t - \mathrm{i}a(t)|\xi|v \end{pmatrix}, \qquad A_1 = \begin{pmatrix} \phi_1 & \overline{b_1} \\ b_1 & \overline{\phi_1} \end{pmatrix},$$

$$b_1 = \overline{b_1} = -\frac{a'(t)}{2a(t)} \quad \text{and} \quad \phi_1 = \frac{a'(t)}{2a(t)} + \mathrm{i}a(t)|\xi|.$$

Here we note that

$$\big|V_1(t,\xi)\big|^2 = 2\mathcal{E}(t,\xi).$$

Let us denote the eigenvalues, and the corresponding eigenvectors of A_1 by $\lambda_{1\pm}$, and $\Theta_{1\pm}$ respectively, then we have

$$\lambda_{1\pm} = \phi_{1\Re} \pm \mathrm{i}\sqrt{\phi_{1\Im}^2 - |b_1|^2}, \qquad \Theta_{1+} = \begin{pmatrix} 1 \\ \theta_1 \end{pmatrix}, \qquad \Theta_{1-} = \begin{pmatrix} \overline{\theta_1} \\ 1 \end{pmatrix},$$

where $\Re\{\phi_1\} = \phi_{1\Re}$, $\Im\{\phi_1\} = \phi_{1\Im}$ and

$$\theta_1 = \frac{\lambda_{1+} - \phi_1}{\overline{b_1}}.$$

Let us denote $\lambda_1 = \lambda_{1+}$ and $M_1 = (\Theta_{1+}\ \Theta_{1-})$. If $|\theta_1|$ is small, then $\det M_1 > 0$, it follows that A_1 is diagonalized as follows:

$$M_1^{-1} A_1 M_1 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \overline{\lambda_1} \end{pmatrix}.$$

Here we introduce symbol classes in Z_H for the convenience of the next step of the proof. For integers $p \ge 0$, q and r the symbol class $S^{(p)}\{q,r\}$ is the set of functions satisfying

$$\left|\partial_t^k f(t,\xi)\right| \le C_k\big(\lambda(t)|\xi|\big)^q \left(\frac{\lambda(t)}{\Theta(t)}\left(\frac{\Theta(t)}{\Lambda(t)}\right)^{1/m}\big(\log\Lambda(t)\big)^{\beta}\right)^{r+k} \tag{30}$$

for $k = 0,\ldots,p$ in Z_H. Moreover, we denote the general functions of the symbol class $S^{(p)}\{q,r\}$ by $\sigma^{(p)}\{q,r\} = \sigma^{(p)}\{q,r\}(t,\xi)$ for convenience. This definition of the symbol class generates the following usual algebraic properties and a hierarchy of symbols in Z_H:

Lemma 3 *The following properties are valid*:

(i) *Let* $p \ge 1$. *If* $f \in S^{(p)}\{q,r\}$, *then* $f \in S^{(p-1)}\{q,r\}$ *and* $\partial_t f \in S^{(p-1)}\{q,r+1\}$.
(ii) *If* $f_1 \in S^{(p)}\{q_1,r_1\}$ *and* $f_2 \in S^{(p)}\{q_2,r_2\}$, *then* $f_1 f_2 \in S^{(p)}\{q_1+q_2, r_1+r_2\}$.

(iii) *If* $f \in S^{(p)}\{q, r\}$, *then* $f \in S^{(p)}\{q+1, r-1\}$.
(iv) $b_1 \in S^{(m-1)}\{0, 1\}$, $1/\phi_{1\Im} \in S^{(m)}\{-1, 0\}$ *and* $\theta_1 \in S^{(m-1)}\{-1, 1\}$.

Proof (i) and (ii) are trivial from the definition of the symbol class. Let $f(t,\xi) \in S^{(p)}\{q+1, r-1\}$. By (13) and the definition of $t_0(\xi)$ we have

$$\begin{aligned}\left|\partial_t^k f(t,\xi)\right| &\le C_k\big(\lambda(t)|\xi|\big)^q\left(\frac{\lambda(t)}{\Theta(t)}\left(\frac{\Theta(t)}{\Lambda(t)}\right)^{1/m}\big(\log \Lambda(t)\big)^{\beta}\right)^{r+k}\\ &\le C_k N^{-1}\big(\lambda(t)|\xi|\big)^{q+1}\left(\frac{\lambda(t)}{\Theta(t)}\big(\log \Lambda(t)\big)^{\beta}\right)^{-1}\\ &\quad\times\left(\frac{\lambda(t)}{\Theta(t)}\left(\frac{\Theta(t)}{\Lambda(t)}\right)^{1/m}\big(\log \Lambda(t)\big)^{\beta}\right)^{r+k}\\ &\le C_k N^{-1}\big(\lambda(t)|\xi|\big)^{q+1}\left(\frac{\lambda(t)}{\Theta(t)}\left(\frac{\Theta(t)}{\Lambda(t)}\right)^{1/m}\big(\log \Lambda(t)\big)^{\beta}\right)^{r-1+k}\end{aligned}$$

in Z_H for $k = 0, \ldots, p$. It follows that $f(t,\xi) \in S^{(p)}\{q+1, r-1\}$; thus (iii) is proved. The first two properties of (iv) are trivial by (11) and (16). By (ii) we have $(|b_1|/\phi_{1\Im})^2 \in S^{(m-1)}\{-2, 2\}$. Hence, we have

$$\begin{aligned}\left(\frac{|b_1|}{\phi_{1\Im}}\right)^2 &\le C\big(\lambda(t)|\xi|\big)^{-2}\left(\frac{\lambda(t)}{\Theta(t)}\left(\frac{\Theta(t)}{\Lambda(t)}\right)^{1/m}\big(\log \Lambda(t)\big)^{\beta}\right)^2\\ &\le CN^{-2}\left(\frac{\Theta(t)}{\Lambda(t)}\right)^{2/m} \le CN^{-2}.\end{aligned}$$

Noting $\phi_{1\Im} > 0$ and the representation $\sqrt{1-\delta} = 1 - \delta/2 + Q(\delta)\delta^2$ for $|\delta| < 1$, where

$$Q(\delta) = \sum_{l=0}^{\infty}\binom{\frac{1}{2}}{l+2}(-\delta)^l = O(1),$$

we have

$$\begin{aligned}\theta_1 &= \frac{\mathrm{i}\phi_{1\Im}}{\overline{b_1}}\left(\sqrt{1-\left(\frac{|b_1|}{\phi_{1\Im}}\right)^2} - 1\right)\\ &= \frac{\mathrm{i}\phi_{1\Im}}{\overline{b_1}}\left(-\frac{1}{2}\left(\frac{|b_1|}{\phi_{1\Im}}\right)^2 + Q\left(\left(\frac{|b_1|}{\phi_{1\Im}}\right)^2\right)\left(\frac{|b_1|}{\phi_{1\Im}}\right)^4\right)\\ &= -\frac{\mathrm{i}b_1}{2\phi_{1\Im}} + \mathrm{i}Q\left(\left(\frac{|b_1|}{\phi_{1\Im}}\right)^2\right)\frac{b_1|b_1|^2}{\phi_{1\Im}^3} \in S^{(m-1)}\{-1, 1\}.\end{aligned}$$

□

By the diagonalizer M_1 of A_1, (29) is reduced to the following system,

$$\partial_t V_2 = A_2 V_2,$$

where

$$V_2 = M_1^{-1} V_1, \qquad A_2 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \overline{\lambda_1} \end{pmatrix} - M_1^{-1}(\partial_t M_1) = \begin{pmatrix} \phi_2 & \overline{b_2} \\ b_2 & \overline{\phi_2} \end{pmatrix},$$

$$b_2 = -\frac{(\theta_1)_t}{1-|\theta_1|^2} \quad \text{and} \quad \phi_2 = \lambda_1 + \frac{\overline{\theta_1}(\theta_1)_t}{1-|\theta_1|^2}.$$

Actually, M_1 is a diagonalizer of A_1, but not of $A_1 - I\partial_t$. However, we observe that $b_1 \in S^{(m-1)}\{0,1\}$ and $b_2 \in S^{(m-2)}\{-1,2\}$; thus M_1 can be a diagonalizer of $A_1 - I\partial_t$ modulo $S^{(m-2)}\{0,1\}$. Moreover, the matrix A_2 has the same structure as A_1; this means that we can continue the same diagonalization procedure to A_2. Indeed, we have the following lemma to carry out further steps of diagonalization procedure, which is called the refined diagonalization procedure introduced in [5] (see also [10]).

Lemma 4 *Let k be a positive integer satisfying $k < m$. Suppose that A_k is given by*

$$A_k = \begin{pmatrix} \phi_k & \overline{b_k} \\ b_k & \overline{\phi_k} \end{pmatrix},$$

and λ_k is an eigenvalue of A_k. If $b_k \in S^{(m-k)}\{-k+1,k\}$, $1/\phi_{\Im k} \in S^{(m-k)}\{-1,0\}$ and $\phi_{\Im k} > 0$, then the matrix M_k which is defined by

$$M_k = \begin{pmatrix} 1 & \overline{\theta_k} \\ \theta_k & 1 \end{pmatrix}, \quad \theta_k = \frac{\lambda_k - \phi_k}{\overline{b_k}},$$

is a diagonalizer of $A_k - \partial_t I$ modulo $S^{(m-k-1)}\{-k+1,k\}$, where $\phi_{k\Re} = \Re\{\phi_k\}$ and $\phi_{k\Im} = \Im\{\phi_k\}$. More precisely, denoting $A_{k+1} = M_k^{-1}(A_k - \partial_t I)M_k$, A_{k+1} is represented as follows:

$$A_{k+1} = \begin{pmatrix} \phi_{k+1} & \overline{b_{k+1}} \\ b_{k+1} & \overline{\phi_{k+1}} \end{pmatrix},$$

where

$$b_{k+1} = -\frac{(\theta_k)_t}{1-|\theta_k|^2} \in S^{(m-k-1)}\{-k,k+1\}, \tag{31}$$

$$\Re\{\phi_{k+1}\} = \phi_{(k+1)\Re} = \phi_{k\Re} - \frac{\partial_t \log(1-|\theta_k|^2)}{2} \tag{32}$$

and

$$\Im\{\phi_{k+1}\} = \phi_{(k+1)\Im} = \sqrt{\phi_{k\Im}^2 - |b_k|^2} - \Im\{\overline{\theta_k} b_{k+1}\}. \tag{33}$$

Proof By the same argument as in the proof of Lemma 3(iv) we have $(|b_k|/\phi_{k\Im})^2 \in S^{(m-k)}\{-2k,2k\}$, it follows that $(|b_k|/\phi_{k\Im})^2 \le CN^{-2k}$. Noting the representations

$$\lambda_k = \phi_{k\Re} + \mathrm{i}\sqrt{\phi_{k\Im}^2 - |b_k|^2}$$

and

$$\theta_k = -\frac{\mathrm{i}b_k}{2\phi_{k\Im}} + \mathrm{i}Q\left(\left(\frac{|b_k|}{\phi_{k\Im}}\right)^2\right)\frac{b_k|b_k|^2}{\phi_{k\Im}^3} \in S^{(m-k)}\{-2k, 2k\},$$

we see that $\det M_k > 0$ uniformly in Z_H. By direct calculations we have the representations (31), (32) and (33). Moreover, noting

$$\frac{1}{1-|\theta_k|^2} = 1 + \sum_{j=1}^{\infty} |\theta_k|^2 = 1 + \sigma^{(m-k)}\{-2k, 2k\},$$

we have $b_{k+1} \in S^{(m-k-1)}\{-k, k+1\}$. By the representation (33) we have

$$\phi_{(k+1)\Im} = \phi_{k\Im} + \sigma^{(m-k-1)}\{-2k+1, 2k\}.$$

Therefore, we have

$$\begin{aligned}\frac{1}{\phi_{(k+1)\Im}} &= \frac{1}{\phi_{k\Im}}\left(1 + \sum_{j=1}^{\infty}\left(-\frac{\sigma^{(m-k-1)}\{-2k+1, 2k\}}{\phi_{k\Im}}\right)^2\right) \\ &= \frac{1}{\phi_{k\Im}}\big(1 + \sigma^{(m-k-1)}\{-2k, 2k\}\big),\end{aligned}$$

it follows that $1/\phi_{(k+1)\Im} \in S^{(m-k-1)}\{-1, 0\}$. □

Actually Lemma 4 can be applied successively to A_1 till $k = m-1$; thus we finally arrive at the following system:

$$\partial_t V_m = A_m V_m,$$

where

$$V_m = M_{m-1}^{-1} \cdots M_1^{-1} V_1, \qquad A_m = \begin{pmatrix} \phi_m & \overline{b_m} \\ b_m & \overline{\phi_m} \end{pmatrix},$$

$$\phi_{m\Re} = \frac{1}{2}\partial_t \log\left(\frac{a(t)}{\prod_{k=1}^{m-1}(1-|\theta_k|^2)}\right), \qquad \phi_{m\Im} = a(t)|\xi| + \sigma^{(0)}\{-1, 2\}$$

and

$$b_m \in S^{(0)}\{-m+1, m\}.$$

Differentiating $|V_m(t,\xi)|^2$ with respect to t we have

$$\begin{aligned}\partial_t|V_m|^2 &= 2\Re(A_m V_m, V_m)_{\mathbb{C}^2} = 2\phi_{m\Re}|V_m|^2 + 4\Re\{b_m V_{m,1}\overline{V_{m,2}}\} \\ &\le 2\big(\phi_{m\Re} + |b_m|\big)|V_m|^2.\end{aligned}$$

Denoting $T = \max\{t_0, T_0\}$, by Gronwall's inequality we have

$$\begin{aligned}\left|V_m(t,\xi)\right|^2 &\le \exp\left(2\int_T^t \left(\phi_{m\Re}(\tau,\xi) + \left|b_m(\tau,\xi)\right|\right)d\tau\right)\left|V_m(T,\xi)\right|^2 \\ &= \frac{a(t)}{a(T)}\prod_{k=1}^{m-1}\left(\frac{1-|\theta_k(T,\xi)|^2}{1-|\theta_k(t,\xi)|^2}\right)\exp\left(2\int_T^t \left|b_m(\tau,\xi)\right|d\tau\right)\left|V_m(T,\xi)\right|^2.\end{aligned}$$

Here we note that the estimate $|\theta_k| \le 1/2$ holds in Z_H for $k = 1, \ldots, m-1$, it follows that the diagonalizer $M_1, \ldots, M_{m-1}$ are uniformly bounded. Consequently, we have the following equivalence:

$$\left|V_m(t,\xi)\right|^2 \approx \left|V_1(t,\xi)\right|^2 \approx \mathcal{E}(t,\xi) \quad \text{in } Z_H.$$

Therefore, noting (11) we have the following estimate in Z_H:

$$\mathcal{E}(t,\xi) \le C\frac{\lambda(t)}{\lambda(T)}\exp\left(2\int_T^t \left|b_m(\tau,\xi)\right|d\tau\right)\mathcal{E}(T,\xi).$$

If $|\xi| \le d$, that is, $T = t_0$, then by (15) we have

$$\begin{aligned}&\exp\left(\int_{t_0}^t \left|b_m(\tau,\xi)\right|d\tau\right) \\ &\le \exp\left(C_k|\xi|^{-m+1}\int_{t_0}^t \frac{\lambda(\tau)}{\Lambda(\tau)}\Theta(\tau)^{-m+1}\left(\log\Lambda(\tau)\right)^{m\beta}d\tau\right) \\ &\le \exp\left(C_k|\xi|^{-m+1}\Lambda(t_0)^{\varepsilon_0/2}\Theta(t_0)^{-m+1}\left(\log\Lambda(t_0)\right)^{m\beta}\int_{t_0}^t \frac{\lambda(\tau)}{\Lambda(\tau)^{1+\varepsilon_0/2}}d\tau\right) \\ &\le \exp\left(\frac{2C_k}{\varepsilon_0}|\xi|^{-m+1}\Theta(t_0)^{-m+1}\left(\log\Lambda(t_0)\right)^{m\beta}\right) \\ &= \exp\left(\frac{2C_k}{\varepsilon_0 N^{m-1}}\left(\log\Lambda(t_0)\right)^{\beta}\right).\end{aligned}$$

It follows that

$$\exp\left(\int_{t_0}^t \left|b_m(\tau,\xi)\right|d\tau\right) \le \Lambda(t_0)^{C/(\log\Lambda(t_0))^{1-\beta}}$$

for $n \ge 2$. Consequently, we have

$$\mathcal{E}(t,\xi) \le C\frac{\lambda(t)}{\lambda(t_0)}\Lambda(t_0)^{\kappa}\mathcal{E}(t_0,\xi), \tag{34}$$

where κ is an arbitrary small positive constant for $n \ge 2$, and $\kappa = 0$ for $n = 1$ by choosing T_0 large enough. On the other hand, if $|\xi| \ge d$, that is, $T = T_0$, then we have

$$\mathcal{E}(t,\xi) \le C\frac{\lambda(t)}{\lambda(T_0)}\exp\left(\frac{C_k}{2\varepsilon_0}d^{-m+1}\frac{(\log\Lambda(T_0))^{m\beta}}{\Theta(T_0)^{m-1}}\right)\mathcal{E}(T_0,\xi)$$
$$\le C\lambda(t)\mathcal{E}(T_0,\xi). \tag{35}$$

Collecting the estimates of Lemma 2, (34) and (35) we have the following lemma:

Lemma 5 *Let $n \ge 2$. For any positive small constant κ there exist positive constants N, T_0 and C such that the following estimates hold*:

$$\mathcal{E}(t,\xi) \le \begin{cases} C\frac{\lambda(t)^2}{\lambda(\tau_0)^2}\Lambda(t_0)^{2\kappa}\mathcal{E}_0(t_0,\xi) & \textit{for } (t,\xi)\in\{(t,\xi)\in Z_H\cup Z_S;\, |\xi|\le d\},\\ C\lambda(t)\mathcal{E}(0,\xi) & \textit{for } (t,\xi)\in\{(t,\xi)\in[0,\infty)\times\mathbb{R}^n;\, |\xi|\ge d\},\end{cases} \tag{36}$$

and

$$\mathcal{E}_0(t,\xi) \le C\mathcal{E}_0(0,\xi) \quad \textit{for } (t,\xi)\in Z_\Psi. \tag{37}$$

If $n = 1$, then (36) and (37) hold for $\kappa = 0$.

7.2.4 Completion of the Proof

By Lemma 5 we have the following estimates:

$$\left|v(t,\xi)\right| \le C\Lambda(t_0)^{\kappa}\left(\left|v_0(\xi)\right| + \tau_0\left|v_1(\xi)\right|\right) \tag{38}$$

for $(t,\xi)\in\{(t,\xi)\in[0,\infty)\times\mathbb{R}^n;\, |\xi|\le d\}$, and

$$\left|v(t,\xi)\right| \le C\left(\left|v_0(\xi)\right| + |\xi|^{-1}\left|v_1(\xi)\right|\right) \tag{39}$$

for $(t,\xi)\in\{(t,\xi)\in[0,\infty)\times\mathbb{R}^n;\, |\xi|\ge d\}$. Noting the inequalities:

$$\left|w(t,x)\right| = (2\pi)^{-n/2}\left|\int_{\mathbb{R}^n} e^{ix\cdot\xi}v(t,\xi)d\xi\right| \le C\left\|v(t,\cdot)\right\|_{L^1}$$

and

$$\left|\langle\xi\rangle^s v_0(\xi)\right| + \left|\langle\xi\rangle^{s-1}v_1(\xi)\right| \le C\left(\left\|\langle D\rangle^s w_0(\cdot)\right\|_{L^1} + \left\|\langle D\rangle^{s-1}w_1(\cdot)\right\|_{L^1}\right),$$

we see that the estimate (20) is concluded if we prove the following inequality:

$$\left\|v(t,\cdot)\right\|_{L^1} \le C\left(\langle\xi\rangle^s\left|v_0(\xi)\right| + \langle\xi\rangle^{s-1}\left|v_1(\xi)\right|\right). \tag{40}$$

Lemma 6 *If $a(t)$ satisfies all the assumptions of Theorem 2, then the estimate (40) is established.*

Proof By (39) we have the following estimates for $|\xi| \geq d$ and $s > n$:

$$\begin{aligned}\int_{|\xi|\geq d} \big|v(t,\xi)\big| d\xi &\leq C\bigg(\int_{|\xi|\geq d} \big|v_0(\xi)\big| d\xi + \int_{|\xi|\geq d} |\xi|^{-1}\big|v_1(\xi)\big| d\xi\bigg) \\ &\leq C \sup_{|\xi|\geq d}\big\{\langle\xi\rangle^s\big|v_0(\xi)\big| + \langle\xi\rangle^{s-1}\big|v_1(\xi)\big|\big\}\int_{|\xi|\geq d}\langle\xi\rangle^{-s} d\xi \\ &\leq C \sup_{|\xi|\geq d}\big\{\langle\xi\rangle^s\big|v_0(\xi)\big| + \langle\xi\rangle^{s-1}\big|v_1(\xi)\big|\big\}.\end{aligned}$$

On the other hand, for $|\xi| \leq d$ and $n \geq 2$, noting

$$|\xi|^{-1} = \frac{\Theta(t_0)}{N(\log \Lambda(t_0))^\beta} \geq C\Lambda(t_0)^{\varepsilon_0/2}$$

and

$$|\xi|^{-1} = N^{-1}\tau_0\lambda(\tau_0) \geq N^{-1}\lambda(0)\tau_0,$$

we have

$$\int_{|\xi|\leq d} \Lambda\big(t_0(\xi)\big)^\kappa \tau_0(\xi) d\xi \leq C\int_{|\xi|\leq d} |\xi|^{-2\kappa/\varepsilon_0 - 1} d\xi < \infty$$

by choosing κ satisfying $\kappa < \varepsilon_0(n-1)/2$. For $n = 1$ noting $\kappa = 0$ and $N \geq \Lambda(\tau_0)|\xi|$ we have

$$\int_{|\xi|<d} \Lambda\big(t_0(\xi)\big)^\kappa \tau_0(\xi) d\xi \leq 2\int_0^1 \Lambda^{-1}\left(\frac{N}{r}\right) dr < \infty$$

by (14). Therefore, by (38) we obtain

$$\begin{aligned}\int_{|\xi|\leq d} \big|v(t,\xi)\big| d\xi &\leq C \sup_{|\xi|\leq d}\big\{\big|v_0(\xi)\big| + \big|v_1(\xi)\big|\big\}\int_{|\xi|\leq d} \Lambda\big(t_0(\xi)\big)^\kappa \tau_0(\xi) d\xi \\ &\leq C \sup_{|\xi|\leq d}\big\{\big|v_0(\xi)\big| + \big|v_1(\xi)\big|\big\} \\ &\leq C \sup_{|\xi|\leq d}\big\{\langle\xi\rangle^s\big|v_0(\xi)\big| + \langle\xi\rangle^{s-1}\big|v_1(\xi)\big|\big\}.\end{aligned}$$

Thus the estimate (40) is proved. □

7.3 Proof of Theorem 3

7.3.1 Division of the Hyperbolic Zone

The estimates of Lemma 2 are proved without any difference from Theorem 2; thus we shall prove corresponding estimates in Z_H.

For a large constant N and $m \in \mathbb{N}$ we define $t_m = t_m(\xi) \in [0, \infty) \times \mathbb{R}^n$ implicitly by

$$\Theta(t_m)\left(\log \frac{\Lambda(t_m)}{\Theta(t_m)}\right)^\nu |\xi| = N(m+1)^\nu \big(\log \Lambda(t_m)\big)^\beta. \tag{41}$$

Then we define the subzones $Z_{H,m}$ of Z_H by

$$Z_{H,m} = \big\{(t, \xi) \in Z_H;\, t_{m-1}(\xi) \le t \le t_m(\xi)\big\}.$$

Here we note that the following lemma is valid:

Lemma 7 *If t_m and t_{m+1} exist for a given $\xi \in \mathbb{R}^n$, then we have $t_m < t_{m+1}$. Moreover, there exists $m_0 \in \mathbb{N}$ satisfying $m_0 \ge 2$ such that $t_0 < t_{m_0}$.*

Proof We note that $\Theta(t)(\log \Lambda(t))^{-\beta}$ is monotone increasing for any large t. By (41) we have

$$\frac{\Theta(t_{m+1})(\log(\Lambda(t_{m+1})/\Theta(t_{m+1})))^\nu (\log \Lambda(t_{m+1}))^{-\beta}}{\Theta(t_m)(\log(\Lambda(t_m)/\Theta(t_m)))^\nu (\log \Lambda(t_m))^{-\beta}} = \left(\frac{m+2}{m+1}\right)^\nu > 1.$$

Here $\Theta(t)(\log(\Lambda(t)/\Theta(t)))^\nu (\log \Lambda(t))^{-\beta}$ is monotone increasing for any large t, it follows that $t_m < t_{m+1}$. If $t_m \le t_0$ holds for any $m \in \mathbb{N}$, then we have

$$\begin{aligned} \frac{N}{|\xi|} &= \frac{\Theta(t_0)}{(\log \Lambda(t_0))^\beta} = \frac{\Theta(t_m)(\log(\Lambda(t_m)/\Theta(t_m)))^\nu}{(m+1)^\nu (\log \Lambda(t_m))^\beta} \\ &< \frac{\Theta(t_0)(\log(\Lambda(t_0)/\Theta(t_0)))^\nu}{(m+1)^\nu (\log \Lambda(t_0))^\beta} \le \frac{\Theta(t_0)}{(\log \Lambda(t_0))^\beta} \end{aligned}$$

for $(m+1)^\nu \ge (\log(\Lambda(t_0)/\Theta(t_0)))^\nu$; however, the inequality is not true because $\lim_{|\xi|\to 0} \log(\Lambda(t_0(\xi))/(\Theta(t_0(\xi)))) = \infty$. Therefore, there exists $m_0 \in \mathbb{N}$ such that $t_0 < t_{m_0}$. On the other hand, noting the estimate

$$\frac{N}{|\xi|} = \frac{\Theta(t_0)}{(\log \Lambda(t_0))^\beta} = \frac{\Theta(t_1)(\log(\Lambda(t_1)/\Theta(t_1)))^\nu}{2^\nu (\log \Lambda(t_1))^\beta} \ge \frac{\Theta(t_1)}{(\log \Lambda(t_1))^\beta},$$

we have $t_1 \le t_0$, and thus $m_0 \ge 2$. Actually, m_0 can be large under the choice of the large constant T_0. □

In the following we denote to a given $\xi \in \mathbb{R}^n$ by $m_0 = m_0(\xi)$ the term

$$m_0(\xi) = \min\big\{m \in \mathbb{N};\, t_0(\xi) < t_m(\xi)\big\}.$$

7.3.2 Gevrey Symbol Class in $Z_{H,k}$

We have introduced the symbol class $S^{(p)}\{q,r\}$ for the proof of Theorem 2; however it is not sufficient for the proof of Theorem 3. In the proof of Theorem 3 we have to derive a benefit of the properties of the Gevrey functions, which is represented by the order of the constants C_k of (30) as $k \to \infty$. Therefore, the new symbol class for the Gevrey functions has to be more precise as follows. Let us fix a positive integer m. For integers p, q and r satisfying $0 \le p \le m$, and positive real numbers K, ρ and N the symbol class $S^{(p)}\{q,r;K,\rho,N\}$ is the set of functions satisfying

$$\left|\partial_t^k f(t,\xi)\right| \le K\frac{(r+k)!^\nu}{(r+k+1)^2}\big(\lambda(t)|\xi|\big)^q\left(\rho\frac{\lambda(t)}{\Theta(t)}\left(\log\frac{\Lambda(t)}{\Theta(t)}\right)^{-\nu}\big(\log\Lambda(t)\big)^\beta\right)^{r+k}$$

for $k = 0,\ldots,p$ in $Z_{H,m}$. Here we use the notations

$$S^{(p)}\{q,r;K,\rho,N\} = S\{q,r;K,\rho,N\} = S\{q,r;K\}$$

without any confusion. Then we have the following properties corresponding to Lemma 3:

Lemma 8 *Let us denote* $\eta = \max\{4\pi^2/3, 3^\nu\rho\}$. *The following properties are valid*:

(i) *If* $f \in S^{(p)}\{q,r;K\}$ *and* $p \ge 1$, *then* $\partial_t f \in S^{(p-1)}\{q,r+1;K\}$.
(ii) *If* $f_1 \in S\{q,r;K_1\}$ *and* $f_2 \in S\{q,r;K_2\}$, *then* $f_1 + f_2 \in S\{q,r;K_1+K_2\}$.
(iii) *If* $f_1 \in S\{q_1,r_1;K_1\}$ *and* $f_2 \in S\{q_2,r_2;K_2\}$, *then* $f_1f_2 \in S\{q_1+q_2,r_1+r_2;\eta K_1K_2\}$.
(iv) *If* $f \in S^{(m)}\{q,r;K\}$, *then for* $l \ge 1$, $r \le 2m$, $q + l \le 0$ *and* $r - l \ge 0$ *we have* $f \in S^{(m)}\{q+l,r-l;K(\eta N^{-1})^l\}$.

Proof Let $(t,\xi) \in Z_{H,m}$, and denote that

$$\mu(t) = \frac{1}{\Theta(t)}\left(\log\frac{\Lambda(t)}{\Theta(t)}\right)^{-\nu}\big(\log\Lambda(t)\big)^\beta.$$

(i) and (ii) are trivial from the definition of the symbol classes.

(iii): Let $k \in \mathbb{N}$ and assume that $r_1 \le r_2$ without loss of generality. By Leibniz rule we have

$$\begin{aligned}&\left|\partial_t^k(f_1f_2)\right|\\ &\quad\le K_1K_2\big(\lambda(t)|\xi|\big)^{q_1+q_2}\big(\rho\lambda(t)\mu(t)\big)^{r_1+r_2+k}\\ &\qquad\times\sum_{l=0}^k\binom{k}{l}\frac{(r_1+l)!^\nu}{(r_1+l+1)^2}\frac{(r_2+k-l)!^\nu}{(r_2+k-l+1)^2}\end{aligned}$$

$$= K_1 K_2 \frac{(r_1+r_2+k)!^\nu}{(r_1+r_2+k+1)^2} \big(\lambda(t)|\xi|\big)^{q_1+q_2} \big(\rho\lambda(t)\mu(t)\big)^{r_1+r_2+k}$$
$$\times \sum_{l=0}^{k} \binom{k}{l} \left(\frac{(r_1+l)!(r_2+k-l)!}{(r_1+r_2+k)!}\right)^\nu \left(\frac{r_1+r_2+k+1}{(r_1+l+1)(r_2+k-l+1)}\right)^2.$$

Then noting

$$\binom{k}{l} \left(\frac{(r_1+l)!(r_2+k-l)!}{(r_1+r_2+k)!}\right)^\nu \le \left(\frac{k!}{l!(k-l)!}\right)^{1-\nu} \le 1$$

and

$$\sum_{l=0}^{k} \left(\frac{r_1+r_2+k+1}{(r_1+l+1)(r_2+k-l+1)}\right)^2 \le 2\left(\frac{2r_2+k+1}{r_2+[(k+1)/2]+1}\right)^2 \sum_{l=0}^{[k/2]} \frac{1}{(l+1)^2}$$
$$\le 8 \sum_{l=1}^{\infty} \frac{1}{l^2} = \frac{4\pi^2}{3},$$

where $[\cdot]$ denotes Gaussian symbol, we have (iii).

(iv): Noting $|\xi| \ge N m^\nu \mu(t)$ for $(t,\xi) \in Z_{H,m}$ we have

$$\big|\partial_t^k f\big| \le K \frac{(k+r)!^\nu}{(k+r+1)^2} \big(\lambda(t)|\xi|\big)^q \big(\rho\lambda(t)\mu(t)\big)^{k+r}$$
$$= \frac{K(k+r)!^\nu}{(k+r-l)!^\nu} \left(\frac{\rho\mu(t)}{|\xi|}\right)^l \left(\frac{k+r-l+1}{k+r+1}\right)^2 \frac{(k+r-l)!^\nu}{(k+r-l+1)^2}$$
$$\times \big(\lambda(t)|\xi|\big)^{q+l} \big(\rho\lambda(t)\mu(t)\big)^{k+r-l}$$
$$\le K \left(\frac{3^\nu \rho}{N}\right)^l \frac{(k+r-l)!^\nu}{(k+r-l+1)^2} \big(\lambda(t)|\xi|\big)^{q+l} \big(\rho\lambda(t)\mu(t)\big)^{k+r-l}$$

for $k \le m$. □

Let us denote the general functions of the symbol class $S^{(p)}\{q,r;K,\rho,N\}$ by $\sigma^{(p)}\{q,r;K,\rho,N\}(t,\xi) = \sigma^{(p)}\{q,r;K,\rho,N\}$. Then for any $f \in S^{(p)}\{q,r;K,\rho,N\}$ we denote

$$f \lesssim \sigma^{(p)}\{q,r;K,\rho,N\}.$$

Moreover, we denote $\sigma^{(p)}\{q,r;K,\rho,N\} = \sigma\{q,r;K\}$, $\sigma\{q,r;1\} = \sigma\{q,r\}$ and thus $\sigma\{q,r;K\} = K\sigma\{q,r\}$ for convenience. Indeed, the properties of Lemma 8(iii) and (iv) are represented as follows:

Lemma 9 *Lemma* 8(iii) *and* (iv) *are represented as follows*:

(iii′) *If* $f_1 \in S\{q_1, r_1; K_1\}$ *and* $f_2 \in S\{q_2, r_2; K_2\}$, *then*

$$f_1 f_2 \lesssim \sigma\{q_1, r_1; K_1\}\sigma\{q_2, r_2; K_2\} \lesssim \sigma\{q_1+q_2, r_1+r_2; \eta K_1 K_2\}$$
$$= \eta K_1 K_2 \sigma\{q_1+q_2, r_1+r_2\}.$$

(iv′) *If* $f \in S^{(m)}\{q, r; K\}$, *then for* $l \geq 1$, $r \leq 2m$, $q+l \leq 0$ *and* $r - l \geq 0$ *we have*

$$f \lesssim \sigma\left\{q+l, r-l; K\left(\eta N^{-1}\right)^l\right\} = K\left(\frac{\eta}{N}\right)^l \sigma\{q+l, r-l\}.$$

Moreover, we have the following properties:

Lemma 10 *The following properties are valid*:

(v) $1 \lesssim \sigma\{0, 0\}$.
(vi) *For* $l \in \mathbb{N}$ *we have* $\sigma\{q, r; K\}^l \lesssim \sigma\{lq, lr; \eta^{l-1}K^l\}$.
(vii) *If* $f \in S\{0, 0; K\}$ *with* $K \leq 1/(2\eta)$, *then* $1/(1-f) \lesssim 1 + 2K\sigma\{0, 0\}$.
(viii) *If* $f \in S\{-q, q; K\}$ *for* $q \geq 1$, *then there exists* $g \in S\{0, 0; 2\}$ *such that*

$$1 - \sqrt{1-f} = fg$$

for $N \geq \eta(K + K\eta)^{1/q}$.

Proof (v) and (vi) are trivial.

(vii): By Lemma 9(iii′) we have

$$\frac{1}{1-f} = 1 + \sum_{l=1}^{\infty} f^l \lesssim 1 + \sum_{l=1}^{\infty}\left(K\sigma\{0,0\}\right)^l \lesssim 1 + \eta^{-1}\sum_{l=1}^{\infty}(K\eta)^l\sigma\{0,0\}$$
$$= 1 + \frac{K}{1-K\eta}\sigma\{0,0\} \lesssim 1 + 2K\sigma\{0,0\}.$$

(viii): By Lemma 9(iv′) we see that $f \lesssim K(\eta N^{-1})^q\sigma\{0,0\}$. Noting $\left|\binom{\frac{1}{2}}{l}\right| \leq 1$ for any positive integer l we have

$$\sum_{l=1}^{\infty}\binom{\frac{1}{2}}{l}(-f)^{l-1} \lesssim \sum_{l=1}^{\infty}\left(K\eta^q N^{-q}\sigma\{0,0\}\right)^{l-1}$$
$$\lesssim \left(1 + \eta^{-1}\sum_{l=1}^{\infty}\left(K\eta^{q+1}N^{-q}\right)^l\right)\sigma\{0,0\} \leq 2\sigma\{0,0\}.$$

Therefore, noting the representation

$$1 - \sqrt{1-f} = f\sum_{l=1}^{\infty}\binom{\frac{1}{2}}{l}(-f)^{l-1},$$

we conclude the proof. □

Let us choose $\tilde{\rho}_0 > \rho_0$ satisfying $\sup_{k\geq 1}\{(k+1)^2(\rho_0/\tilde{\rho}_0)^k\} \leq 1$, where ρ_0 is the constant of (18). Then we see that

$$\begin{aligned}\left|a^{(k)}(t)\right| &\leq (k+1)^2\left(\frac{\rho_0}{\tilde{\rho}_0}\right)^k \frac{k!^\nu}{(k+1)^2}\lambda(t)\big(\tilde{\rho}_0\lambda(t)\mu(t)\big)^k \\ &\leq \frac{k!^\nu}{(k+1)^2}\lambda(t)\big(\tilde{\rho}_0\lambda(t)\mu(t)\big)^k\end{aligned}$$

for $k \in \mathbb{N}$, it follows that $a(t)|\xi| \in S\{1, 0; \max\{1, C_0\}, \tilde{\rho}_0, N\}$. Moreover, there exist constants $\gamma_1 \geq \max\{1, C_0\}$ and $\rho_1 > \tilde{\rho}_0$ such that

$$\frac{1}{a(t)|\xi|} = \frac{1}{\phi_{1\Im}} \in S\{-1, 0; \gamma_1, \rho_1, N\}. \tag{42}$$

Indeed, if we assume that

$$\left|\partial_t^j \frac{1}{a(t)|\xi|}\right| \leq \gamma_1 \frac{j!^\nu}{(j+1)^2}\big(\lambda(t)|\xi|\big)^{-1}\big(\rho_1\lambda(t)\mu(t)\big)^j \tag{43}$$

for any $j = 1, \ldots, k$, then by the equality

$$0 = \partial_t^{k+1}\left(a(t)\frac{1}{a(t)}\right) = a(t)\partial_t^{k+1}\frac{1}{a(t)} + \sum_{j=0}^{k}\binom{k+1}{j}\big(\partial_t^{k+1-j}a(t)\big)\left(\partial_t^j\frac{1}{a(t)}\right),$$

we have

$$\begin{aligned}\left|\partial_t^{k+1}\frac{1}{a(t)|\xi|}\right| &\leq \frac{\gamma_1}{C_0}\big(\lambda(t)|\xi|\big)^{-1}\big(\lambda(t)\mu(t)\big)^{k+1} \\ &\qquad \times \sum_{j=0}^{k}\binom{k+1}{j}\tilde{\rho}_0^{k+1-j}\rho_1^j\frac{(k+1-j)!^\nu}{(k-j+2)^2}\frac{j!^\nu}{(j+1)^2} \\ &\leq \frac{4\pi^2\tilde{\rho}_0}{3C_0\rho_1}\gamma_1\big(\lambda(t)|\xi|\big)^{-1}\big(\rho_1\lambda(t)\mu(t)\big)^{k+1} \\ &\leq \gamma_1\big(\lambda(t)|\xi|\big)^{-1}\big(\rho_1\lambda(t)\mu(t)\big)^{k+1}\end{aligned}$$

for $\rho_1 \geq 4\pi^2\tilde{\rho}_0/(3C_0)$. Thus (43) is valid for any $j \geq 0$.

We can suppose that the diagonalization procedure in Z_H for the proof of Theorem 2 is also applicable if we replace the assumption (16) in (18) for any fixed $m \in \mathbb{N}$. Then we have the following lemma corresponding to the properties of Lemma 4, which is crucial for the proof of Theorem 3:

Lemma 11 *Let $m \in \mathbb{N}$ and $(t, \xi) \in Z_{H,m}$. There exist positive constants γ and N depending only on ν and ρ_0 such that $b_j \in S\{-j+1, j; \gamma^j, \rho_1, N\}$ for $j = 1, \ldots, m$, and $\theta_j \in S\{-j, j; \gamma^j, \rho_1, N\}$ for $j = 1, \ldots, m-1$.*

For the preparation to prove Lemma 11 we introduce the following lemma:

Lemma 12 *If* $b_j \in S\{-j+1, j; \gamma^j, \rho_1, N\}$ *and* $\theta_j \in S\{-j, j; \gamma^j, \rho_1, N\}$ *for* $j = 1, \ldots, k$*, then there exists a positive constant* N *independent of* k *such that* $b_{k+1} \in S\{-k, k+1; 2\eta\gamma^k, \rho_1, N\}$ *and* $\phi_{1\Im}/\phi_{(k+1)\Im} - 1 \in S\{0, 0; 1, \rho_1, N\}$ *for* $\gamma \geq \gamma_1$*, where* $\eta = \max\{4\pi^2/3, 3^\nu\rho_1\}$.

Proof Let $N \geq \sqrt{2}\eta^2\gamma$. It follows that $\eta(\eta\gamma/N)^{2k} \leq 1/(2\eta)$. Noting

$$|\theta_k|^2 \lesssim \sigma\left\{-2k, 2k; \eta\gamma^{2k}\right\} \lesssim \sigma\left\{0, 0; \eta(\eta\gamma/N)^{2k}\right\},$$

by Lemma 10(vii) we have

$$\frac{1}{1-|\theta_k|^2} \lesssim 1 + 2\eta\left(\frac{\eta\gamma}{N}\right)^{2k}\sigma\{0,0\} \lesssim 1 + \frac{1}{\eta}\sigma\{0,0\} \lesssim 2\sigma\{0,0\}.$$

Therefore, using Lemma 8(i) and the representation

$$b_{k+1} = -\frac{(\theta_k)_t}{1-|\theta_k|^2},$$

we have $b_{k+1} \lesssim \sigma\{-k, k+1; \gamma^k\}\sigma\{0,0;2\} \lesssim \sigma\{-k, k+1; 2\eta\gamma^k\}$. Moreover, we have

$$\frac{\Im\{\overline{\theta_k}(\theta_k)_t\}}{1-|\theta_k|^2} = -\Im\{\overline{\theta_k}b_{k+1}\} \lesssim \sigma\left\{-2k, 2k+1; 2\eta^2\gamma^{2k}\right\}.$$

Noting $1/(1+|\theta_j|^2) \lesssim \sigma\{0,0;2\}$ we see that

$$\frac{|b_j|^2}{\phi_{j\Im}^2} = \left(\frac{2|\theta_j|}{1+|\theta_j|^2}\right)^2 \lesssim \sigma\left\{-2j, 2j; 16\eta^3\gamma^{2j}\right\}.$$

Therefore, for $N \geq 4\eta^2\gamma(\eta+\eta^2)^{1/2}$, which ensures $N \geq \eta(K_1 + K_1\eta)^{1/2j}$ with $K_1 = 16\eta^3\gamma^{2j}$, there exists $g_j \in S\{0,0;2\}$ such that

$$-1 + \sqrt{1 - \frac{|b_j|^2}{\phi_{j\Im}^2}} = -\frac{|b_j|^2}{\phi_{j\Im}^2}g_j \lesssim \sigma\left\{-2j, 2j; 32\eta^3\gamma^{2j}\right\}$$

by Lemma 10(viii). Let us define

$$p_j = -1 + \sqrt{1 - \frac{|b_j|^2}{\phi_{j\Im}^2}} \quad \text{and} \quad q_j = -\frac{\Im\{\overline{\theta_j}b_{j+1}\}}{\phi_{1\Im}}.$$

Then denoting $K_2 = \max\{32\eta^5\gamma^2, 2\eta^6\gamma^3\}$ we have

$$p_j \lesssim \left(\frac{\eta\gamma}{N}\right)^{2j}\sigma\left\{0,0;32\eta^3\right\} \lesssim \left(\frac{K_2}{N}\right)^j\sigma\{0,0\}$$

and

$$q_j \lesssim \sigma\left\{-2j-1, 2j+1; 2\eta^3\gamma_1\gamma^{2j}\right\} \lesssim \frac{2\eta^4\gamma}{N}\left(\frac{\eta\gamma}{N}\right)^{2j}\sigma\{0,0\}$$
$$\lesssim \left(\frac{K_2}{N}\right)^j \sigma\{0,0\}.$$

We can suppose that $\phi_{j\Im} > 0$ for any $j = 1, \ldots, k$. Recalling the representations

$$\phi_{(j+1)\Im} = \sqrt{\phi_{j\Im}^2 - |b_j|^2} - \Im\{\overline{\theta_j} b_{j+1}\}$$

for any $j \geq 1$ we have

$$\phi_{(k+1)\Im} = \phi_{1\Im}\psi_k,$$

where

$$\psi_k = \prod_{l=1}^{k}(1+p_l) + \sum_{j=1}^{k-1} q_j \prod_{l=j+1}^{k}(1+p_l) + q_k.$$

Denoting $\sigma\{0,0\} = \sigma$ and $K_2/N = \delta$, for $\eta\delta \leq 1/2$ we have

$$\psi_k \lesssim \left(1 + \sum_{j=1}^{k}\delta^j\sigma\right)\prod_{j=1}^{k}\left(1+\delta^j\sigma\right) \lesssim \left(1 + \sum_{j=1}^{k}\delta^j\sigma\right)\left(1 + \sum_{j=1}^{\infty}(\delta\sigma)^j\right)^2$$
$$\lesssim \left(1 + \sum_{j=1}^{k}\delta^j\sigma\right)\left(1 + \sum_{j=1}^{\infty}(\eta\delta)^j\sigma\right)^2 \lesssim \left(1 + \frac{\eta\delta}{1-\eta\delta}\sigma\right)^3$$
$$\lesssim (1+2\eta\delta\sigma)^3 = 1 + 6\eta\delta\sigma + 12\eta^2\delta^2\sigma^2 + 8\eta^3\delta^3\sigma^3$$
$$\lesssim 1 + \delta\left(6\eta + 12\eta^3\delta^2 + 8\eta^5\delta^3\right)\sigma \lesssim 1 + \delta\left(9\eta + \eta^2\right)\sigma.$$

Consequently, for $\delta \leq 1/(2\eta(9\eta+\eta^2))$ we have

$$\frac{\phi_{1\Im}}{\phi_{(k+1)\Im}} - 1 = \frac{1-\psi_k}{\psi_k} = \sum_{j=1}^{\infty}(1-\psi_k)^j \lesssim \sum_{j=1}^{\infty}\left(\frac{1}{2\eta}\sigma\right)^j \lesssim \sum_{j=1}^{\infty}\left(\frac{1}{2}\right)^j\sigma = \sigma.$$

Thus the proof is concluded. □

Proof of Lemma 11 By Lemma 9(iii′) and (42) we have

$$b_1 = -\frac{\partial_t a(t)|\xi|}{2a(t)|\xi|} \lesssim \sigma\left\{0, 1; \frac{\eta\gamma_1}{2}\right\}$$

and

$$\left(\frac{|b_1|}{\phi_{1\Im}}\right)^2 \lesssim \sigma\{-2, 2; K_2\},$$

where $K_2 = \eta^5\gamma_1^4/4$. By using Lemma 10(viii), there exists $g_1 \in S\{0,0;2\}$ such that

$$\theta_1 = \frac{\mathrm{i}\phi_{1\Im}}{\overline{b_1}}\left(\sqrt{1-\left(\frac{|b_1|}{\phi_{1\Im}}\right)^2}-1\right) = \frac{\mathrm{i}\phi_{1\Im}}{\overline{b_1}}\left(\frac{|b_1|}{\phi_{1\Im}}\right)^2 g_1 = \frac{\mathrm{i}b_1}{\phi_{1\Im}}g_1$$
$$\lesssim \sigma\left\{-1,1;\eta^3\gamma_1^2\right\}$$

for $N \geq \eta(K_2 + K_2\eta)^{1/2}$. Therefore, for $\gamma \geq \eta^3\gamma_1^2$ we have $b_1 \in S\{0,1;\gamma\}$ and $\theta_1 \in S\{-1.1;\gamma\}$. Let us suppose that $b_j \in S\{-j+1, j;\gamma^j\}$ and $\theta_j \in S\{-j,j;\gamma^j\}$ for any $j = 1,\dots,k$. By Lemma 12 we have

$$b_{k+1} \in S\left\{-k,k+1;2\eta\gamma^k\right\},$$
$$\frac{b_{k+1}}{\phi_{(k+1)\Im}} = \frac{b_{k+1}}{\phi_{1\Im}}\left(1+\sigma\{0,0\}\right) \lesssim \sigma\left\{-k-1,k+1;4\eta^3\gamma_1\gamma^k\right\},$$

and thus

$$\left(\frac{|b_{k+1}|}{\phi_{(k+1)\Im}}\right)^2 \lesssim \sigma\left\{-2(k+1),2(k+1);K_{k+1}\right\},$$

where $K_{k+1} = 16\eta^7\gamma_1^2\gamma^{2k}$. Therefore, by Lemma 10(viii), there exists $g_{k+1} \in S\{0,0;2\}$ such that

$$\theta_{k+1} = \frac{\mathrm{i}\phi_{(k+1)\Im}}{\overline{b_{k+1}}}\left(\sqrt{1-\left(\frac{|b_{k+1}|}{\phi_{(k+1)\Im}}\right)^2}-1\right)$$
$$= \frac{\mathrm{i}\phi_{(k+1)\Im}}{\overline{b_{k+1}}}\left(\frac{|b_{k+1}|}{\phi_{(k+1)\Im}}\right)^2 g_{k+1} = \frac{\mathrm{i}b_{k+1}}{\phi_{(k+1)\Im}}g_{k+1}$$
$$\lesssim \sigma\left\{-k-1,k+1;8\eta^4\gamma_1\gamma^k\right\}$$

for $N \geq 4\eta^3\gamma$, which ensures $N \geq \eta(K_{k+1} + K_{k+1}\eta)^{1/2(k+1)}$. Therefore, choosing the constant γ by

$$\gamma = \max\left\{\gamma_1, \eta^3\gamma_1, 2\eta, 8\eta^4\gamma_1\right\} = 8\eta^4\gamma_1,$$

we have

$$b_{k+1} \in S\left\{-k,k+1,\gamma^{k+1}\right\} \quad \text{and} \quad \theta_{k+1} \in S\left\{-k-1,k+1,\gamma^{k+1}\right\}$$

for sufficiently large N as we chosen above. Here we remark that all the above choices for N depend only on ν, γ_1 and ρ_1, here γ_1 and ρ_1 are determined by ρ_0 of (18). □

7.3.3 Uniform Estimate in $Z_{H,k}$

Let us fix $\xi \in \{\xi \in \mathbb{R}^n; |\xi| \le d\}$ and $t > t_0(\xi)$. Then there exist $m_0, m \in \mathbb{N}$ satisfying $2 \le m_0 \le m$ such that $t_{m_0-1} < t_0 \le t_{m_0}$ and $t_{m-1} < t \le t_m$; thus $(t, \xi) \in Z_{H,m}$. We denote

$$\beta_j = \beta_j(\xi) = \int_{t_{j-1}}^{t_j} \left|b_j(\tau, \xi)\right| d\tau$$

for $k = m_0, \ldots, m$. By the definition of $V_j(t) = V_j(t, \xi)$ we have

$$\begin{aligned}\left|V_j(t)\right|^2 &= \frac{1}{(1-|\theta_{j-1}(t)|^2)^2} \\ &\quad \times \left(\left(1+\left|\theta_{j-1}(t)\right|^2\right)\left|V_{j-1}(t)\right|^2 - 4\Re\left\{\theta_{j-1}(t)V_{j-1,1}(t)\overline{V_{j-1,2}(t)}\right\}\right) \\ &\lesseqgtr \frac{1}{(1 \mp |\theta_{j-1}(t)|)^2}\left|V_{j-1}(t)\right|^2\end{aligned}$$

for any $j \ge 2$. Then, by the same way for the proof of Theorem 2 we have

$$\begin{aligned}\left|V_m(t)\right|^2 &\le e^{\beta_m} \frac{a(t)}{a(t_{m-1})} \frac{(1+|\theta_{m-1}(t_{m-1})|)}{(1-|\theta_{m-1}(t_{m-1})|)} \frac{\prod_{k=1}^{m-2}(1-|\theta_k(t_{m-1})|^2)}{\prod_{k=1}^{m-1}(1-|\theta_k(t)|^2)} \\ &\quad \times \left|V_{m-1}(t_{m-1})\right|^2, \\ \left|V_j(t_j)\right|^2 &\le e^{\beta_j} \frac{a(t_j)}{a(t_{j-1})} \frac{\prod_{k=1}^{j-1}(1-|\theta_k(t_{j-1})|^2)}{\prod_{k=1}^{j-1}(1-|\theta_k(t_j)|^2)} \left|V_j(t_{j-1}, \xi)\right|^2 \\ &\le e^{\beta_j} \frac{a(t_j)}{a(t_{j-1})} \frac{(1+|\theta_{j-1}(t_{j-1})|)}{(1-|\theta_{j-1}(t_{j-1})|)} \frac{\prod_{k=1}^{j-2}(1-|\theta_k(t_{j-1})|^2)}{\prod_{k=1}^{j-1}(1-|\theta_k(t_j)|^2)} \left|V_{j-1}(t_{j-1})\right|^2\end{aligned}$$

for $j = m_0 + 1, \ldots, m-1$,

$$\begin{aligned}\left|V_{m_0}(t_{m_0})\right|^2 &\le e^{\beta_{m_0}} \frac{a(t_{m_0})}{a(t_0)} \frac{(1+|\theta_{m_0-1}(t_0)|)}{(1-|\theta_{m_0-1}(t_0)|)} \frac{\prod_{k=1}^{m_0-2}(1-|\theta_k(t_0)|^2)}{\prod_{k=1}^{m_0-1}(1-|\theta_k(t_{m_0})|^2)} \left|V_{m_0-1}(t_0)\right|^2 \\ &\le e^{\beta_{m_0}} \frac{a(t_{m_0})}{a(t_0)} \frac{(1+|\theta_{m_0-1}(t_0)|)}{(1-|\theta_{m_0-1}(t_0)|)} \frac{\prod_{k=1}^{m_0-2}(1-|\theta_k(t_0)|^2)}{\prod_{k=1}^{m_0-1}(1-|\theta_k(t_{m_0})|^2)} \\ &\quad \times \frac{1}{\prod_{k=1}^{m_0-2}(1-|\theta_k(t_0)|)^2} \left|V_1(t_0)\right|^2\end{aligned}$$

and

$$\left|V_m(t)\right|^2 \ge \frac{1}{\prod_{k=1}^{m-1}(1+|\theta_k(t)|)^2} \left|V_1(t)\right|^2.$$

It follows that

$$\begin{aligned}
\left|V_m(t)\right|^2 &\le \exp\left(\sum_{k=m_0}^{m} \beta_k\right) \frac{a(t)}{a(t_0)} \frac{(1+|\theta_{m_0-1}(t_0)|)}{(1-|\theta_{m_0-1}(t_0)|)} \frac{\prod_{k=m_0}^{m-1}(1+|\theta_k(t_k)|)}{\prod_{k=m_0}^{m-1}(1-|\theta_k(t_k)|)} \\
&\quad \times \frac{\prod_{k=1}^{m_0-2}(1-|\theta_k(t_0)|^2)}{\prod_{k=1}^{m-1}(1-|\theta_k(t)|^2)} \frac{1}{\prod_{k=1}^{m_0-2}(1-|\theta_k(t_0)|)^2} \left|V_1(t_0)\right|^2 \\
&= \exp\left(\sum_{k=m_0}^{m} \beta_k\right) \frac{a(t)}{a(t_0)} \frac{\prod_{k=m_0}^{m-1}(1+|\theta_k(t_k)|)}{\prod_{k=m_0}^{m-1}(1-|\theta_k(t_k)|)} \frac{\prod_{k=1}^{m_0-1}(1+|\theta_k(t_0)|)}{\prod_{k=1}^{m-1}(1-|\theta_k(t)|^2)} \\
&\quad \times \frac{1}{\prod_{k=1}^{m_0-1}(1-|\theta_k(t_0)|)} \left|V_1(t_0)\right|^2,
\end{aligned}$$

and thus

$$\begin{aligned}
\left|V_1(t)\right|^2 &\le \exp\left(\sum_{k=m_0}^{m} \beta_k\right) \frac{a(t)}{a(t_0)} \frac{\prod_{k=m_0}^{m-1}(1+|\theta_k(t_k)|)}{\prod_{k=m_0}^{m-1}(1-|\theta_k(t_k)|)} \frac{\prod_{k=1}^{m_0-1}(1+|\theta_k(t_0)|)}{\prod_{k=1}^{m-1}(1-|\theta_k(t)|)} \\
&\quad \times \frac{\prod_{k=1}^{m-1}(1+|\theta_k(t)|)}{\prod_{k=1}^{m_0-1}(1-|\theta_k(t_0)|)} \left|V_1(t_0)\right|^2.
\end{aligned}$$

By Lemma 11 we have $|\theta_k(t)| \le 2^{-k}$ for $N \ge 2\eta\gamma$. Moreover, we have the following lemma:

Lemma 13 *Let $m \in \mathbb{N}$ and $\xi \in \mathbb{R}^n \setminus \{0\}$ satisfying $0 < |\xi| \le d$. If $b_k(\tau, \xi) \in S\{-k+1, k; \gamma^k, \rho_1, N\}$ for $k = m_0, \ldots, m$, then there exists a positive constant C independent of m such that*

$$\sum_{k=m_0}^{m} \int_{t_{k-1}}^{t_k} b_k(\tau, \xi) d\tau \le C.$$

Proof For the positive constant ε_0 from (15) we can assume that $\varepsilon_0 m_0 > 1$ by choosing T_0 from (22) large enough. Here we note that $\Lambda(t)^{\varepsilon_0}\mu(t)$ is monotone decreasing for $t \ge t_{m_0-1}$. Therefore, we have

$$\begin{aligned}
\sum_{k=m_0}^{m} \beta_k &\le \sum_{k=m_0}^{m} \frac{k!^\nu}{(k+1)^2} |\xi|^{-k+1} (\gamma\rho_1)^k \int_{t_{k-1}}^{t_k} \frac{\lambda(\tau)}{\Lambda(\tau)^{\varepsilon_0 k}} \left(\Lambda(\tau)^{\varepsilon_0}\mu(\tau)\right)^k d\tau \\
&\le \sum_{k=m_0}^{m} \frac{k!^\nu}{(k+1)^2} |\xi|^{-k+1} (\gamma\rho_1)^k \left(\Lambda(t_{k-1})^{\varepsilon_0}\mu(t_{k-1})\right)^k \int_{t_{k-1}}^{t_k} \frac{\lambda(\tau)}{\Lambda(\tau)^{\varepsilon_0 k}} d\tau
\end{aligned}$$

$$
\begin{aligned}
&= N \sum_{k=m_0}^{m} \left(\frac{k!^{\nu}}{k^k}\right)^{\nu} \frac{k^{\nu}}{(k+1)^2} \left(\frac{\gamma\rho_1}{N}\right)^{k} \Lambda(t_{k-1})^{\varepsilon_0} \mu(t_{k-1}) \int_{t_{k-1}}^{t_k} \frac{\lambda(\tau)}{\Lambda(\tau)^{\varepsilon_0 k}} d\tau \\
&\le \frac{N}{\varepsilon_0 m_0 - 1} \sum_{k=m_0}^{m} \left(\frac{k!^{\nu}}{k^k}\right)^{\nu} \frac{k^{\nu}}{(k+1)^2} \left(\frac{\gamma\rho_1}{N}\right)^{k} \Lambda(t_{k-1}) \mu(t_{k-1}).
\end{aligned}
$$

Here we note that the following inequalities hold for $k \ge m_0$:

$$
\begin{aligned}
&\frac{k^{\nu}}{\Theta(t_{k-1})} \left(\log \frac{\Lambda(t_{k-1})}{\Theta(t_{k-1})}\right)^{-\nu} \big(\log \Lambda(t_{k-1})\big)^{\beta} \\
&\quad = \frac{|\xi|}{N} = \frac{(\log \Lambda(t_0))^{\beta}}{\Theta(t_0)} \ge \frac{(\log \Lambda(t_{k-1}))^{\beta}}{\Theta(t_{k-1})},
\end{aligned}
$$

it follows that

$$
\mathrm{e}^k \ge \frac{\Lambda(t_{k-1})}{\Theta(t_{k-1})}.
$$

Hence, we have

$$
\sum_{k=m_0}^{m} \beta_k \le \frac{N}{\varepsilon_0 m_0 - 1} \sum_{k=m_0}^{m} \left(\frac{k!^{\nu}}{k^k}\right)^{\nu} \frac{k^{\nu}}{(k+1)^2} \left(\frac{\mathrm{e}\rho_1}{N}\right)^{k} \le C
$$

for $N > \mathrm{e}\gamma\rho_1$. □

Consequently, noting the estimates

$$
\begin{aligned}
&\frac{\prod_{k=m_0}^{m-1}(1+|\theta_k(t_k)|)}{\prod_{k=m_0}^{m-1}(1-|\theta_k(t_k)|)} \frac{\prod_{k=1}^{m_0-1}(1+|\theta_k(t_0)|)}{\prod_{k=1}^{m-1}(1-|\theta_k(t)|)} \frac{\prod_{k=1}^{m-1}(1+|\theta_k(t)|)}{\prod_{k=1}^{m_0-1}(1-|\theta_k(t_0)|)} \\
&\quad \le \left(\prod_{k=1}^{\infty} \frac{1+2^{-k}}{1-2^{-k}}\right)^2 \le \exp\left(2\sum_{k=1}^{\infty} \frac{2^{-k}}{1-2^{-k}}\right) \le \mathrm{e}^4,
\end{aligned}
$$

we obtain the following estimates in Z_H:

$$
\big|V_1(t)\big|^2 \le \mathrm{e}^C \frac{a(t)}{a(t_0)} \left(\prod_{k=1}^{\infty} \frac{1+2^{-k}}{1-2^{-k}}\right)^2 \big|V_1(t_0)\big|^2 \le \mathrm{e}^{C+4} \frac{a(t)}{a(t_0)} \big|V_1(t_0)\big|^2,
$$

which is a corresponding estimate to the estimate (34) with $\kappa = 0$. Thus we can conclude the proof by the same way as in the proof of Theorem 2.

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Chapter 8
Filippov Solutions to Systems of Ordinary Differential Equations with Delta Function Terms as Summands

Uladzimir Hrusheuski

Abstract This paper is devoted to the investigation of the Cauchy problem for the system of ordinary differential equations

$$\dot{y}(t) = f\bigl(t, y(t)\bigr) + A\delta^{(s)}(t), \quad y(-1) = y_0 \in \mathbb{R}^n, \tag{1}$$

with a vector containing derivatives of the delta function and a possibly discontinuous function $f : [-1, T_0] \times \mathbb{R}^n \to \mathbb{R}^n$, $T_0 > 0$, and a constant matrix A on the right-hand side. In our approach, the components of $\delta^{(s)}$ are replaced by derivatives of different δ-sequences and the limiting behavior of approximating solutions is examined. Filippov's notion of solution to a differential equation with discontinuous right-hand side is used.

Mathematics Subject Classification Primary 34A36 · 34A37 · 34A60 · 46T30 · Secondary 34A26

8.1 Introduction

The present paper was deeply influenced by the article [11] and it is a natural continuation of the work started there, where the scalar problem (1) with a continuous nonlinearity $f(t, y)$ was considered. The authors of paper [11] replaced the delta functions by δ-sequences and examined under what conditions a limiting solution exists. It was also noted there that some peculiarities arise in the multidimensional case.

In this article, we will consider problem (1) in the light of the theory of differential equations with discontinuous right-hand sides which has been being developed during the past decades (see [3]) and, due to increased demand in applied problems, it is attracting even more interest nowadays. More precisely, we ask ourselves the same question about the existence of a limiting solution but for problem (1) with

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a right-hand side f discontinuous in (t, y) and in the multidimensional case. All solutions to differential equations with discontinuous right-hand sides are understood in the sense of Filippov [3]. We will also employ different δ-sequences to replace the components of $\delta(t)$.

A substantiation of the relevance of the subject as well as a survey of papers and monographs on differential equations with distributions, particularly with distributions as additive terms, are given in [11], therefore we do not duplicate it here. The only details we would like to add is to mention the monographs [13] and [16], and say some words about the theory of differential equations with distributions and discontinuous nonlinearitites.

In general, the study of this subject goes back to the problem of multiplication of distributions (see [12]) and, currently, it is being successfully examined mainly in the framework of the theory of new generalized functions [2, 4] currently. An investigation of scalar autonomous differential equations with products of distributions and functions f having a finite number of jumps was done in [9, 10]. Scalar nonautonomous problems with products of distributions and a function f discontinuous on a C^1-curve was considered in [5]. The case of systems was investigated in [6]. It should be emphasized that the associated (limiting) solutions in the form of sliding modes [15] were not ignored in these articles. We do not face the product problem here since distributions are entering in the right-hand side additively and sliding mode limiting solutions are admissible.

The paper can be divided into two parts. In the first part, it is proved that when the arbitrary piecewise continuous function $f(t, y)$ is sublinear of order $r < 1/\|s\|_\infty$, $s \in \mathbb{N}^n$ with respect to the variable y, uniformly with respect to the variable t, the limiting solution exists and its non-distribution part is a continuous function on $[-1, T_0]$. In the second part, we, firstly, determine a class of piecewise continuous functions f we will work with, namely, piecewise Lipschitz continuous functions which are discontinuous at a finite number of relatively simple hypersurfaces. It will be shown that the Filippov set-valued maps F_f of functions f from this class satisfy a global Lipschitz-like condition which implies linear growth of F_f but does not coincide with the classical Lipschitz condition for set-valued maps. Then the results on existence of limiting solutions for problems with such right-hand sides are formulated. Both the cases of bounded and unbounded sets of discontinuities are considered. It is shown that in both cases the non-distribution part of the limiting solution is discontinuous at $t = 0$ and the values of the jumps are obtained.

8.2 Preliminaries

Let ∂G, $\overline{G}$ and $\mu_k(G)$ be the boundary, closure and Lebesgue measure of the (Lebesgue measurable) set $G \subset \mathbb{R}^k$, $k = \overline{1, n+1}$, $n \in \mathbb{N}$, respectively; $B + F := \{b + \eta | b \in B, \eta \in F\}$, $B, F \subset \mathbb{R}^n$; $[x, y] := \{p \in \mathbb{R}^n | p = (1-\theta)x + \theta y, \theta \in [0, 1] \subset \mathbb{R}\}$, $x, y \in \mathbb{R}^n$; $G^{t_0} = G^{t=t_0} := \{y \in \mathbb{R}^n | (t_0, y) \in G \subset \mathbb{R}^{n+1}\}$; $\|y\| = \sqrt{\sum_{i=1}^n y_i^2}$, $\|y\|_\infty = \max_{i=\overline{1,n}} |y_i|$, $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$; $\|A\| = \max_{i,j=\overline{1,n}} |a_{ij}|$, $A = [a_{ij}]_{i,j=\overline{1,n}} \in \mathbb{R}^{n \times n}$.

In the sequel, the symbol $\sqcup$ means the disjoint union, a domain is a connected open set, P is the hyperplane $t = \text{const}$ in the space $\mathbb{R}^{n+1}$ of the variables (t, x), $D_0 = [-1, T_0]$, $T_0 > 0$, $D_k = D_0 \times \mathbb{R}^k$, $C(D_k)$ is the set of all continuous functions on D_k, $k = \overline{1, n}$ and $\mathcal{B}(\mathbb{R}^n)$ is the power set of $\mathbb{R}^n$.

For self-sufficiency reason let us recall the following definitions which will be used in the sequel.

Definition 1 A function $f : D_n \to \mathbb{R}^n$ is called piecewise continuous (see [3]) if for any bounded domain $G \subset D_n$ there exist domains G_i, $i = \overline{1, m}$, $m \in \mathbb{N}$ such that

$$\overline{G} = G_1 \sqcup \ldots \sqcup G_m \sqcup M_f,$$

$$M_f = \partial G_1 \cup \ldots \cup \partial G_m, \qquad \mu_{n+1}(M_f) = 0,$$

f is continuous in G_i and can be continuously extended to ∂G_i.

Definition 2 A function $f : D_n \to \mathbb{R}^n$, $(t, y) \mapsto f(t, y)$ is called sublinear of order $r \in [0, 1)$ (and of linear growth when $r = 1$) with respect to the variable y, uniformly with respect to the variable t if the following condition holds

$$\exists C > 0 : \forall (t, y) \in D_n : \left\| f(t, y) \right\| \le C\big(1 + \|y\|^r\big).$$

Definition 3 A set-valued map $F : D_n \to \mathcal{B}(\mathbb{R}^n)$, $(t, y) \mapsto F(t, y)$ is called upper semicontinuous (see [1]) at $(t_0, y_0) \in D_n$ if for any open $V \subset \mathbb{R}^n$ such that $F(t_0, y_0) \subset V$ there exists a neighborhood U of (t_0, y_0) such that

$$\bigcup_{(t,y)\in U\cap D_n} F(t, y) \subset V.$$

We say that F is upper semicontinuous if it is so at every point of D_n. In the sequel, we will also write $F : D_n \to \mathbb{R}^n$, simply saying that it is a set-valued map.

Definition 4 A set-valued map $F : D_n \to \mathbb{R}^n$ will be called sublinear of order $r \in [0, 1)$ (and of linear growth when $r = 1$) with respect to the variable y, uniformly with respect to the variable t if the following condition holds

$$\exists C > 0 : \forall (t, y) \in D_n, \forall \eta \in F(t, y) : \|\eta\| \le C\big(1 + \|y\|^r\big).$$

Definition 5 An n-dimensional continuous hypersurface Ψ_n in D_n is called relatively simple of order $N \in \mathbb{N}$ if for μ_1-almost all $t \in D_0$ it holds that $\forall x, y \in D_n^t$ there exists a decomposition

$$[x, y] = [x, p_1) \sqcup [p_1, p_2) \sqcup \ldots \sqcup [p_{\bar{N}}, y], \quad \bar{N} \le N, \tag{2}$$

where the interior of each interval in the decomposition (2) consists either of points of Ψ_n^t or points of $D_n^t \backslash \Psi_n^t$. The class of all relatively simple hypersurfaces of order $N \in \mathbb{N}$ will be denoted by the symbol RS_N.

Example 1 Let Ψ_2^1 and Ψ_2^2 be defined by equations $y_1^2 + y_2^2 = 1$ and $b_1 y_1 + b_2 y_2 = 0$, $b_1, b_2 \in \mathbb{R}$, $t \in D_0$ respectively. Then Ψ_2^1 and Ψ_2^2 are relatively simple hypersurfaces of orders 2 and 1, respectively.

Example 2 Let Ψ_2 be defined by equation

$$y_2 = \begin{cases} y_1 \sin(1/y_1), & y_1 \in \mathbb{R}\backslash\{0\}, \\ 0, & y_1 = 0, \end{cases}$$

$t \in D_0$. Then $\forall N \in \mathbb{N} : \Psi_2 \notin RS_N$. Indeed, fix any $t \in D_0$ and take $x = (-1, 0)$ and $y = (1, 0)$. Then the segment $[x, y]$ cannot be represented in the form (2) since Ψ_2 intersects it in an infinite number of points.

In the sequel, we will say that a hypersurface Ψ_n in D_n is called bounded with respect to the variable y_i, $i \in \{1, \dots, n\}$ if $\exists \kappa > 0 : \Psi_n \subset D_{i-1} \times [-\kappa, \kappa] \times \mathbb{R}^{n-i}$.

Finally, let us recall several facts from the theory of differential equations with discontinuous right-hand sides.

Definition 6 We will say that the domain of continuity G_i of a piecewise continuous function f satisfies condition Γ if for μ_1-almost all $t \in D_0$ it holds that

$$(\partial G_i)^t = \partial G_i^t.$$

Definition 7 An absolutely continuous function $y(t)$ defined on $[b_1, b_2] \subset \mathbb{R}$ is called a Filippov solution to the equation

$$\dot{y}(t) = f\big(t, y(t)\big) \tag{3}$$

with the piecewise continuous function $f : D_n \to \mathbb{R}^n$ on the interval $[b_1, b_2]$ if it is a solution to the differential inclusion

$$\dot{y}(t) \in F_f\big(t, y(t)\big), \tag{4}$$

i.e. if it satisfies (4) for μ_1-almost all $t \in [b_1, b_2]$. The set-valued map F_f is called the Filippov set-valued map of the function f and is defined in the following way: $F_f(t, y)$ is the smallest convex set containing the accumulation points $f(t, y^*)$ as $y^* \to y$, $y^* \notin M_f^t$.

Remark 1 It should be emphasized that the Filippov set-valued map F_f may be not defined for all $t \in D_0$ but only for μ_1-almost all $t \in D_0$. Indeed, since $\mu_{n+1}(M_f) = 0$ it holds that $\mu_n(M_f^t) = 0$ for μ_1-almost all $t \in D_0$.

Example 3 Let the function $f : D_2 \to \mathbb{R}^2$ be discontinuous at the set

$$M_f = \Psi_2^1 \sqcup \Psi_2^2 \sqcup \Psi_2^3,$$

where $\Psi_2^1 = [-1, 0) \times \mathbb{R} \times \{0\}$, $\Psi_2^2 = \{0\} \times \mathbb{R} \times [0, 1]$, $\Psi_2^3 = (0, T_0] \times \mathbb{R} \times \{1\}$. Then the map F_f is not defined on the strip $\{0\} \times \mathbb{R} \times (0, 1)$ since all these points are unreachable from $\{0\} \times \mathbb{R} \times ((-\infty, 0) \sqcup (1, +\infty))$. Moreover, $\mu_2(M_f^{t=0}) = +\infty$.

8.3 Statement of the Problem

We consider the system of scalar ordinary differential equations

$$\begin{cases} \dot{y}^1(t) = f_1(t, y^1(t), \dots, y^n(t)) + \alpha_1 \delta_1^{(s_1)}(t), \\ \dot{y}^2(t) = f_2(t, y^1(t), \dots, y^n(t)) + \alpha_2 \delta_2^{(s_2)}(t), \\ \dots \\ \dot{y}^n(t) = f_n(t, y^1(t), \dots, y^n(t)) + \alpha_n \delta_n^{(s_n)}(t), \end{cases} \tag{5}$$

with initial data $y^i(-1) = y_i^0$, $i = \overline{1, n}$ or, in vector notation,

$$\dot{y}(t) = f\big(t, y(t)\big) + A\delta^{(s)}(t), \quad y(-1) = y_0,$$

where $f : D_n \to \mathbb{R}^n$ is a piecewise continuous function, $A = \operatorname{diag}(\alpha_1, \dots, \alpha_n)$, $\delta^{(s)}(t) = (\delta_1^{(s_1)}(t), \dots, \delta_n^{(s_n)}(t))^T$, $\delta_i^{(s_i)}(t)$ denotes the s_i-order derivative of the Dirac delta function $\delta_i(t)$, $i = \overline{1, n}$, $s = (s_1, \dots, s_n) \in \mathbb{N}^n$, $y_0 = (y_1^0, \dots, y_n^0)^T \in \mathbb{R}^n$, $\alpha = (\alpha_1, \dots, \alpha_n)^T \in \mathbb{R}^n$.

We distinguish exemplars $\delta_i(t)$, $i = \overline{1, n}$ of the Dirac delta function because we are going to investigate the limiting behavior of the Filippov solution $y_\varepsilon(t) = (y_\varepsilon^1(t), \dots, y_\varepsilon^n(t))$ to the problem (5) as $\varepsilon \to 0$ when $\delta_i(t)$, $i = \overline{1, n}$ are replaced by different δ-sequences $\phi_\varepsilon^i(t) = (1/\varepsilon)\phi^i(t/\varepsilon)$, $\varepsilon \in (0, 1]$, where $\phi^i \in C^\infty(\mathbb{R})$, $\operatorname{supp} \phi^i = [-a, b]$, $\int_{-a}^{b} \phi^i(t) dt = 1$, $\mu_1(\{t \in [-a, b] | \phi^i(t) = 0\}) = 0$, $i = \overline{1, n}$, $a, b > 0$.

In the sequel we will use the notations

$$\phi(t) := \big(\phi^1(t), \dots, \phi^n(t)\big)^T, \qquad \phi_\varepsilon(t) := \big(\phi_\varepsilon^1(t), \dots, \phi_\varepsilon^n(t)\big)^T,$$

the vector of derivatives of the functions ϕ_ε^i will be denoted by the symbol

$$\phi_\varepsilon^{(s)}(t) := \Big(\big(\phi_\varepsilon^1\big)^{(s_1)}(t), \dots, \big(\phi_\varepsilon^n\big)^{(s_n)}(t)\Big)^T,$$

and in the case $s_i = 1$, $i = \overline{1, n}$ we will use the notation $\dot{\phi}_\varepsilon(t) := (\dot{\phi}_\varepsilon^1(t), \dots, \dot{\phi}_\varepsilon^n(t))^T$.

We will also need the following assumptions.

Assumption A Equation (3) with the piecewise continuous function f has at most one Filippov solution on D_0 for any initial data $y(-1) = y_0 \in \mathbb{R}^n$.

Assumption B Equation (3) with the piecewise continuous function f has at most one Filippov solution on $[0, T_0]$ for any initial data $y(0) = y_{00} \in \mathbb{R}^n$.

8.4 Main Results

Theorem 1 *Let $f(t, y) : D_n \to \mathbb{R}^n$ be a piecewise continuous function, such that each subdomain of continuity satisfies condition Γ, and sublinear of order $r \in [0, 1)$ with respect to the variable y, uniformly with respect to the variable t. Suppose that Assumption A is made and $\|s\|_\infty < 1/r$ (i.e. s is arbitrary if f is bounded). Then the Filippov solutions $y_\varepsilon(t)$ to the equation*

$$\dot{y}_\varepsilon(t) = f\big(t, y_\varepsilon(t)\big) + A\phi_\varepsilon^{(s)}(t), \quad y_\varepsilon(-1) = y_0, \tag{6}$$

converge to $y(t) = \overline{y}(t) + A\delta^{(s-1)}(t)$ in $D'(D_0)$, where $\overline{y}(t)$ is the Filippov solution to the problem

$$\dot{y}(t) = f\big(t, y(t)\big), \quad y(-1) = y_0. \tag{7}$$

on D_0 and $s - 1 := (s_1 - 1, \ldots, s_n - 1)$.

Remark 2 Due to the existence theorem Chap. 2, Sect. 7, Theorem 1 in [3], the Cauchy problem (7) with piecewise continuous right-hand side, each of whose domains of continuity satisfies condition Γ, has a local solution. However, one can show that, due to the sublinearity of the function f, any global Filippov solution $y(t)$ to the problem (7) in the case of its existence does not leave the compact set $D_0 \times B$, $B = \{y | \|y\| \le C(1 + \|y_0\|^r)\}$. Therefore, the existence of a solution on the whole segment D_0 is guaranteed by the extension theorem Chap. 2, Sect. 7, Theorem 2 in [3]. Assumption A guarantees the uniqueness of this solution.

Remark 3 The function $g_\varepsilon(t, y) = f(t, y) + A\phi_\varepsilon^{(s)}(t)$, as ε is fixed, is also piecewise continuous with $M_g = M_f$. Therefore, each subdomain of continuity of g_ε satisfies condition Γ. Moreover, $g_\varepsilon(t, y)$ is sublinear of the same order r with respect to the variable y, uniformly with respect to the variable t. Reasoning by analogy we get the existence of a Filippov solution to problem (6) on the whole segment D_0, but it can be nonunique. Theorem 1 states that the limit of $y_\varepsilon(t)$ does not depend on the choice of solution to problem (6) as ε is fixed.

Lemma 1 *Suppose that the conditions of Theorem 1 hold. Then $\exists C > 0$ such that for μ_1-almost all $t \in D_0$ it holds that $\forall y \in D_n^t$ and $\forall \eta \in F_f(t, y)$ the following inequality holds*

$$\|\eta\| \le C\big(1 + \|y\|^r\big), \tag{8}$$

where F_f is the Filippov set-valued map of the function f.

Proof Fix $t \in D_0$ such that the map F_f is defined for all $y \in D_n^t$ (see Remark 1). Then fix $y \in D_n^t$ and $\eta \in F_f(t, y)$. If (t, y) is a point of continuity of the function f, then $\eta = f(t, y) = F_f(t, y)$ and inequality (8) holds automatically. If $(t, y) \in M_f$, then by the definition of a piecewise continuous function there exist not more than a finite number m of domains G_i, $i = \overline{1, m}$ such that $y \in \partial G_i^t$.

Let $\overline{f}^i$ be the continuous extension of the function f from G_i to $\overline{G_i}$, $i = \overline{1, m}$. Then taking the limit in the inequality $\|f(t, y^*)\| \leq C(1 + \|y^*\|^r)$ as $y^* \to y$, $y^* \in G_i^t$, we get

$$\left\|\overline{f}^i(t, y)\right\| \leq C\left(1 + \|y\|^r\right), \quad i = \overline{1, m}.$$

Since the least convex set containing a finite collection of points in $\mathbb{R}^n$ is the set of all convex combination of these points and it is closed we have

$$\|\eta\| \leq \sum_{i=1}^{m} \beta_i \left\|\overline{f}^i(t, y)\right\| \leq C\left(1 + \|y\|^r\right), \quad \sum_{i=1}^{m} \beta_i = 1.$$

□

Proof of Theorem 1 By Definition 7, the Filippov solution to the problem (6) is defined as a solution to the problem

$$\dot{y}_\varepsilon(t) \in F_{g_\varepsilon}\big(t, y_\varepsilon(t)\big), \quad y_\varepsilon(-1) = y_0. \tag{9}$$

Fix an arbitrary point $(t, y) \in M_f$ and let $\overline{g}_\varepsilon^i$ be the continuous extension of the function g_ε from G_i to $\overline{G_i}$, $i = \overline{1, m}$. Then we have

$$\overline{g}_\varepsilon^i(t, y) = \overline{f}^i(t, y) + A\phi_\varepsilon^{(s)}(t), \quad i = \overline{1, m},$$

and, consequently,

$$\begin{aligned} F_{g_\varepsilon}(t, y) &= \left\{\eta | \eta = \sum_{i=1}^{m} \beta_i^\eta \overline{g}_\varepsilon^i(t, y), \sum_{i=1}^{m} \beta_i^\eta = 1\right\} \\ &= \left\{\eta | \eta = \sum_{i=1}^{m} \beta_i^\eta \overline{f}^i(t, y) + A\phi_\varepsilon^{(s)}(t), \sum_{i=1}^{m} \beta_i^\eta = 1\right\} \\ &= F_f(t, y) + A\phi_\varepsilon^{(s)}(t). \end{aligned}$$

Therefore, we can rewrite problem (9) in the form

$$\dot{y}_\varepsilon(t) \in F_f\big(t, y_\varepsilon(t)\big) + A\phi_\varepsilon^{(s)}(t), \quad y_\varepsilon(-1) = y_0. \tag{10}$$

Any solution to problem (10) can be represented as a sum $y_\varepsilon(t) = x_\varepsilon(t) + A\phi_\varepsilon^{(s-1)}(t)$, where $x_\varepsilon(t)$ is a solution of the problem

$$\dot{x}_\varepsilon(t) \in F_f\big(t, x_\varepsilon(t) + A\phi_\varepsilon^{(s-1)}(t)\big), \quad x_\varepsilon(-1) = y_0. \tag{11}$$

At the same time the differential inclusion (11) is equivalent to the integral equation

$$x_\varepsilon(t) = y_0 + \int_{-1}^{t} \eta_\varepsilon(u) du,$$

where $\eta_\varepsilon(u) \in F_f(u, x_\varepsilon(u) + A\phi_\varepsilon^{(s-1)}(u))$ for μ_1-almost all $u \in D_0$.

The initial data in (10) and (11) are the same due to the fact that $\operatorname{supp}(\phi_\varepsilon^i)^{(s_i)}(t) \subseteq [-a\varepsilon, b\varepsilon]$, $i = \overline{1,n}$. By the same reason any solution $x_\varepsilon(t)$ to (11) is equal to $\overline{y}(t)$ for $t \in [-1, -a\varepsilon]$, in particular $x_\varepsilon(-a\varepsilon) = \overline{y}(-a\varepsilon)$. Moreover, by the continuity of the Filippov solution $\overline{y}(t)$ we have $x_{0\varepsilon} := \overline{y}(-a\varepsilon) \to \overline{y}(0)$ as $\varepsilon \to 0$. However, this does not imply that $x_\varepsilon(0) \to \overline{y}(0)$ uniformly with respect to the choice of $x_\varepsilon(\cdot)$ as $\varepsilon \to 0$, though it will be shown below.

Fix ε and let $t \in [-a\varepsilon, b\varepsilon]$, $d_\varepsilon = a\varepsilon + b\varepsilon$ and $x_\varepsilon(\cdot)$ is any solution to the problem (11). Then we have

$$\begin{aligned}
\left\|x_\varepsilon(t) - x_{0\varepsilon}\right\| &\le \int_{-a\varepsilon}^{t} \left\|\eta_\varepsilon(u)\right\| du \le C \int_{-a\varepsilon}^{t} \left(1 + \left\|x_\varepsilon(u) + A\phi_\varepsilon^{(s-1)}(u)\right\|^r\right) du \\
&= C d_\varepsilon + C \int_{-a\varepsilon}^{t} \left\|x_\varepsilon(u) + A\phi_\varepsilon^{(s-1)}(u) - x_{0\varepsilon} + x_{0\varepsilon}\right\|^r du \\
&\le C d_\varepsilon \left(1 + \|x_{0\varepsilon}\|^r\right) + C\|A\|^r \int_{-a\varepsilon}^{b\varepsilon} \left\|\phi_\varepsilon^{(s-1)}(u)\right\|^r du \\
&\quad + C \int_{-a\varepsilon}^{t} \left\|x_\varepsilon(u) - x_{0\varepsilon}\right\|^r du \le C d_\varepsilon \left(1 + \|x_{0\varepsilon}\|^r\right) \\
&\quad + C\|A\|^r \varepsilon^{1-r\|s\|_\infty} \int_{-a}^{b} \left\|\phi^{(s-1)}(u)\right\|^r du \\
&\quad + C \int_{-a\varepsilon}^{t} \left\|x_\varepsilon(u) - x_{0\varepsilon}\right\|^r du. \qquad (12)
\end{aligned}$$

Since $\|u\|^r \le \max\{1, \|u\|\}$, we get that

$$\begin{aligned}
\left\|x_\varepsilon(t) - x_{0\varepsilon}\right\| \le C d_\varepsilon \left(1 + \|x_{0\varepsilon}\|^r\right) & \\
&+ C\|A\|^r \varepsilon^{1-r\|s\|_\infty} \int_{-a}^{b} \left\|\phi^{(s-1)}(u)\right\|^r du \\
&+ C \int_{-a\varepsilon}^{t} \max\left\{1, \left\|x_\varepsilon(u) - x_{0\varepsilon}\right\|\right\} du.
\end{aligned}$$

Adding 1 to the right-hand side of the inequality, we get that

$$\begin{aligned}
\max\left\{1, \left\|x_\varepsilon(t) - x_{0\varepsilon}\right\|\right\} \le 1 + C d_\varepsilon \left(1 + \|x_{0\varepsilon}\|^r\right) & \\
&+ C\|A\|^r \varepsilon^{1-r\|s\|_\infty} \int_{-a}^{b} \left\|\phi^{(s-1)}(u)\right\|^r du \\
&+ C \int_{-a\varepsilon}^{t} \max\left\{1, \left\|x_\varepsilon(u) - x_{0\varepsilon}\right\|\right\} du.
\end{aligned}$$

Applying Gronwall's inequality we get

$$\max\left\{1, \left\|x_\varepsilon(t) - x_{0\varepsilon}\right\|\right\} \le \left(1 + Cd_\varepsilon\left(1 + \|x_{0\varepsilon}\|^r\right) + C\|A\|^r\varepsilon^{1-r\|s\|_\infty}\int_{-a}^{b}\left\|\phi^{(s-1)}(u)\right\|^r du\right)e^{Cd_\varepsilon},$$

which implies

$$\left\|x_\varepsilon(t) - x_{0\varepsilon}\right\| \le \left(1 + Cd_\varepsilon\left(1 + \|x_{0\varepsilon}\|^r\right) + C\|A\|^r\varepsilon^{1-r\|s\|_\infty}\int_{-a}^{b}\left\|\phi^{(s-1)}(u)\right\|^r du\right)e^{Cd_\varepsilon}. \tag{13}$$

Since $Cd_\varepsilon(1 + \|x_{0\varepsilon}\|^r) \to 0$ and $C\|A\|^r\varepsilon^{1-r\|s\|_\infty}\int_{-a}^{b}\|\phi^{(s-1)}(u)\|^r du \to 0$ as well as $e^{Cd_\varepsilon} \to 1$ as $\varepsilon \to 0$ we have for all $\varepsilon < \varepsilon_0$

$$\left\|x_\varepsilon(t) - x_{0\varepsilon}\right\| \le 2, \quad t \in [-a\varepsilon, b\varepsilon], \tag{14}$$

where $x_\varepsilon(\cdot)$ is any solution to the problem (11).

Finally, substituting estimate (14) into inequality (12) we get

$$\begin{aligned}\left\|x_\varepsilon(t) - x_{0\varepsilon}\right\| \le\ & Cd_\varepsilon\left(1 + 2^r + \|x_{0\varepsilon}\|^r\right) \\ & + C\|A\|^r\varepsilon^{1-r\|s\|_\infty}\int_{-a}^{b}\left\|\phi^{(s-1)}(u)\right\|^r du \\ \to\ & 0 \quad \text{as } \varepsilon \to 0\end{aligned}$$

uniformly with respect to the choice of $x_\varepsilon(\cdot)$. Particularly, $x_\varepsilon(0) \to \overline{y}(0)$ and $x_\varepsilon(b\varepsilon) \to \overline{y}(0)$ uniformly with respect to the choice of $x_\varepsilon(\cdot)$ as $\varepsilon \to 0$.

Now, we prove that $(x_\varepsilon(t))_{\varepsilon\in(0,1]}$ converges pointwise with respect to t and uniformly with respect to the choice of $x_\varepsilon(\cdot)$ to the solution $\overline{y}(t)$ to problem (7) on $(0, T_0]$ as $\varepsilon \to 0$, i.e., $\forall t > 0\ \forall\triangle > 0\ \exists\varepsilon_1(t, \triangle) > 0$ such that $\forall\varepsilon \in (0, \varepsilon_1)$ it holds that $\|x_\varepsilon(t) - \overline{y}(t)\| < \triangle$ for any solution $x_\varepsilon(\cdot)$ to problem (11).

Fix an arbitrary $t > 0$ and $\triangle > 0$. Since the set-valued map $F(u, y + A \times \phi_\varepsilon^{(s-1)}(u)) = F(u, y)$ for $u \ge b\varepsilon$, $\varepsilon \in (0, 1]$, each solution $x_\varepsilon(\cdot)$ to problem (11), when ε is fixed, can be represented on $[b\varepsilon, T_0]$ as a solution $z_\varepsilon(\cdot)$ to the problem

$$\dot{z}_\varepsilon(u) \in F_f\left(u, z_\varepsilon(u)\right), \quad z_\varepsilon(b\varepsilon) = x_\varepsilon(b\varepsilon). \tag{15}$$

Therefore, $x_\varepsilon(t) = z_\varepsilon(t)$ for all ε such that $b\varepsilon \le t$. At the same time $\overline{y}(u) = y(u)$, $u \in [0, T_0]$, where $y(\cdot)$ is the unique solution to the problem

$$\dot{y}(u) \in F_f\left(u, y(u)\right), \quad y(0) = \overline{y}(0). \tag{16}$$

By the theorem on the continuous dependence of solutions to problem (16) on initial data Chap. 2, Sect. 8, Theorem 1 in [3] there exists $\gamma > 0$ such that the inequalities

$$\left\| x_\varepsilon(b\varepsilon) - \overline{y}(0) \right\| \leq \gamma, \quad b\varepsilon \leq \gamma,$$

imply the inequality

$$\left\| z_\varepsilon(t) - y(t) \right\| < \triangle,$$

where $z_\varepsilon(\cdot)$ is any of the solutions to the problem (15) on $[0, T_0]$. Note that $x_\varepsilon(u)$ is not necessarily equal to $z_\varepsilon(u)$ for $u \in [0, b\varepsilon)$. Setting

$$\varepsilon_1 := \sup\left\{\varepsilon : \varepsilon \leq \min\{t/b, \gamma/b\}, \left\| x_\varepsilon(b\varepsilon) - \overline{y}(0) \right\| < \gamma \right\},$$

we get that $\forall t \in D_0 : x_\varepsilon(t) \to \overline{y}(t)$ as $\varepsilon \to 0$.

Reasoning in the same way as the estimate (14) was obtained one can show that for all $\varepsilon \in (0, 1]$, any solution $x_\varepsilon(t)$ to (11) does not go out of some compact set. Therefore, it follows from Lebesgue's dominated convergence theorem that $x_\varepsilon(t) \to \overline{y}(t)$ as $\varepsilon \to 0$ in $D'(D_0)$. □

Theorem 2 *Let the following conditions hold*:

(1) *the function* $f : D_n \to \mathbb{R}^n$ *is piecewise continuous*, *the set* M_f *has the form*

$$M_f = \Psi_n^1 \cup \ldots \cup \Psi_n^q, \tag{17}$$

where the hypersurfaces $\Psi_n^i \in RS_{N_i}$, $i = \overline{1, q}$, $q \in \mathbb{N}$ *are bounded and each domain of continuity* G_i, $i = \overline{1, m}$, $m \in \mathbb{N}$ *of the function* f *satisfies condition* Γ. *Moreover*, *let Assumptions* A *and* B *hold*;

(2) *the function* f *is Lipschitz with respect to* y *in each domain of continuity*, *i.e.* $\forall G_i$, $i = \overline{1, m}$, $\exists C_i > 0$ *such that for* μ_1*-almost all* $t \in \{t | P \cap G_i \neq \emptyset\}$ *the following condition holds*:

$$\forall x \in G_i^t, \forall y \in G_i^t : \left\| f(t, x) - f(t, y) \right\| \leq C_i \|x - y\|;$$

(3) $\alpha_{i_j} \neq 0$ *for all* $j = \overline{1, \tau}$, *where* $\tau \in \{1, 2, \ldots, n\}$ *and for each* $w = (w_1, \ldots, w_\tau) \in \{-1, 1\}^\tau$ *there exists the limit*

$$M_{k,(w_1,\ldots,w_\tau)}^{i_\varsigma,(i_1,\ldots,i_\tau)} := \lim_{\substack{t \to 0 \\ \hat{y}_{i_1} \to w_1 \cdot \infty \\ \cdots \\ \hat{y}_{i_\tau} \to w_\tau \cdot \infty}} \frac{f_k(t, \hat{y}_1, \ldots, \hat{y}_{i_\varsigma}, \ldots, \hat{y}_n)}{\hat{y}_{i_\varsigma}}, \quad k = \overline{1, n},$$

for some $i_\varsigma = i_\varsigma(w, k)$, $\varsigma \in \{1, 2, \ldots, \tau\}$, *where* f_k *is the k-th component of the function* f *and* $\hat{y}_i = y_i$ *if* $\alpha_i \neq 0$ *and* $\hat{y}_i = 0$ *otherwise*, $i = \overline{1, n}$.

Then the Filippov solutions $y_\varepsilon(t)$ to the equations

$$\dot{y}_\varepsilon(t) = f\big(t, y_\varepsilon(t)\big) + A\dot{\phi}_\varepsilon(t), \quad \varepsilon \in (0, 1], \tag{18}$$

with initial data $y_\varepsilon(-1) = y_0$ converge to $y(t) = \overline{y}(t) + A\delta(t)$, where $\overline{y}$ restricted to $[-1, 0)$ is the Filippov solution to the equation

$$\dot{y}(t) = f\big(t, y(t)\big), \tag{19}$$

with initial data $y(-1) = y_0$, and $\overline{y}$ restricted to $(0, T]$ is the Filippov solution to (19) with initial data $y(0) = \overline{y}(0) + \beta$, $\beta = (\beta^1, \dots, \beta^n)$ on $(0, T]$, where

$$\beta^k = \sum_{w \in \{-1,1\}^\tau} \alpha_{i_\varsigma(w,k)} M^{i_\varsigma(w,k),(i_1,\dots,i_\tau)}_{k,(\mathrm{sign}(\alpha_{i_1})\cdot w_1,\dots,\mathrm{sign}(\alpha_{i_\tau})\cdot w_\tau)} \int_{-a}^{b} \phi_w^{i_\varsigma(w,k)}(u)du,$$

$$\phi_w^{i_\varsigma(w,k)}(u) = \begin{cases} \phi^{i_\varsigma(w,k)}(u), & u \in \bigcap_{j=1}^{\tau} A_w^j, \\ 0, & u \notin \bigcap_{j=1}^{\tau} A_w^j, \end{cases}$$

$$A_w^j := \begin{cases} \{u|\phi^{i_j}(u) > 0\}, & w_j = 1, \\ \{u|\phi^{i_j}(u) < 0\}, & w_j = -1. \end{cases}$$

Remark 4 It is worth emphasizing that a piecewise continuous function in the sense of Definition 1 can have an infinite number m of domains of continuity. However, for a piecewise continuous function with bounded set M_f of the form (17) it holds that $m < \infty$. Indeed, boundedness of the set M_f implies the existence of a rectangle $\Omega = D_0 \times [-\kappa, \kappa]^n$, $\kappa > 0$ outside of which there are no discontinuities of the function f. At the same time, by the definition of a piecewise continuous function, there is a finite number of domains of continuity of the function f in Ω.

However, the number of domains of continuity of the function f does not depend on the number of the hypersurfaces Ψ_n^i as the following example illustrates.

Example 4 Let the function $f : D_1 \to \mathbb{R}$ be discontinuous on the two curves $y = \sin \frac{\pi m}{T_0+1}(t+1)$ and $y = -\sin \frac{\pi m}{T_0+1}(t+1)$. Then f has $m+2$ domains of continuity.

Remark 5 It will be shown below that under the assumptions of Theorem 2 the set-valued function F_f has linear growth with respect to the variable y, uniformly with respect to the variable t. Therefore, reasoning by analogy as in Remarks 2 and 3 we get that problem (18) has a solution on the whole segment D_0 and (19) with initial data $y(-1) = y_0$ and $y(0) = \overline{y}(0) + \beta$ has solutions on the whole segments $[-1, 0]$ and $[0, T_0]$ respectively. Moreover, both solutions are unique. As above, Theorem 2 states that the limit of $y_\varepsilon(t)$ does not depend on the choice of solution to problem (18) as ε is fixed.

An example illuminating the conditions of Theorem 2 will be given after its proof.

Lemma 2 *Suppose the function $f : D_n \to \mathbb{R}^n$ satisfies conditions (1)–(2) of Theorem 2. Then $\exists C > 0$ such that for μ_1-almost all $t \in D_0$ it holds that $\forall x, y \in D_n^t$, $\forall \eta \in F_f(t, x)$, $\forall \upsilon \in F_f(t, y)$ the following inequality holds*

$$\|\eta - \upsilon\| \le C\big(1 + \|x - y\|\big), \tag{20}$$

where F_f is the Filippov set-valued map of the function f.

Proof Let $L := \max_{i=\overline{1,m}} C_i$ and $N := \sum_{i=1}^q N_i$, where C_i is the Lipschitz constant in G_i and N_i is the order of the relative simplicity of the hypersurface Ψ_n^i. Fix $t \in D_0$ such that the map F_f is defined for all $y \in D_n^t$ (see Remark 1), representation (2) holds for all hypersurfaces Ψ_n^i, $i = \overline{1,q}$ and the Lipschitz inequality holds for all G_i such that $P \cap G_i \ne \emptyset$. Then fix $x, y \in D_n^t$, $\eta \in F_f(t, x)$, $\upsilon \in F_f(t, y)$. Then the relative simplicity of each Ψ_n^i implies

$$\begin{aligned}
[x, y] &= \big[x, p_1^1\big) \sqcup \big[p_1^1, p_2^1\big) \sqcup \ldots \sqcup \big[p_{\bar N_1}^1, y\big], \quad \bar N_1 \le N_1,\\
&\ldots\\
[x, y] &= \big[x, p_1^q\big) \sqcup \big[p_1^q, p_2^q\big) \sqcup \ldots \sqcup \big[p_{\bar N_q}^q, y\big], \quad \bar N_q \le N_q.
\end{aligned}$$

Renumbering all points p_k^l, $l = \overline{1,q}$, $k = \overline{1, \bar N_l}$ in ascending order we get the decomposition

$$[x, y] = [x, p_1) \sqcup [p_1, p_2) \sqcup \ldots \sqcup [p_{\bar N}, y], \quad \bar N \le \sum_{i=1}^q \bar N_i \le N, \tag{21}$$

that implies $\|x - y\| = \sum_{l=0}^{\bar N} \|p_l - p_{l+1}\|$, where $p_0 := x$, $p_{\bar N+1} := y$.

Since $\overline{G_i} \cap \Omega$, $i = \overline{1,m}$ is compact, we have

$$K := \max_{i=\overline{1,m}} \sup_{(t,y)\in \overline{G_i}\cap\Omega} \big\|\overline{f}^i(t, y)\big\| < \infty.$$

Denote

$$f(t, p_l+) := \begin{cases} \lim\limits_{\substack{p^*\to p_l\\ p^*\in[p_l, p_{l+1}]}} f(t, p^*), & \text{if } \exists G_i : (p_l, p_{l+1}) \subset G_i^t,\\ \lim\limits_{\substack{p^*\to p_l\\ p^*\in G_i^t\\ p_l, p_{l+1}\in\partial G_i^t}} f(t, p^*), & \text{if } (p_l, p_{l+1}) \subset M_f^t, l = \overline{0, \bar N}, \end{cases}$$

$$f(t, p_l-) := \begin{cases} \lim\limits_{\substack{p^*\to p_l\\ p^*\in[p_{l-1}, p_l]}} f(t, p^*), & \text{if } \exists G_i : (p_{l-1}, p_l) \subset G_i^t,\\ \lim\limits_{\substack{p^*\to p_l\\ p^*\in G_i^t\\ p_l, p_{l-1}\in\partial G_i^t}} f(t, p^*), & \text{if } (p_{l-1}, p_l) \subset M_f^t, l = \overline{1, \bar N + 1}, \end{cases}$$

and

$$f(t, p_{\bar{N}+1}+) := v, \qquad f(t, p_0-) := \eta.$$

It should be emphasized that in the case $(p_l, p_{l+1}) \subset G_i^t$ the inequality $\|f(t, p_l+) - f(t, p_{l+1}-)\| \leq C_i \|p_l - p_{l+1}\|$ holds due to condition (2) of Theorem 2. If $(p_l, p_{l+1}) \subset M_f^t$ and there exist G_i such that $p_l, p_{l+1} \in \partial G_i^t$, then again the inequality $\|f(t, p_l+) - f(t, p_{l+1}-)\| \leq C_i \|p_l - p_{l+1}\|$ holds. It should be emphasized that several domains G_i can exist such that $p_l, p_{l+1} \in \partial G_i^t$. Therefore, when doing the calculation of $f(t, p_{l+1}-)$, one has to take the same G_i as was taken for $f(t, p_l+)$ in the previous step. In the case when $(p_l, p_{l+1}) \subset M_f^t$ and there is no such G_i that $p_l, p_{l+1} \in \partial G_i^t$, we always can split the segment $[p_l, p_{l+1}]$ into a finite number of parts in the way that for both endpoints of any segment of the partition there exists G_i such that these endpoints belong to ∂G_i^t.

It is worth noting also that if (t, x) (or (t, y)) is a point of continuity then $\eta = f(t, p_0+)$ (and respectively $v = f(t, p_{\bar{N}+1}-)$). Otherwise $\|\eta\| \leq K$ ($\|v\| \leq K$) since η and v are convex combinations of the limiting values of the function f, each of which is less or equal to K.

Thus, we have

$$\begin{aligned}
\|\eta - v\| &\leq \big\|\eta - f(t, p_0+)\big\| + \big\|f(t, p_0+) - f(t, p_1-)\big\| \\
&\quad + \big\|f(t, p_1-) - f(t, p_2+)\big\| + \ldots + \big\|f(t, p_{\bar{N}}+) - f(t, p_{\bar{N}+1}-)\big\| \\
&\quad + \big\|f(t, p_{\bar{N}+1}-) - v\big\| \\
&\leq \sum_{k=0}^{\bar{N}} L\|p_k - p_{k+1}\| + \sum_{l=0}^{\bar{N}+1} \big\|f(t, p_l+)\big\| + \sum_{l=0}^{\bar{N}+1} \big\|f(t, p_l-)\big\| \\
&\leq L\|x - y\| + 2K(N+2).
\end{aligned}$$

Thus, the inequality (20) holds with the constant $C = \max\{L, 2K(N+2)\}$. □

Lemma 3 *Let $f : D_n \to \mathbb{R}^n$ be a piecewise continuous function, each of whose domains of continuity satisfies condition Γ, and F_f be the Filippov set-valued map of f. Then for all $\varepsilon \in (0, 1]$ there exists a μ_1-integrable function $v_\varepsilon(t)$ such that for μ_1-almost all $t \in [-a\varepsilon, b\varepsilon]$ it holds that*

$$v_\varepsilon(t) \in H_\varepsilon(t),$$

where $H_\varepsilon(t) := F_f(t, A\phi_\varepsilon(t))$.

Proof It is a well known fact (see Chap. 2, Sect. 6, Lemma 3 in [3]) that the Filippov set-valued map $F_f(t, y)$ is upper semicontinuous with respect to the variable y. Moreover, due to condition Γ there exists a set-valued map $F_f^0(t, y)$ such that $F_f(t, y) = F_f^0(t, y)$ for μ_1-almost all $t \in D_0$ and $F_f^0(t, y)$ is upper semicontinuous with respect to the variables (t, y) (see Chap. 2, Sect. 6, Lemma 4 in [3]).

From its definition one can see that the set-valued map $H_\varepsilon(t)$ is upper semicontinuous in t and, consequently (see [7]), has a Borel selection v (see [8, 14]). Since F_f^0 is semicontinuous in (t, y), it is bounded (see Chap. 2, Sect. 5, Lemma 15 in [3]) on the compact set containing the graph of $A\phi_\varepsilon(t)$. Therefore, the selection v is bounded and, consequently, μ_1-integrable. □

Corollary 1 *The differential inclusion*

$$\dot{z}_\varepsilon(t) \in F_f\big(t, A\phi_\varepsilon(t)\big), \quad z_\varepsilon(-a\varepsilon) = x_{0\varepsilon}, \tag{22}$$

has a solution on the segment $[-a\varepsilon, b\varepsilon]$, *but it can be nonunique.*

Proof of Theorem 2 It follows from inequality (20) that the set-valued function F_f has linear growth with respect to the variable y, uniformly with respect to the variable t. Therefore, estimate (13) transforms into the estimate

$$\begin{aligned}\big\|x_\varepsilon(t) - x_{0\varepsilon}\big\| &\le \left(1 + Cd_\varepsilon\big(1 + \|x_{0\varepsilon}\|\big) + C\|A\| \int_{-a}^{b} \big\|\phi(u)\big\| du\right) e^{Cd_\varepsilon} \\ &=: \lambda_\varepsilon, \quad t \in [-a\varepsilon, b\varepsilon],\end{aligned} \tag{23}$$

which holds for all $\varepsilon \in (0, 1]$ and all solutions $x_\varepsilon(\cdot)$ to problem (11).

We estimate the difference between any two solutions to problem

$$\dot{x}_\varepsilon(t) \in F_f\big(t, x_\varepsilon(t) + A\phi_\varepsilon(t)\big), \quad x_\varepsilon(-a\varepsilon) = x_{0\varepsilon},$$

and to the problem

$$\dot{z}_\varepsilon(t) \in F_f\big(t, A\phi_\varepsilon(t)\big), \quad z_\varepsilon(-a\varepsilon) = x_{0\varepsilon}.$$

It holds that

$$\big\|x_\varepsilon(t) - z_\varepsilon(t)\big\| \le \int_{-a\varepsilon}^{t} \big\|\eta_\varepsilon(u) - v_\varepsilon(u)\big\| du,$$

where $\eta_\varepsilon(u) \in F_f(u, x_\varepsilon(u) + A\phi_\varepsilon(u))$, $v_\varepsilon(u) \in F_f(u, A\phi_\varepsilon(u))$ for μ_1-almost all $u \in [-a\varepsilon, b\varepsilon]$.

Using inequalities (20) and (23) we have

$$\begin{aligned}\big\|x_\varepsilon(t) - z_\varepsilon(t)\big\| &\le C \int_{-a\varepsilon}^{t} \big(1 + \big\|x_\varepsilon(u)\big\|\big) du \\ &= C \int_{-a\varepsilon}^{t} \big(1 + \big\|x_\varepsilon(u) - x_{0\varepsilon} + x_{0\varepsilon}\big\|\big) du \le Cd_\varepsilon\big(1 + \lambda_\varepsilon + \|x_{0\varepsilon}\|\big).\end{aligned}$$

Consequently, $x_\varepsilon(b\varepsilon) - z_\varepsilon(b\varepsilon) \to 0$ uniformly with respect to the choice of $x_\varepsilon(\cdot)$ and $z_\varepsilon(\cdot)$ as $\varepsilon \to 0$.

For each $\varepsilon \in (0,1]$ we choose any solution $z_\varepsilon(\cdot)$ to problem (22) and determine the jump of the limiting solution. Define the sets

$$A_w^j := \begin{cases} \{u | \phi^{i_j}(u) > 0\}, & w_j = 1, \\ \{u | \phi^{i_j}(u) < 0\}, & w_j = -1, \end{cases} \qquad j = \overline{1,\tau},$$

$$A_w := \bigcap_{j=1}^{\tau} A_w^j.$$

Then ignoring the sets where at least one of the function $\phi^{i_j}(u)$ equals to zero we have

$$\begin{aligned} z_\varepsilon^k(b\varepsilon) &= x_{0\varepsilon}^k + \int_{-a\varepsilon}^{b\varepsilon} v_\varepsilon^k(u)du = x_{0\varepsilon}^k + \varepsilon \int_{-a}^{b} v_\varepsilon^k(\varepsilon\xi)d\xi \\ &= x_{0\varepsilon}^k + \sum_{w\in\{-1,1\}^\tau} \alpha_{i_\varsigma(w)} \int_{[-a,b]\cap A_w} \frac{v_\varepsilon^k(\varepsilon\xi)}{\alpha_{i_\varsigma(w)}\varepsilon^{-1}\phi^{i_\varsigma(w)}(\xi)} \phi^{i_\varsigma(w)}(\xi)d\xi, \end{aligned}$$

where $v_\varepsilon(\varepsilon\xi) = (v_\varepsilon^1(\varepsilon\xi), \dots, v_\varepsilon^n(\varepsilon\xi)) \in F_f(\varepsilon\xi, \varepsilon^{-1}A\phi(\xi))$.

For fixed $\xi \in A_w$ and sufficiently small ε we have

$$F_f\big(\varepsilon\xi, \varepsilon^{-1}A\phi(\xi)\big) = f\big(\varepsilon\xi, \varepsilon^{-1}A\phi(\xi)\big) = v_\varepsilon(\varepsilon\xi).$$

Consequently,

$$\frac{v^k(\varepsilon\xi)}{\alpha_{i_\varsigma(w)}\varepsilon^{-1}\phi^{i_\varsigma(w)}(\xi)}\phi^{i_\varsigma(w)}(\xi) \to M_{k,(\mathrm{sign}(\alpha_{i_1})\cdot w_1,\dots,\mathrm{sign}(\alpha_{i_\tau})\cdot w_\tau)}^{i_\varsigma(w),(i_1,\dots,i_\tau)}\phi^{i_\varsigma(w)}(\xi)$$

as $\varepsilon \to 0$.

At the same time

$$\begin{aligned} \left|\frac{v^k(\varepsilon\xi)}{\alpha_{i_\varsigma(w)}\varepsilon^{-1}\phi^{i_\varsigma(w)}(\xi)}\phi^{i_\varsigma(w)}(\xi)\right| &\le \frac{\varepsilon}{|\alpha_{i_\varsigma(w)}|}\big|v^k(\varepsilon\xi)\big| \\ &\le \frac{C\varepsilon}{\min_{j=\overline{1,\tau}}|\alpha_{i_j}|}\big(1+\varepsilon^{-1}\big\|A\phi(\xi)\big\|\big) \\ &\le \frac{C}{\min_{j=\overline{1,\tau}}|\alpha_{i_j}|}\Big(1+\|A\|\max_{\xi\in[-a,b]}\big\|\phi(\xi)\big\|\Big). \end{aligned}$$

Let

$$\phi_w^{i_\varsigma(w)}(\xi) := \begin{cases} \phi^{i_\varsigma(w)}(\xi), & \xi \in [-a,b]\cap A_w, \\ 0, & \xi \notin [-a,b]\cap A_w. \end{cases}$$

Then an application of Lebesgue's dominated convergence theorem finishes the proof. □

To illustrate the conditions of Theorem 2, we discuss the following

Example 5 Let be $T_0 = 3$, $n = 2$, $y_0 = (0, 0)^T$, $\alpha_1 = 0$, $\alpha_2 \neq 0$,

$$f_1(t, y_1, y_2) = H\Big(|y_2| - \sqrt{4 - t^2 - y_1^2}\Big),$$

$$f_2(t, y_1, y_2) = H\Bigg(\sqrt{\frac{1}{4} - t^2 - y_1^2} - |y_2|\Bigg),$$

where $H(\cdot)$ is the Heaviside function. Then

$$\Psi_2^1 : t^2 + y_1^2 + y_2^2 = 4, \qquad \Psi_2^2 : t^2 + y_1^2 + y_2^2 = \frac{1}{4},$$

$N_1 = N_2 = 2$ and, consequently, $q = 2$, $m = 3$.

Since there is only one coefficient $\alpha_2 \neq 0$, we have $\tau = 1$, $i_j = i_1 = 2$, $w \in \{(-1, -1), (-1, 1), (1, -1), (1, 1)\}$, $\hat{y}_1 = 0$, $\hat{y}_2 = y_2$, $i_\varsigma = i_1 = 2$.

Therefore,

$$M^{2,(2)}_{1,(-1,\pm 1)} = M^{2,(2)}_{1,(1,\pm 1)} = \lim_{\substack{t \to 0 \\ y_2 \to \pm\infty}} \frac{f_1(t, 0, y_2)}{y_2} = \lim_{y_2 \to \pm\infty} \frac{1}{y_2} = 0,$$

$$M^{2,(2)}_{2,(-1,\pm 1)} = M^{2,(2)}_{2,(1,\pm 1)} = \lim_{\substack{t \to 0 \\ y_2 \to \pm\infty}} \frac{f_2(t, 0, y_2)}{y_2} = \lim_{y_2 \to \pm\infty} \frac{0}{y_2} = 0.$$

Solving the equations (3) with initial value $y_0 = (0, 0)^T$ in the domains of continuity of the function f, one can get that the function

$$y(t) = \begin{cases} (0, 0)^T, & t \in [-1, -\frac{1}{2}], \\ (0, t + \frac{1}{2})^T, & t \in [-\frac{1}{2}, 0], \\ (0, \frac{1}{2})^T, & t \in [0, \frac{\sqrt{15}}{2}], \\ (t - \frac{\sqrt{15}}{2}, \frac{1}{2})^T, & t \in [\frac{\sqrt{15}}{2}, 3] \end{cases}$$

is the unique Filippov solution on D_0. Therefore Assumption A holds. Moreover, it is immediate to see that Assumption B holds as well. Thus, the non-distribution part of the limiting solution does not have a jump in this case.

Remark 6 The assumption

$$\mu_1\big(\{t \in [-a, b] | \phi^i(t) = 0\}\big) = 0, \quad i = \overline{1, n},$$

can be replaced with the requirement that additional limits as in condition (3) of Theorem 2 have to exist.

The Theorem 2 can be easily modified to the case of an unbounded set M_f.

Theorem 3 *Assume that*

(1) *the function* $f : D_n \to \mathbb{R}^n$ *is piecewise continuous having a finite number of domains of continuity and the set* M_f *has the form*

$$M_f = \Psi_n^1 \cup \ldots \cup \Psi_n^q, \tag{24}$$

where hypersurfaces $\Psi_n^i \in RS_{N_i}$, $i = \overline{1, q}$, $q \in \mathbb{N}$, *given by equalities* $\psi_i(t, y_1, \ldots, y_n) = 0$, *are bounded at least with respect to one of the variables* y_j, $j \in \{1, \ldots, n\}$ *and each domain of continuity of the function* f *satisfies condition* Γ. *Moreover, let Assumptions* A *and* B *be made*;

(2) *the functions*

$$K_i : \Psi_i \to \mathbb{R}^n, (t, y) \mapsto \max_{j: y \in \partial G_j^t} \lim_{\substack{y^* \to y \\ y^* \in G_j^t}} \left\| f\left(t, y^*\right) \right\|, \quad i = \overline{1, q}$$

are bounded.

Further, suppose that the conditions (2)–(3) *of Theorem* 2 *hold. Then the assertion of Theorem* 2 *remains valid.*

Lemma 4 *Suppose the function* $f : D^n \to \mathbb{R}^n$ *satisfies conditions* (1)–(2) *of Theorem* 3 *and condition* (2) *of Theorem* 2. *Then the statement of the Lemma* 2 *remains valid.*

Proof The proof of Lemma 1 remains the same if one replaces the constant K by $\bar{K}$, where $\bar{K}$ is the maximal constant among all constants bounding the functions K_i, $i = \overline{1, q}$. □

Proof of the Theorem 3 The proof of the Theorem 3 remains the same but with application of Lemma 4 instead of Lemma 2. □

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Chapter 9
Resolvent Estimates and Scattering Problems for Schrödinger, Klein-Gordon and Wave Equations

Kiyoshi Mochizuki

Abstract We survey some basic problems of Schrödinger, Klein-Gordon and wave equations in the framework of general scattering theory. The following topics are treated under suitable decay and/or smallness conditions on the perturbation term: Growth estimates of generalized eigenfunctions, Resolvent estimates, Scattering direct and inverse problems, Smoothing properties and Strichartz estimates. Due to our formulation of the weighted energy method, some topics are naturally extended to time-dependent and/or non-selfadjoint perturbations.

Mathematics Subject Classification 81Q10 · 81Uxx · 35R30

9.1 Introduction

This article will summarize with some addendum and modification the following four papers which remain to the author as a personal history of participation in ISAAC: [13] (August 2001, Berlin), [14] (July 2005, Catania), [15] (August 2007, Ankara), [16] (July 2009, London).

In the first 3 sections we deal with the selfadjointness, growth estimates, the principle of limiting absorption, spectral representations, and the existence and completeness of the Møller wave operators for magnetic Schrödinger operators with singular, short range potentials. A most important ingredient for these problems is the growth estimate of the generalized eigenfunctions. As in the previous results the proof is given by formulating a differential inequality for a functional of solutions. The functional is adopted in [14] to include an approximate phase of the operator. Then the principle of limiting absorption yields directly the estimates. In Sect. 9.5 an inverse scattering problem of [13] is generalized to wave equations with both "dissipation" and potential terms. We give a reconstruction procedure of both coefficients from the scattering amplitude with a fixed energy.

Smoothing properties for magnetic Schrödinger operators are treated in Sect. 9.6 based on the uniform resolvent estimates. As it is seen in [16], smallness con-

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ditions are required on the perturbation terms for these purposes. The smoothing properties are used in Sect. 9.8 and Sect. 9.9 to treat scattering and Strichartz estimates, respectively, for Schrödinger, Klein-Gordon and wave equations under time dependent small perturbations. In [15] we did not enter into the Strichartz estimates and excluded to treat Klein-Gordon equations there. Moreover, in Sect. 9.7 decay-nondecay properties of solutions in L^2 are illustrated to dissipative Schrödinger evolution equations.

9.2 Selfadjointness of Magnetic Schrödinger Operators

Let Ω be an exterior domain in $\mathbf{R}^n$ with smooth compact boundary $\partial\Omega$ (the case $\Omega = \mathbf{R}^n$ is not excluded). We consider in Ω the Schrödinger operator

$$Lu = -\sum_{j=1}^{n}\left\{\partial_j + ib_j(x)\right\}^2 u + c(x)u, \tag{1}$$

where $x = (x_1, \ldots, x_n) \in \mathbf{R}^n$, $\partial_j = \partial/\partial x_j$, $i = \sqrt{-1}$, $b_j(x)$ are real valued C^1-functions of $x \in \mathbf{R}^n$ and $c(x)$ is a real valued continuous function of $x \in \mathbf{R}^n\backslash\{0\}$. $b(x) = (b_1(x), \ldots, b_n(x))$ represents a magnetic potential. Thus the magnetic field is defined by its rotation $\nabla \times b(x)$. As it is seen in ($A1$), the external potential $c(x)$ may have a singularity like $O(|x|^{-2})$ at $x = 0$ when $\Omega = \mathbf{R}^n$.

In the following we put $\nabla_b = \nabla + b(x)$, $\Delta_b = \nabla_b \cdot \nabla_b$, $r = |x|$, $\tilde{x} = x/r$ and $\partial_r = \tilde{x} \cdot \nabla$. The inner product and norm of the Hilbert space $L^2 = L^2(\Omega)$ are defined by

$$(f, g) = \int f(x)\overline{g(x)}dx \quad \text{and} \quad \|f\| = \sqrt{(f, f)}.$$

Here we specify by $\int dx$ the integration over Ω. For a function $\mu = \mu(r) > 0$ let L^2_μ be the weighted L^2-space with norm $\|f\|^2_\mu = \int \mu(r)|f(x)|^2 dx < \infty$.

We assume

$$\exists c_\infty(x) \in L^\infty \quad \text{such that} \quad c(x) - c_\infty(x) \geq \frac{\beta}{r^2} \text{ with } \beta > -\frac{(n-2)^2}{4}. \tag{$A1$}$$

Theorem 1 *Under ($A1$) let L be defined by*

$$\begin{cases} Lu = -\Delta_b u + c(x)u \quad \text{for } u \in \mathcal{D}(L), \\ \mathcal{D}(L) = \{u \in L^2 \cap H^2_{\text{loc}}(\overline{\Omega}\backslash\{0\}); (-\Delta_b + c)u, r^{-1}u \in L^2, u|_{\partial\Omega} = 0\}. \end{cases} \tag{2}$$

Then it gives a lower semibounded selfadjoint operator in L^2.

To show that (2) determines the Friedrichs extension of the operator (1) initially defined on $C_0^\infty(\Omega\backslash\{0\})$, the following lemma plays a crucial role (cf. Kalf et al. [8] where is treated the case $b(x) \equiv 0$).

Lemma 1

(i) *If* $u \in \mathcal{D}(L)$, *then we have* $\nabla_b u \in [L^2]^n$.
(ii) (*the Hardy inequality*) *If* $\nabla_b u \in [L^2]^n$, *then*

$$\int \frac{(n-2)^2}{4r^2}|u|^2 dx \leq \int |\tilde{x} \cdot \nabla_b u|^2 dx.$$

The essential spectrum $\sigma_e(L)$ of L is included in the half line $[0, \infty)$ if $c(x) \to 0$ as $|x| \to \infty$. To investigate further properties of the essential spectrum we prepare a quadratic identity for solutions u to the Schrödinger equation

$$-\Delta_b u + c(x)u - \kappa^2 u = f(x), \tag{3}$$

where $\kappa \in \Pi_\pm = \{\kappa \in \mathbf{C}; \pm \mathrm{Re}\,\kappa > 0, \mathrm{Im}\,\kappa \geq 0\}$ and $f \in L^2$.

We put $v = e^{-i\kappa r} r^{(n-1)/2} e^{\sigma(r)} u$, $g = e^{-i\kappa r} r^{(n-1)/2} e^{\sigma(r)} f$ and rewrite (3) as follows:

$$\begin{aligned} -\nabla_b \cdot \nabla_b v &+ \left(-2i\kappa + \frac{n-1}{r} + 2\sigma'\right)\tilde{x} \cdot \nabla_b v \\ &+ \left(c + \frac{(n-1)(n-3)}{4r^2} + \sigma'' - \sigma'^2 + 2i\kappa\sigma'\right)v = g. \end{aligned}$$

Multiply by $\phi \overline{\tilde{x} \cdot \nabla_b v}$, where $\phi = \phi(r) = e^{-2\,\mathrm{Im}\,\kappa r} r^{-n+1} \varphi(r)$, this equation and integrate the real part over $B_{R,t}$, where for $0 < s < t$ we put $B_{s,t} = \{x; s < |x| < t\}$, $B_t = \{x; |x| < t\}$, $B'_t = \mathbf{R}^n \backslash B_t$ and $S_t = \{x; |x| = t\}$. Then noting

$$\nabla_b v = e^{-i\kappa r} r^{(n-1)/2} \left\{\nabla_b\left(e^\sigma u\right) + \tilde{x}\left(\frac{n-1}{2r} - i\kappa\right)\left(e^\sigma u\right)\right\},$$

$$\phi'(r) = \phi(r)\left(-2\,\mathrm{Im}\,\kappa - \frac{n-1}{r} + \frac{\varphi'}{\varphi}\right),$$

we obtain (cf., [12] or [16])

Proposition 1 *Let* $\varphi = \varphi(r)$ *and* $\sigma = \sigma(r)$ *be a smooth nonnegative function of* $r > 0$. *Put* $u_\sigma = e^\sigma u$, $f_\sigma = e^\sigma f$ *and*

$$\theta_\sigma = \theta_\sigma(x, \kappa) = \nabla_b u_\sigma + \tilde{x}\left(\frac{n-1}{2r} - i\kappa\right)u_\sigma.$$

Then

$$\begin{aligned} &\left[\int_{S_t} - \int_{S_R}\right]\varphi\left\{-|\tilde{x} \cdot \theta_\sigma|^2 + \frac{1}{2}|\theta_\sigma|^2\right\}dS + \int_{B_{R,t}} \varphi\left\{\left(\frac{\varphi'}{\varphi} - \frac{1}{r}\right)|\tilde{x} \cdot \theta_b|^2\right. \\ &\quad + \left(\mathrm{Im}\,\kappa - \frac{\varphi'}{2\varphi} + \frac{1}{r}\right)|\theta_\sigma|^2 + 2\sigma'|\tilde{x} \cdot \theta_\sigma|^2 + \mathrm{Re}\, J_\sigma(x, \kappa) \\ &\quad \left. + \mathrm{Re}\left[\left(\sigma'' - \sigma'^2 + 2i\kappa\sigma'\right)u_\sigma \overline{\tilde{x} \cdot \theta_\sigma}\right]\right\}dx = \mathrm{Re} \int_{B_{R,t}} \varphi f_\sigma \overline{\tilde{x} \cdot \theta_\sigma} dx; \end{aligned} \tag{4}$$

$$J_\sigma(x,\kappa) = -(\tilde{x}\times\theta_\sigma)\cdot\overline{(\nabla\times ib)u_\sigma} + \left(c + \frac{(n-1)(n-3)}{4r^2}\right)u_\sigma\overline{\tilde{x}\cdot\theta_\sigma}. \quad (5)$$

The main calculation is in the equality

$$-\nabla_b\cdot\nabla_b v\phi\overline{\tilde{x}\cdot\nabla_b v} = -\nabla\cdot\{\nabla_b v\phi\overline{\tilde{x}\cdot\nabla_b v}\} + (\nabla\phi\cdot\nabla_b v)\overline{\tilde{x}\cdot\nabla_b v} + \phi\nabla_b v\cdot\overline{\nabla_b(\tilde{x}\cdot\nabla_b v)}.$$

The last term of the right applied by

$$(\partial_j + ib_j)\big\{\tilde{x}_k(\partial_k + ib_k)v\big\} = \tilde{x}_k\partial_k\big\{(\partial_j + ib_j)v\big\} + \frac{\delta_{jk} - \tilde{x}_j\tilde{x}_k}{r}(\partial_k + ib_k)v + i\tilde{x}_k(\partial_j b_k - \partial_k b_j)v$$

brings the term

$$-i\varphi\sum_{j=1}^{n}\theta_{\sigma j}\sum_{k=1}^{n}\tilde{x}_k\{\partial_j b_k - \partial_k b_j\}\overline{u_\sigma} = -\varphi(\tilde{x}\times\theta_\sigma)\cdot\overline{(\nabla\times ib)u_\sigma}.$$

9.3 Growth Estimate of Generalized Eigenfunctions and Principle of Limiting Absorption

First consider the homogeneous equation

$$-\Delta_b u + c(x)u - \lambda u = 0, \quad \lambda > 0, \quad (6)$$

with $b(x)$ and $c(x)$ satisfying the additional condition

$$\max\big\{\big|\nabla\times b(x)\big|, \big|c(x)\big|\big\} \le \mu(r), \quad r = |x| > \exists R_0, \quad (A2)$$

where $\mu = \mu(r)$ is a smooth, positive, non-increasing L^1-function of $r \in \mathbf{R}_+ = (0,\infty)$.

Theorem 2 *Under* ($A1$), ($A2$) *let* $u \in H^2_{\mathrm{loc}}(\overline{\Omega}\backslash\{0\})$ *solve* (6). *If the support of* u *is not compact*, *then*

$$\liminf_{t\to\infty}\int_{S_t}\big|\tilde{x}\cdot\theta(x,\pm\sqrt{\lambda})\big|^2 dS \neq 0,$$

where $\theta(x,\kappa) = \nabla_b u + \tilde{x}(\frac{n-1}{2r} - i\kappa)u$ *for* $\kappa \in \mathbf{C}$.

If we additionally require the following condition ($A3$), then the unique continuation property is applicable to show the non-existence of positive eigenvalues of L from this theorem. The condition ($A3$) reads as follows:

$$\nabla b_j(x) (j = 1,\dots,n) \text{ and } c(x) \text{ are locally Hölder continuous.} \quad (A3)$$

The following identities are used to show Theorem 2.

Lemma 2 *Let u be a solution of the homogeneous equation* (6). *Then for each* $\lambda > 0$ *and* $r > 0$ *we have*

$$\mathrm{Im}\left[\int_{S_r} (\tilde{x} \cdot \nabla_b u_\sigma)\overline{u_\sigma} dS\right] = 0,$$

$$\int_{S_r} \left\{\left|\tilde{x} \cdot \nabla_b u_\sigma + \frac{n-1}{2r} u_\sigma\right|^2 + \lambda |u_\sigma|^2\right\} dS = \int_{S_r} \left|\tilde{x} \cdot \theta_\sigma(x, \pm\sqrt{\lambda})\right|^2 dS.$$

We define

$$F(r) = \frac{1}{2}\int_{S_r} \left\{2|\tilde{x} \cdot \theta|^2 - |\theta|^2\right\} dS,$$

$$F_{\sigma,\tau} = \frac{1}{2}\int_{S_r} \left\{2|\tilde{x} \cdot \theta_\sigma|^2 - |\theta_\sigma|^2 + \left(\sigma'^2 - \tau\right)|u_\sigma|^2\right\} dS,$$

where $\tau = \tau(r)$ is another weight function. The proof of Theorem 2 is divided into two parts. The identity (4) with $\kappa^2 = \lambda$, $f \equiv 0$, $\sigma = 0$ and $\varphi \equiv 1$ is used if there exists a sequence $r_k \to \infty$ such that $F(r_k) > 0$. On the other hand, if $F(r) \leq 0$ for $r \geq R_1$ $(\geq R_0)$ and u does not have a compact support, we use (4) choosing $\varphi = r$ and

$$\sigma(r) = \frac{m}{1-\varepsilon} r^{1-\varepsilon} \quad (m \geq 1, 1/3 < \varepsilon < 1/2), \qquad \tau(r) = r^{-2\varepsilon} \log r.$$

In the following we choose the weight function $\mu(r)$ to satisfy also

$$\int_r^\infty \mu(s) ds \geq r\mu(r) \quad \text{for } r \geq R_0. \tag{7}$$

Typical examples are $C(1+r)^{-1-\delta}$, $C(1+r)^{-1}[\log(1+r)]^{-1-\delta}$ $(C > 0, 0 < \delta < 1)$. These examples also satisfy (32) which is given later in Sect. 9.7.

We put $\varphi_1(r) = (\int_r^\infty \mu(s)ds)^{-1}$. Then $\varphi_1'(r) = \mu(r)\varphi_1(r)^2$. So $\mu\varphi_1$ and, hence, $\varphi_1' = \mu\varphi_1^2$ are not in $L^1(\mathbf{R}_+)$. Moreover,

$$\frac{\varphi_1'(s)}{\varphi_1(s)} = \mu(r)\varphi_1(r) \leq \frac{1}{r} \quad \text{for } r \geq R_0.$$

Lemma 3 *Let* $u = R(\kappa^2)f$ *with* $\kappa \in K_\pm$ *and* $f \in L^2_{\mu_1^{-1}}$. *Then* $\exists R_5 \geq R_0$ *and* $C = C(K_\pm, R_5) > 0$ *such that for* $R \geq R_5$,

$$\|\theta\|^2_{\varphi_1', B_R'} \leq C\left\{\|u\|^2_\mu + \|f\|^2_{\mu^{-1}}\right\},$$

$$a\|u\|^2_{\mu, B_R'} \leq C\varphi_1(R)^{-1}\left\{\|\tilde{x} \cdot \theta\|^2_{\varphi_1', B_R'} + \|u\|^2_\mu + \|f\|^2_{\mu^{-1}}\right\}.$$

The first inequality of this lemma is derived from Proposition 1 with $\sigma = 0$ and $\varphi = \varphi_1$. On the other hand, the second inequality is a result of the Gauss formula.

With these inequalities and Theorem 2, the Rellich compactness criterion shows the following assertion by contradiction.

Theorem 3 *Assume (A1), (A2) with μ satisfying also (7) and (A3). Then for any $0 < a < b < \infty$, the resolvent $R(\kappa^2) \in \mathcal{B}(L^2_{\mu^{-1}}, L^2_{\mu})$ restricted to $\kappa \in K_\pm = \{\kappa; a \leq \pm \operatorname{Re}\kappa \leq b, 0 < \operatorname{Im}\kappa \leq 1\}$ is continuously extended to $K_\pm \cup [a, b]$ as an operator from $L^2_{\mu^{-1}}$ to L^2_{μ}. Thus, the positive spectrum of L is absolutely continuous with respect to the Lebesgue measure.*

Theorem 2 is a real generalization of Rellich [20] (cf. Kato [9]). Theorem 3 states the principle of limiting absorption, the proof of which is originated by Eidus [3]. A more general oscillating long-range potential is treated in [14] (also Jäger-Rejto [7]).

9.4 Spectral Representations and Scattering

The Fourier transform $\hat{f}(\xi) = (2\pi)^{-n/2} \int e^{-ix\cdot\xi} f(x)dx$ determines the spectral representation of L_0. Namely, put

$$\big[F_0(\sigma)f\big](\omega) = \sigma^{(n-1)/2} \hat{f}(\sigma\omega), \quad \omega \in S^{n-1},$$

$$\big[F_0^*(\sigma)h\big](x) = \sigma^{(n-1)/2}(2\pi)^{-n/2} \int_{S^{n-1}} e^{i\sigma x\cdot\omega} h(\omega)dS_\omega, \quad h \in L^2\big(S^{n-1}\big).$$

Then $[F_0 f](\sigma, \omega) = [F_0(\sigma)f](\omega)$ gives a unitary operator from $L^2(\mathbf{R}^n)$ to $L^2(\mathbf{R}_+ \times S^{n-1})$ and its adjoint F_0^* is given by

$$\big[F_0^* h\big](x) = \int_0^\infty \big[F_0^*(\sigma)h(\sigma, \cdot)\big](x)d\sigma \quad \text{for } h(\sigma, \omega) \in L^2\big(\mathbf{R}_+ \times S^{n-1}\big).$$

In this section we require

$$\max\big\{\big|b(x)\big|, \big|\nabla b(x)\big|, \big|c(x)\big|\big\} \leq \mu(r), \quad r = |x| > \exists R_0 > 0. \tag{A4}$$

The decay condition for $b(x)$ itself is used to compare L with the free Laplacian $-\Delta$ in $L^2(\mathbf{R}^n)$. Let $j(r)$ be a C^∞-function of $r > 0$ such that $j(r) = 0$ $(r < R_0)$ and $= 1$ $(r > R_0 + 1)$, and define the operator $J : L^2(\mathbf{R}^n) \to L^2 = L^2(\Omega)$ and its adjoint J^* by

$$[Jf](x) = j(r)f(x), \quad x \in \Omega,$$

$$\big[J^* g\big](x) = j(r)g(x) \quad (x \in \Omega) \quad \text{and} \quad = 0 \quad \big(x \in \mathbf{R}^n \backslash \Omega\big).$$

Let $R_0(\kappa^2) = (L_0 - \kappa^2)^{-1}$. Then we have the following resolvent equation

$$R(\kappa^2)J = \{J - R(\kappa^2)V\}R_0(\kappa^2), \quad V = LJ - JL_0,$$
$$J^* R(\kappa^2) = R_0(\kappa^2)\{J^* - V^* R(\kappa^2)\}, \quad V^* = J^* L - L_0 J^*.$$

For each $\sigma \in \mathbf{R}_+$ we define

$$F_\pm(\sigma) = F_0(\sigma)\{J^* - V^* R(\sigma^2 \pm i0)\},$$
$$F_\pm^*(\sigma) = \{J - R(\sigma^2 \mp i0)V\}F_0^*(\sigma).$$

Theorem 4 *Assume* ($A1$), ($A3$) *and* ($A4$). *Then the operator*

$$[F_\pm f](\sigma, \omega) = [F(\pm\sigma) f](\omega), \quad (\sigma, \omega) \in \mathbf{R}_+ \times S^{n-1},$$

is extended to a unitary operator from $\{I - P\}L^2$ *onto* $L^2(\mathbf{R}_+ \times S^{n-1})$:

$$F_\pm^* F_\pm = I - P \quad in\ L^2 \quad (completeness),$$
$$F_\pm F_\pm^* = I \quad in\ L^2(\mathbf{R}_+ \times S^{n-1}) \quad (orthogonality),$$

where P *is the orthogonal projection onto the eigenspace of* L.

We define the operators $U_\pm$ and $\mathcal{S}$ by

$$U_\pm = F_\pm^* F_0, \qquad \mathcal{S} = U_+^* U_- = F_0^* F_+ F_-^* F_0.$$

Proposition 2 *The operators* $U_\pm : L^2(\mathbf{R}^n) \to (I - P)L^2(\Omega)$ *are unitary operators which intertwine* L_0 *and* L:

$$LU_\pm f = U_\pm L_0 f, \quad f \in \mathcal{D}(L_0).$$

The operator $\mathcal{S}$ *is a unitary operator in* $L^2(\mathbf{R}^n)$ *which commutes with* L_0.

Now, let us consider the Schrödinger evolution operators e^{-itL} and e^{-itL_0}. Theorem 4 implies that for $f \in (I - P)L^2(\Omega)$ and $f_0 \in L^2(\mathbf{R}^n)$,

$$e^{-itL} f = F_\pm e^{-i\sigma^2 t} F_\pm f, \qquad e^{-itL_0} f_0 = F_0 e^{-i\sigma^2 t} F_0 f_0.$$

Theorem 5 *Assume* ($A1$), ($A3$) *and* ($A4$). *Then the Møller wave operator exists and coincides with* $U_\pm$:

$$s - \lim_{t \to \pm\infty} e^{itL} J e^{-itL_0} = U_\pm.$$

Thus, $S = U_+^* U_-$ *defines the Møller scattering operator, which representation is given in the momentum space* $L^2(\mathbf{R}_+ \times S^{n-1})$ *by*

$$F_0 S F_0^* = \hat{I} - \hat{T}, \qquad [\hat{T} \hat{f}](\sigma, \omega) = \frac{1}{2\sigma}[F_+(\sigma) V F_0^* \hat{f}(\sigma, \cdot)](\omega).$$

The kernel of $\hat{T}$ *is called the scattering amplitude.*

A survey of the classical stationary approach on short-range scattering is given above. We can find a detailed description e.g. in Mochizuki [12].

9.5 Inverse Scattering for Small Nonselfadjoint Perturbations of Wave Equations

We consider the wave equation of the form

$$w_{tt} + b(x)w_t - \Delta w + c(x)w = 0, \quad (x,t) \in \mathbf{R}^n \times \mathbf{R}, \tag{8}$$

where $n \geq 3$ and $b(x)$ and $c(x)$ are real, continuous functions satisfying

$$\big|b(x)\big| \leq \varepsilon_0 \mu(r), \qquad \frac{\beta}{r^2} < c(x) \leq \mu(r) \tag{A5}$$

with $\varepsilon_0 > 0$ (small) and $\beta > -\frac{(n-2)^2}{4}$. Here $\mu(r)$ is a positive L^1-function satisfying (7).

We rewrite (8) in the form

$$i\partial_t u = \Lambda u \equiv \Lambda_0 u + Vu, \quad u = \{w, w_t\};$$
$$\Lambda_0 = i\begin{pmatrix} 0 & 1 \\ \Delta & 0 \end{pmatrix} \quad \text{and} \quad V = -i\begin{pmatrix} 0 & 0 \\ c(x) & b(x) \end{pmatrix}.$$

Let $\mathcal{H}_E = \dot{H}^1 \times L^2$ be the Hilbert space with energy norm

$$\|f\|_E^2 = \frac{1}{2}\big\{\|\nabla f_1\|^2 + \|f_2\|^2\big\}, \quad f = \{f_1, f_2\}.$$

The operator Λ_0 is selfadjoint in $\mathcal{H}_E$, and its spectral representation is determined by

$$\mathcal{F}_0(\lambda) = \frac{1}{2}F_0\big(|\lambda|\big)\begin{pmatrix} 1 & i\lambda^{-1} \\ -i\lambda & 1 \end{pmatrix} \quad (\lambda \neq 0).$$

The spectral representation of Λ is then given by

$$\mathcal{F}_\pm(\lambda) = \mathcal{F}_0(\lambda)\big\{I - V\mathcal{R}(\lambda \pm i0)\big\}, \qquad \mathcal{F}_\pm^{(*)}(\lambda) = \big\{I - \mathcal{R}(\lambda \mp i0)V\big\}\mathcal{F}_0^*(\lambda),$$

where $\mathcal{R}(\zeta)$ is the resolvent of Λ. Since the coefficient $b(x)$ of the nonselfadjoint part is small, $\mathcal{R}(\zeta) \in \mathcal{B}(\mathcal{H}_{E,\mu^{-1}}, \mathcal{H}_{E,\mu})$ is extended continuously to $\zeta = \lambda \pm i0$ $(\lambda \in \mathbf{R}\backslash\{0\})$.

Proposition 3 *There exists the strong limit*

$$\mathcal{W}_\pm = s - \lim_{t\to\pm\infty} e^{itA}e^{-itA_0}.$$

It is expressed as $\mathcal{W}_\pm = \mathcal{F}_\pm^{(*)}\mathcal{F}_0$, *and defines a bijection in* $\mathcal{H}_E$. *The scattering operator exists and is given by*

$$\mathcal{S} = \mathcal{W}_+^{-1}\mathcal{W}_- = \mathcal{F}_0^*\mathcal{F}_+^{(*)-1}\mathcal{F}_-^{(*)}\mathcal{F}_0.$$

The last assertion gives us $\mathcal{F}_0(I-\mathcal{S})\mathcal{F}_0^* = \mathcal{F}_+(\mathcal{F}_+^{(*)} - \mathcal{F}_-^{(*)})$. Thus the scattering amplitude $\mathcal{A}(\lambda)$ with energy $\lambda \neq 0$ is expressed as

$$2\pi i\mathcal{A}(\lambda) \equiv \mathcal{F}_0(\lambda)\big\{I - \mathcal{S}(\lambda)\big\}\mathcal{F}_0^* = \frac{\pi i}{2}\hat{T}(\lambda)\begin{pmatrix} 1 & i\lambda^{-1} \\ -i\lambda & 1 \end{pmatrix},$$

where $\hat{T}(\lambda)$ is the scalar amplitude given by

$$\hat{T}(\lambda) = \lambda^{-1}F_0\big(|\lambda|\big)\big\{1 + q(\cdot,\lambda)R\big(\lambda^2 - i0,\lambda\big)\big\}q(\cdot,\lambda)F_0^*\big(|\lambda|\big)$$

with $R(\zeta^2,\alpha) = (-\Delta + c - i\alpha b - \zeta^2)^{-1}$ and $q(x,\alpha) = c - i\alpha b$.

The operator $\hat{T}(\lambda)$ is an integral operator on S^{n-1} with kernel

$$a\big(\lambda,\omega,\omega'\big) = (2\pi)^{-n}\lambda^{n-2}\bigg[\int e^{-i\lambda(\omega-\omega')\cdot x}q(x,\lambda)dx + \int e^{-i\lambda\omega\cdot x}q(x,\lambda)R\big(\lambda^2 - i0,\lambda\big)\big\{q(\cdot,\lambda)e^{i\lambda\theta'\cdot}\big\}(x)dx\bigg]. \quad (9)$$

Our aim is to derive a reconstruction procedure of $b(x)$ and $c(x)$ from the kernel $a(\lambda,\omega,\omega')$.

The following result is well known as the high energy Born approximation.

Theorem 6 *In the case* $b(x) \equiv 0$, *if we further require* $c(x) \in L^1(\mathbf{R}^n)$, *then for any* $\xi \in \mathbf{R}^n$ *we can choose* $\omega(\lambda), \omega'(\lambda) \in S^{n-1}$ *to satisfy* $\lambda\{\omega(\lambda) - \omega'(\lambda)\} = \xi$ *and*

$$\lim_{\lambda\to\infty}(2\pi)^n\lambda^{-n+2}a\big(\lambda,\omega(\lambda),\omega'(\lambda)\big) = \int e^{-i\xi\cdot x}c(x)dx.$$

In the case $b(x) \not\equiv 0$, however, $\|R(\lambda^2 - i0,\lambda)\|_{L^2_{\mu^{-1}},L^2_\mu}$ does in general not decay as $|\lambda| \to \infty$. To fill up, we restrict $b(x)$, $c(x)$ to exponentially decreasing functions, and introduce the so called nonphysical Faddeev resolvent ([5]).

Let $k \in \mathbf{R}^n$, $\gamma \in S^{n-1}$, $\varepsilon \geq 0$. We simply write $\zeta^2 = \zeta\cdot\zeta$ for $\zeta \in \mathbf{C}^n$, and both the resolvent and its kernel by $R_0(\kappa^2)$. Then since

$$R_0\big((k + i\varepsilon\gamma)^2\big) = (2\pi)^{-n}\int \frac{e^{i(x-y)\cdot\xi}}{\xi^2 - k^2 + \varepsilon^2 - 2i\varepsilon\gamma\cdot k}d\xi,$$

choosing γ to satisfy $t=\gamma\cdot k\geq 0$ and putting $\xi=\eta+t\gamma$, we have

$$R_0\big((k+i\varepsilon\gamma)^2\big)=(2\pi)^{-n}\int\frac{e^{i(x-y)\cdot(\eta+t\gamma)}}{\eta^2+2t\gamma\cdot\eta-(k^2-\varepsilon^2-t^2)-2i\varepsilon\gamma\cdot k}d\eta.$$

We let $\varepsilon\to+0$ and define the Faddeev unperturbed resolvent depending on γ by

$$\begin{aligned}R_{\gamma,0}\big(k^2,t\big)&=e^{it\gamma\cdot x}G_{\gamma,0}\big((k-t\gamma)^2,t\big)e^{-it\gamma\cdot x},\\ G_{\gamma,0}\big(\sigma^2,t\big)&=(2\pi)^{-n}\int\frac{e^{i(x-y)\cdot\eta}}{\eta^2+2t\gamma\cdot\eta-\sigma^2-i0}d\eta.\end{aligned}\tag{10}$$

Lemma 4 (See Isozaki [6]) *Let $\Phi_\gamma(t)=\chi(\gamma\cdot\theta\geq t/\lambda)$ (defining function of $\theta\in S^{n-1}$). Then*

$$R_{\gamma,0}(\lambda,t)=R_0\big((\lambda+i0)^2\big)-2\pi F_0(\lambda)^*\Phi_\gamma(t)F_0(\lambda).$$

Lemma 5 (See Weder [21]) *In the expression of $G_{\gamma,0}(\sigma^2,t)$ we replace t by $z\in\mathbf{C}_+$. Then*

(i) *$G_{\gamma,0}(\sigma^2,z)$ is continuous in $\{|\sigma|,\gamma\}\in\mathbf{R}_+\times S^{n-1}$ and analytic in $z\in\overline{\mathbf{C}_+}$.*
(ii) *$\forall\varepsilon_0>0$, $\exists C>0$ such that*

$$\big\|G_{\gamma,0}\big(\sigma^2,z\big)\big\|_{\mathcal{B}(L^2_{\mu-1},L^2_\mu)}\leq C\big(|\sigma|+|z|\big)^{-1}\quad \textit{for } |\sigma|+|z|>\varepsilon_0.$$

For $a\in\mathbf{R}$ let $\mathcal{H}_a=\{f;e^{a|x|}f(x)\in L^2\}$, and for $\varepsilon>0$ let $D_\varepsilon=\{z\in\mathbf{C}_+;|\operatorname{Re}z|<\varepsilon/2\}$.

Lemma 6 (See Eskin-Ralston [4]) *There exists an operator $U_{\gamma,0}(\lambda^2,z)$ satisfying the following properties.*

(i) *$\forall\delta>0$, $\exists\varepsilon>0$ such that $U_{\gamma,0}(\lambda^2,z)\in\mathcal{B}(\mathcal{H}_\delta,\mathcal{H}_{\delta-1})$ is analytic in $z\in D_\varepsilon$.*
(ii) *As $z\to t\in(-\varepsilon/2,\varepsilon/2)$ the operator $U_{\gamma,0}(\lambda^2,z)$ has a boundary value $G_{\gamma,0}(\lambda^2-t^2,t)$, and $U_{\gamma,0}(\lambda^2,i\tau)=G_{\gamma,0}(\lambda^2+\tau^2,i\tau)$ for $\tau>0$.*

The perturbed Faddeev resolvent is defined for a.e. $t\in(-\varepsilon/2,\varepsilon/2)$ as follows:

$$R_\gamma(\lambda,t)=\big\{I-R_{\gamma,0}(\lambda,t)(c-i\lambda b)\big\}^{-1}R_{\gamma,0}(\lambda,t).$$

Then $U_\gamma(\lambda,t)=e^{-it\gamma\cdot x}R_\gamma(\lambda,t)e^{it\gamma\cdot x}$ has a unique meromorphic continuation on D_ε and

$$\big\|U_\gamma(\lambda,i\tau)\big\|_{B(L^2_\mu,L^2_{\mu-1})}\leq C/\tau\quad\text{for large }\tau.\tag{11}$$

Theorem 7 *Assume (A5) and also*

$$b(x),c(x)=O\big(e^{-\delta_0|x|}\big)\quad\big(|x|\to\infty\big)\quad \textit{for some } \delta_0>0.\tag{A6}$$

Then $a(\lambda, \omega, \omega')$ with a fixed energy $\lambda \neq 0$ determines uniquely the functions $b(x)$ and $c(x)$.

Proof In (9) we replace $R(\lambda^2 - i0, \lambda)$ by the Faddeev resolvent $R_\gamma(\lambda, t)$, and define the kernel of the Faddeev scattering amplitude by

$$\begin{aligned} &a_\gamma\big(\lambda, \theta, \theta'; t\big) \\ &\quad = (2\pi)^{-n}\lambda^{n-1}\bigg[\int e^{-i\lambda(\theta-\theta')\cdot x}\big\{\lambda^{-1}c(x) - ib(x)\big\}dx; \\ &\qquad + \lambda \int e^{-i\lambda\theta\cdot x}\big\{\lambda^{-1}c(x) - ib(x)\big\} \\ &\qquad \times R_\gamma(\lambda, t)\big\{\big(\lambda^{-1}c - ib\big)e^{i\lambda\theta'\cdot}\big\}(x)dx\bigg]. \end{aligned} \tag{12}$$

Lemma 1 implies that this expression is rewritten by use of the physical scattering amplitude (9).

We choose $\omega, \omega' \in S^{n-1}$ to satisfy $\omega \cdot \gamma = \omega' \cdot \gamma = 0$ and put

$$\lambda\theta = \sqrt{\lambda^2 - t^2}\omega + t\gamma, \qquad \lambda\theta' = \sqrt{\lambda^2 - t^2}\omega' + t\gamma.$$

Then (12) is reduced to

$$\begin{aligned} &(2\pi)^n\lambda^{-n+1}a_\gamma\big(\lambda, \theta, \theta'; t\big) \\ &\quad = \int e^{-i\sqrt{\lambda^2-t^2}(\omega-\omega')\cdot x}\big\{\lambda^{-1}c(x) - ib(x)\big\}dx \\ &\qquad + \lambda \int e^{-i\sqrt{\lambda^2-t^2}\omega\cdot x}\big\{\lambda^{-1}c(x) - ib(x)\big\} \\ &\qquad \times U_\gamma(\lambda, t)\big\{\big(\lambda^{-1}c - ib\big)e^{i\sqrt{\lambda^2-t^2}\omega'\cdot}\big\}(x)dx. \end{aligned}$$

The analytic continuation makes it possible to replace t by $i\tau$ in this equation. Then it follows from (11) that

$$(2\pi)^n\lambda^{-n+1}a_\gamma\big(\lambda, \theta, \theta'; i\tau\big) \simeq \int e^{-i\sqrt{\lambda^2+\tau^2}(\omega-\omega')\cdot x}\big\{\lambda^{-1}c(x) - ib(x)\big\}dx \tag{13}$$

as $\tau \to \infty$. For any $\xi \in \mathbf{R}^n$ we choose $\gamma, \eta \in S^{n-1}$ to satisfy $\xi \cdot \gamma = \xi \cdot \eta = \gamma \cdot \eta = 0$, and put

$$\omega(\tau) = \big(1 - |\xi|^2/4\tau^2\big)^{1/2}\eta + \xi/2\tau, \qquad \omega'(\tau) = \big(1 - |\xi|^2/4\tau^2\big)^{1/2}\eta - \xi/2\tau.$$

Then $\omega(\tau), \omega'(\tau) \in S^{n-1}$ and

$$\sqrt{\lambda^2 + \tau^2}\big(\omega(\tau) - \omega'(\tau)\big) = \sqrt{(\lambda/\tau)^2 + 1}\xi \simeq \xi \quad (\tau \to \infty).$$

Thus, from (13) it is concluded that

$$\lim_{\tau\to\infty}(2\pi)^n\lambda^{-n+1}a_\gamma\big(\lambda,\theta(\tau),\theta'(\tau);i\tau\big)=\int e^{-i\xi\cdot x}\big\{\lambda^{-1}c(x)-ib(x)\big\}dx.$$

This completes the proof. □

9.6 Uniform Resolvent Estimates and Smoothing Properties

We return to the magnetic Schrödinger operator (1). In the following we restrict ourselves to the case $n\ge 3$ and $\mathbf{R}^n\backslash\Omega$ being empty or starshaped with respect to the origin $x=0$.

Theorem 8

(i) *Assume that* $\exists\varepsilon>0$ *small such that*

$$\max\big\{\big|\nabla\times b(x)\big|,\big|c(x)\big|\big\}\le\varepsilon_0 r^{-2}\quad \textit{in } \Omega. \tag{$A7$}$$

Then there exists $C_1>0$ *such that* $u=R(\kappa^2)f$ *satisfies*

$$\int\frac{1}{r^2}|u|^2dx\le C_1^2\int r^2|f|^2dx\quad \textit{for each } \kappa\in\Pi_\pm.$$

(ii) *Assume that*

$$\max\big\{\big|\nabla\times b(x)\big|,\big|c(x)\big|\big\}\le\varepsilon_0\min\big\{\mu(r),r^{-2}\big\}\quad \textit{in } \Omega, \tag{$A8$}$$

where $\mu(r)$ *is a smooth, positive, non-increasing* L^1*-function of* $r\in\mathbf{R}_+$. *Then there exists* $C_2>0$ *such that for each* $\kappa\in\Pi_\pm$ *it holds*

$$\int\Big\{\mu\big(|\nabla_b u|^2+|\kappa u|^2\big)-\mu'\frac{n-1}{2r}|u|^2\Big\}dx\le C_2^2\int\max\big\{\mu^{-1},r^2\big\}|f|^2dx.$$

We can choose $0<\varepsilon_0<1/4\sqrt{3}$ $(n=3)$, $<\sqrt{(n-1)(n-3)/8}$ $(n\ge 4)$ in ($A7$) and ($A8$) (see [16]).

As a corollary of Theorem 8 we are able to obtain space-time weighted estimates (smoothing properties) for the Schrödinger, and relativistic Schrödinger evolution equations

$$i\partial_t u+Lu=0,\quad u(0)=f\in L^2, \tag{14}$$

$$i\partial_t u+\sqrt{L+m^2}u=0\quad (m\ge 0),\quad u(0)=f\in L^2. \tag{15}$$

For an interval $I\subset\mathbf{R}$ and a Banach space X, we denote by $L_t^p(I,X)$ the space of X-valued L^p-functions of t, and simply write L_t^pX for $L^p(\mathbf{R},X)$.

Theorem 9

(i) *Under* (*A*7) *we have for* $h(t) \in L^2_t L^2_{r^2}$, *and* $f \in L^2$,

$$\left\| \int_0^t e^{-i(t-\tau)L} h(\tau) d\tau \right\|_{L^2_t L^2_{r^{-2}}} \le C_1 \|h\|_{L^2_t L^2_{r^2}}, \tag{16}$$

$$\|e^{itL} f\|_{L^2_t L^2_{r^{-2}}} \le \sqrt{2C_1} \|f\|. \tag{17}$$

(ii) *Under* (*A*8) *put* $\tilde{\mu}(r) = \min\{r^{-2}, \mu(r)\}$. *Then we have for* $g \in L^2$,

$$\|e^{it\sqrt{L+m^2}} g\|_{L^2_t L^2_{\tilde{\mu}}} \le \sqrt{mC_1 + C_2} \|g\|. \tag{18}$$

The above two theorems are the main part of [16].

9.7 Decay-Nondecay Problems for Time Dependent Complex Potential

Consider the Schrödinger evolution equation in $L^2(\mathbf{R}^n)$,

$$i\partial_t u - \Delta u + c_1(x,t)u = 0, \quad u(x,0) = f(x), \tag{19}$$

where $c_1(x,t) = c(1+t)^{-\alpha}(1+r)^{-\beta}$ with some $c \in \mathbf{C}$ and $\alpha, \beta \ge 0$. We denote by $U(t,s)$ the evolution operator which maps solutions at time s to those at time t.

Theorem 10

(i) (L^2 *decay*) *If* $\operatorname{Im} c > 0$ *and* $\alpha + \beta \le 1$, *then*

$$\|u(t)\|^2 \le \varphi(t)^{-1}\{\|\sqrt{\varphi(r)} f\|^2 + C\|f\|^2_{H^1}\}; \qquad \varphi(\sigma) = \int_0^\sigma (1+s)^{-\alpha-\beta} ds.$$

(ii) (L^2 *nondecay*) *If* $\operatorname{Im} c \ge 0$ *and* $\alpha + \beta > 1$, *then for each* $f \in L^2 \cap Ł^q$ *with* $2n/(n+\alpha+\beta) < q < 2n/(n+1)$, $\exists s_0 > 0$ *such that for* $\forall s \ge s_0$,

$$U(t,s)e^{-is\Delta} f \not\to 0 \quad \text{as } t \to \infty.$$

(iii) (*existence of the scattering states*) *If* $\mathrm{c} \ge 0$ *and* $\alpha + \frac{\beta}{2} > 1$, *then for any* $s \ge 0$ *and* $f \in L^2$, $\exists f_0 \in L^2$ *such that*

$$\lim_{t\to\infty} \|U(t,s) f - e^{-i(t-s)\Delta} f_0\| = 0.$$

See Mochizuki-Motai [17] for details. Similar properties are also proved for wave equations (e.g., Mochizuki-Nakazawa [18]).

Assertions (i) and (iii) are shown by using the equations

$$\frac{1}{2}\|u(t)\|^2 - \frac{1}{2}\|f\|^2 + \int_0^t \int \operatorname{Im} c_1(x,\tau)|u(\tau)|^2 dx d\tau = 0, \tag{20}$$

and

$$\big(u(t), u_0(t)\big) - \big(u(s), u_0(s)\big) - i\int_s^t \big(c_1(\cdot,\tau)u(\tau), u_0(\tau)\big)d\tau = 0, \tag{21}$$

where $u_0(t) = e^{-t\Delta} f_0$, respectively. Thus, the same results hold for a more general equation with free Laplacian $-\Delta$ replaced by the magnetic Schrödinger operator L satisfying ($A7$). In fact, under the conditions of (iii), $c_1(x,t)$ satisfies

$$|c_1(x,t)| \le |c|\left\{\frac{2-\beta}{2}(1+t)^{-2\alpha/(2-\beta)} + \frac{\beta}{2}(1+r)^{-2}\right\}.$$

Here, without loss of generality, we have assumed $\alpha + \beta \le 2$. Since $(1 + t)^{-2\alpha/(2-\beta)} \in L^1(\mathbf{R}_+)$, Theorem 9(i) is applied to generalize the result.

On the other hand, (21) does not work well under assumptions on (ii). We have used in [17] the L^p-estimate

$$\|u_0\|_{L^p} \le \big(4\pi|t|\big)^{n/p-n/2}\|f_0\|_{L^{p'}} \tag{22}$$

with $2 \le p \le \infty$ and $1/p + 1/p' = 1$, to show assertion (ii).

9.8 Scattering for Time Dependent Perturbations

Let $\mathcal{H}$ be a Hilbert space with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$, and consider in $\mathcal{H}$ the evolution equation

$$i\partial_t u + \Lambda_0 u + V(t)u = 0, \quad u(s) = f \in \mathcal{H}, \tag{23}$$

with initial time $s \in \mathbf{R}$, where Λ_0 is a selfadjoint operator in $\mathcal{H}$ with dense domain $\mathcal{D}(\Lambda_0)$ and $V(t)$ is a Λ_0-bounded operator which depends continuously on $t \in \mathbf{R}$. Let $e^{it\Lambda_0}$ be the unitary group in $\mathcal{H}$ which represents the solution of the free equation $i\partial_t u_0 + \Lambda_0 u_0 = 0$. Then the perturbed problem (23) reduces to the integral equation

$$u(t,s) = e^{i(t-s)\Lambda_0} f + \int_s^t e^{i(t-\tau)\Lambda_0} V(\tau)u(\tau,s)d\tau. \tag{24}$$

($H1$) There exist a Banach space X and $C_3 > 0$ such that

$$\|e^{i(t-s)\Lambda_0} f_0\|_{L_t^2 X} \le C_3\|f_0\| \quad \text{for any } (s, f_0) \in \mathbf{R}\times\mathcal{H}.$$

$(H2)$ There exist $\varepsilon_0 > 0$ and a nonnegative L^1-function $\eta(t)$ such that

$$\big|\big(V(t)u, v\big)\big| \le \eta(t)\|u\|\|v\| + \varepsilon_0\|u\|_X\|v\|_X.$$

$(H3)$ There exists $\varepsilon_1 > 0$ satisfying the following properties: If $\varepsilon_0 < \varepsilon_1$ in $(H2)$, (24) has a unique solution $u(t,s) = U(t,s)f \in C(\mathbf{R}, \mathcal{H})$, which also satisfies

$$\big\|U(t,s)f\big\|_{L_t^2 X} \le C_4\|f\|,$$

where $C_4 = C_4(\varepsilon_1) > 0$ is independent of $(s, f) \in \mathbf{R} \times \mathcal{H}$.

Theorem 11 *Assume* $(H1)$, $(H3)$ *with* $0 < \varepsilon_0 < \varepsilon_1$. *Then we have*

(i) $\{U(t,s)\}_{t,s\in\mathbf{R}}$ *is a family of uniformly bounded operators in* $\mathcal{H}$.
(ii) *For every* $s \in \mathbf{R}_\pm = \{t : \pm t > 0\}$, *there exits the strong limit*

$$Z^\pm(s) = s - \lim_{t\to\pm\infty} e^{i(-t+s)\Lambda_0}U(t,s).$$

(iii) *The operator* $Z^\pm = Z^\pm(0)$ *satisfies*

$$w - \lim_{s\to\pm\infty} Z^\pm U(0,s)e^{is\Lambda_0} = I \quad (\textit{weak limit}).$$

(iv) *If* ε_0 *can be chosen smaller to satisfy* $C_3C_4\varepsilon_0 < 1$, *then* $Z^\pm : \mathcal{H} \longrightarrow \mathcal{H}$ *is a bijection on* $\mathcal{H}$. *Moreover, the scattering operator* $S = Z^+(Z^-)^{-1}$ *is also a bijection.*

A typical example is the Schrödinger equation

$$i\partial_t u + Lu + c_1(x,t)u = 0, \quad u|_{t=s} = f \in L^2, \tag{25}$$

where L is the selfadjoint operator in Sect. 9.7 and $c_1(x,t)$ is a complex function satisfying

$$\big|c_1(x,t)\big| \le \eta(t) + \varepsilon_0 r^{-2} \quad \text{with small } \varepsilon_0 > 0. \tag{A9}$$

We choose $\mathcal{H} = L^2(\Omega)$, $\Lambda_0 = L$, $V(t) = c_1(s,t)$ and $X = L^2_{r^{-2}}$. Then $(H1)$ with $C_3 = \sqrt{2C_1}$ and $(H2)$ are obvious from (17) and $(A9)$, respectively. To verify $(H3)$, put $Y(I) = L_t^\infty(I; L^2) \cap L_t^2(I; L^2_{r^{-2}})$. Then after using (16) and $(A9)$ we have

Proposition 4 *For* $I_{+,s} = (s,T)$ $(s < T \le \infty)$ *or* $I_{-,s} = (T,s)$ $(-\infty \le T < s)$ *let*

$$\Phi_{\pm,s}v(t) = \int_s^t e^{i(t-\tau)L}c_1(\tau)v(\tau)d\tau, \quad v(t) \in Y(I_{\pm,s}).$$

Then we have

$$\|\Phi_{\pm,s}v\|_{Y(I_{\pm,s})} \le C_5\|v\|_{Y(I_{\pm,s})}, \quad C_5 = \max\big\{\|\eta\|_{L^1(s,t)}, \varepsilon_0 C_1\big\}.$$

We choose $|T-s|$ so small or $|s|$ so large, and ε_1 so small that $\max\{\|\eta\|_{L^1(s,t)}, \varepsilon_1 C_1\} < 1$. Then this lemma guarantees the solvability of (25) in $Y(I_{\pm,s})$ and we have

$$\|U(t,s)f\|_{Y(I_{\pm,s})} \le C_6\|f\|, \quad C_6 = \frac{1+C_3}{1-C_5}.$$

Note that $\mathbf{R}$ is covered by a finite number $2N$ of such $I_{\pm,s}$. Then we see that (24) with $s=0$ has a unique global solution satisfying $(H3)$:

$$\|U(t,0)f\|_{L_t^\infty L^2} + \|U(t,0)f\|_{L_t^2 L^2_{r-2}} \le C_4\|f\|, \quad C_4 = 2\sum_{k=1}^{N} C_6^k. \tag{26}$$

As we see, the inhomogeneous smoothing property (16) plays an important role to establish the scattering theory for time dependent perturbations. As for Klein-Gordon equations we have the homogeneous smoothing property (18). However, it is insufficient to develop the scattering theory. So, we restrict ourselves to the simpler problem in the whole $\mathbf{R}^n$:

$$\begin{gathered} \partial_t^2 w - \Delta w + m^2 w + \sum_{j=1}^{n} b_j(x,t)\partial_j w + b_0(x,t)\partial_t w + c(x,t)w = 0, \\ w|_{t=s} = f_1(x), \qquad \partial_t w|_{t=s} = f_2(x). \end{gathered} \tag{27}$$

Here $m>0$, $b_j(x,t)$ $(j=0,1,\ldots,n)$ and $c(x,t)$ are complex functions satisfying

$$\max\{|b_j(x,t)|, m^{-1}|c(x,t)|\} \le \eta(t) + \varepsilon_0\tilde{\mu}(r), \quad \tilde{\mu} = \min\{\mu(r), r^{-2}\}. \tag{A10}$$

Let $\mathcal{H}_E$ and X_E be the spaces with norms

$$\|\{f_1,f_2\}\|_E^2 = \frac{1}{2}\int \{|\nabla f_1|^2 + m^2|f_1|^2 + |f_2|^2\}dx < \infty,$$

$$\|f\|_{X_E}^2 = \frac{1}{2}\{\|\nabla f_1\|_X^2 + m^2\|f_1\|_X^2 + \|f_2\|_X^2\} < \infty,$$

where $X = L^2_{\tilde{\mu}}$. Then as an evolution equation in $\mathcal{H}_E$, the wave (27) is rewritten to the integral equation

$$\begin{gathered} u(t,s) = e^{i(t-s)\Lambda_0} f + \int_s^t e^{i(t-\tau)\Lambda_0} V(\tau)u(\tau,s)d\tau, \quad f = \{f_1,f_2\} \in \mathcal{H}_E; \\ \Lambda_0 = i\begin{pmatrix} 0 & 1 \\ \Delta - m^2 & 0 \end{pmatrix} \quad \text{and} \\ V(t) = -i\begin{pmatrix} 0 & 0 \\ b(x,t)\cdot\nabla + c(x,t) & b_0(x,t) \end{pmatrix}. \end{gathered} \tag{28}$$

For $\kappa \in \mathbf{C}\backslash\mathbf{R}$ let $R_{0m}(\kappa^2) = (-\Delta + m^2 - \kappa^2)^{-1}$. Then the resolvent of Λ_0 is given by

$$\mathcal{R}_0(\kappa) = \begin{pmatrix} -\kappa & i \\ i(\Delta - m^2) & -\kappa \end{pmatrix} R_{0m}(\kappa^2)$$

and, hence, we have for $f, g \in X'_E$ the estimate

$$\begin{aligned} &\left|\left(\mathcal{R}_0(\kappa) f, g\right)_E\right| \\ &\quad\le \sum_{j=1}^n \left\{\left\|\kappa R_{0m}(\kappa^2)\partial_j f_1\right\|_X + \left\|\partial_j R_{0m}(\kappa^2) f_2\right\|_X\right\}\|\partial_j g_1\|_{X'} \\ &\qquad + m^2\left\{\left\|\kappa R_{0m}(\kappa^2) f_1\right\|_X + \left\|R_{0m}(\kappa^2) f_2\right\|_X\right\}\|g_1\|_{X'} \\ &\qquad + \left\{\sum_{j=1}^n \left\|\partial_j R_{0m}(\kappa^2)\partial_j f_1\right\|_X + m^2\left\|R_{0m}(\kappa^2) f_1\right\|_X + \left\|\kappa R_{0m}(\kappa^2) f_2\right\|_X\right\} \\ &\qquad \times \|g_2\|_{X'}. \end{aligned}$$

Both inequalities of Theorem 8 imply that

$$\left\|\nabla R_{0m}(\kappa^2) h\right\|_X^2 + \left(1 + \left|\sqrt{\kappa^2 - m^2}\right|^2\right)\left\|R_{0m}(\kappa^2) h\right\|_X^2 \le C\|h\|_{X'}^2$$

for any $h \in X'$ and κ^2 in the resolvent set of $-\Delta + m^2$. Thus, we conclude the existence of a suitable $C_8 > 0$ verifying

$$\left|\left(\mathcal{R}_0(\kappa) f, g\right)_E\right| \le C_8\|f\|_{X'_E}\|g\|_{X'_E}, \tag{29}$$

or equivalently, we obtain the inhomogeneous smoothing property

$$\left\|\int_0^t e^{i(t-\tau)\Lambda_0} h(\tau)\, d\tau\right\|_{L_t^2 X_E} \le C_8\|h\|_{L^2 X'_E}.$$

Then as in the case of the Schrödinger equation, this and the smallness assumption ($A10$) show the unique existence of solutions to (27) with $s = 0$ satisfying

$$\left\|U(t,0) f\right\|_{L_t^\infty \mathcal{H}_E} + \left\|U(t,0) f\right\|_{L_t^2 X_E} \le C_9\|f\|_E. \tag{30}$$

The above treatment is also possible in the mass less case $m = 0$ if $n \ge 4$. However, more general results in exterior domains including the 3-dimensional problem, are guaranteed if we apply weighted energy methods. We consider in Ω the wave (27) with $m = 0$ and the initial-boundary conditions

$$w|_{t=s} = f_1(x), \qquad w_t|_{t=s} = f_2(x), \qquad w|_{\partial\Omega} = 0, \tag{31}$$

where $b_j(x,t)$ $(j = 0, 1, \dots, n)$ and $c(x,t)$ are real functions satisfying

$$\max\left\{\left|b_j(x,t)\right|, \frac{2r}{n-2}\left|c(x,t)\right|\right\} \le \eta(t) + \varepsilon_0\mu(r). \tag{A11}$$

Here $\mu(r) \in L^1(\mathbf{R})$ is chosen to satisfy also

$$\mu(r) > 0, \qquad \mu'(r) \le 0, \qquad \mu'(r)^2 \le 2\mu(r)\mu''(r). \tag{32}$$

We choose $m = 0$, $\tilde{\mu}(r) = \mu(r)$ and f_1 (the first component) verifying the zero boundary condition $f_1|_{\partial\Omega} = 0$ in the definition of $\mathcal{H}_E$ and X_E. Then $(A11)$ and the following proposition verify $(H1)$, $(H2)$ since the unique existence of solution in $C(\mathbf{R}; \mathcal{H}_E)$ is evident.

Proposition 5 *Under* $(A11)$ *with sufficiently small* $\varepsilon_0 > 0$, *let* $u(t) = \{w(t), w_t(t)\}$ *be the solution of* (27) *with* $m = 0$ *and* (31). *Then*

$$\left\|u(t)\right\|_E \le C_{10}\|f\|_E, \; f = \{f_1, f_2\},$$
$$\frac{1}{2}\int_s^t \int_\Omega \left\{\mu\left(|\nabla w|^2 + w_t^2\right) - \mu'\frac{n-1}{2r}w^2\right\} dx d\tau \le C_{11}^2\|u\|_E^2,$$

where $C_{10} > 0$ *and* $C_{11} > 0$ *are independent of* $(s, f) \in \mathbf{R} \times \mathcal{H}_E$.

For the proof of Theorem 11 and Proposition 5 see [15]. Schrödinger equations (19) with $c_1(x,t) \in L_t^\nu L^r$ $(0 < 1/r \le 2/n,\ 1/\nu = 1 - n/2r)$ and the above wave equations are studied as examples there. But Klein-Gordon equations are not treated there.

9.9 Strichartz Estimates

In the rest of this article we discuss the so called Strichartz estimates. As we will see Strichartz estimates of free equations and smoothing properties of perturbed solutions (i.e., $(H3)$) lead us to the Strichartz estimates for perturbed equations.

First consider Schrödinger equations in $\mathbf{R}^n$. Let $p \ge 2$, q be the admissible exponents $\frac{2}{p} + \frac{n}{q} = \frac{n}{2}$. Then as it is well known, there exists a constant $C > 0$ such that

$$\left\|e^{-it\Delta} f(x)\right\|_{L_t^p L^q} \le C\|f\|. \tag{33}$$

More precisely, the end point estimate is given by

$$\left\|\int_0^t e^{-i(t-s)\Delta} h(x,s) ds\right\|_{L_t^2 L^{2n/(n-2),2}} \le C\|h\|_{L_t^2 L^{2n/(n+2),2}}, \tag{34}$$

where $L^{\alpha,\beta}$ denote the Lorentz spaces.

Theorem 12 *Under* $(A9)$ *with* $\eta(t) \equiv 0$ *let* $u(t) \in C(\mathbf{R}; L^2)$ *be the solution of* (19). *Then for any admissible exponents* p *and* q *there exists* $C > 0$ *such that*

$$\|u\|_{L_t^p L^q} \le C\|f\| \quad \forall f \in L^2.$$

Proof It follows from (26) that $r^{-1}u \in L_t^2 L_x^2$, while by assumption $rc_1(\cdot, t) \in L^{n,\infty}$. Then by the Hölder inequality for Lorentz space (see O'Neil [19])

$$\|c_1 u\|_{L_t^2 L^{2n/(n+2),2}} \le C\|rc_1\|_{L^{n,\infty}} \left\|r^{-1}u\right\|_{L_t^2 L^2}.$$

Thus, it follows from (34) and (26) that

$$\left\|\int_0^t e^{-i(t-s)\Delta} c_1(s)u(s)ds\right\|_{L_t^2 L^{2n/(n-2),2}}$$
$$\le C\|c_1 u\|_{L_t^2 L^{2n/(n+2),2}} \le C\|rc_1\|_{L^{n,\infty}} \left\|r^{-1}u\right\|_{L_t^2 L^2} \le C\|f\|.$$

This and (34) prove the Strichartz estimate at the end point:

$$\|u\|_{L_t^2 L^{2n/(n-2),2}} \le C\|f\|.$$

Interpolation between this and the uniform boundedness of $u(t)$ in L^2 (cf. (26))

$$\|u\|_{L_t^\infty L^2} \le C\|f\|$$

gives the full range of the estimates in Theorem 12. □

Next, the solution $w(t)$ of the Klein-Gordon equation (27) satisfies

$$w(t) = \dot{W}(t)f_1 + W(t)f_2 + \int_0^t W(t-s)\left[V(s)u(s)\right]_2 ds, \tag{35}$$

where $W(t) = \sqrt{-\Delta + m^2}^{-1} \sin(t\sqrt{-\Delta + m^2})$ with $m > 0$ and

$$\left[V(t)u(t)\right]_2 = b_0(x,t)w_t + b(x,t)\cdot\nabla w + c(x,t)w.$$

Let p and q be any admissible exponents for Schrödinger equations, and $\gamma = \frac{1}{p} + \frac{1}{2} - \frac{1}{q}$. Then the following estimate holds for the free solution (see e.g., D'Ancona-Fanelli [2]):

$$\left\|e^{it\sqrt{-\Delta+m^2}} g\right\|_{L_t^p H_q^{-\gamma}} \le C\|g\|. \tag{36}$$

The following is the well known Christ-Kiselev lemma ([1]).

Lemma 7 *Let X, Y be Banach spaces and let $Tf(t) = \int_0^\infty K(t,s)f(s)ds$ be a bounded operator from $L^\alpha(\mathbf{R}; X)$ to $L^\beta(\mathbf{R}; Y)$. If $\alpha < \beta$, then $\tilde{T}f(t) = \int_0^t K(t,s)f(s)ds$ is also a bounded operator, and we have $\|\tilde{T}\| \le C(\alpha,\beta)\|T\|$.*

Theorem 13 *Under assumption (A10) with $\eta(t) \equiv 0$ let $w(t) \in C^1(\mathbf{R}; H^1)$ be the solution of (35). Then for any Schrödinger admissible exponents p, q satisfying also $p > 2$, there exists $C > 0$ such that*

$$\left\|\sqrt{-\Delta + m^2} w\right\|_{L_t^p L^q} + \|w_t\|_{L_t^p L^q} \le C\left\{\|f_1\|_{H^\gamma} + \|f_2\|_{H^{\gamma-1}}\right\}.$$

Proof Let $h(t) \in L_t^2 L_{\tilde{\mu}}^2$. Then it follows from (36) and the above lemma that

$$\left\| \int_0^t e^{i(t-s)\sqrt{-\Delta+m^2}} h(s)ds \right\|_{L_t^p H_q^{-\gamma}}$$
$$\leq C \left\| \int_0^\infty e^{-is\sqrt{-\Delta+m^2}} h(s)ds \right\| \leq C \|h\|_{L_t^2 L_{\tilde{\mu}-1}^2}.$$

In the last inequality we have applied the dual formula of (18) of Theorem 9. Put $h(t) = [V(t)u]_2$. Then as it is seen in (30)

$$\left\| \left[V(t)u \right]_2 \right\|_{L_t^2 L_{\tilde{\mu}-1}^2} \leq C \|f\|_E.$$

Combining these inequalities and (36) we conclude the assertion. □

Finally, we consider the solution $w(t)$ of the wave equation (27) with 0-boundary condition requiring $\mathbf{R}^n \backslash \Omega$ is convex. Let $p \geq 2$, q be any admissible exponents for wave equations satisfying $\frac{2}{p} + \frac{n-1}{q} = \frac{n-1}{2}$ $(q \neq \infty)$, and $\gamma = \frac{1}{p} + \frac{1}{2} - \frac{1}{q}$. Then the following estimate is known to hold (see e.g., Metcalfe [11]):

$$\left\| e^{it\sqrt{-\Delta_D}} g \right\|_{L_t^p H_q^{-\gamma}} \leq C \|g\|. \tag{37}$$

Theorem 14 *Under assumption (A11) with $\eta(t) \equiv 0$ let $w(t) \in C^1(\mathbf{R}; \dot{H}^1)$ be the solution of (35) with $W(t) = \sqrt{-\Delta_D}^{-1} \sin(t\sqrt{-\Delta_D})$. Then for any wave admissible exponents p, q satisfying also $p > 2$, there exists $C > 0$ such that*

$$\|\sqrt{-\Delta_D} w\|_{L_t^p L^q} + \|w_t\|_{L_t^p L^q} \leq C \left\{ \|f_1\|_{\dot{H}^\gamma} + \|f_2\|_{\dot{H}^{\gamma-1}} \right\}.$$

Proof With (37) and the second inequality of Proposition 5 we can follow the above arguments to obtain

$$\left\| \int_0^t e^{i(t-s)\sqrt{-\Delta_D}} \left[V(s)u(s) \right]_2 ds \right\|_{L_t^p(\dot{H}_q^{-\gamma})}$$
$$\leq C \left\| |\partial_t w| + |\nabla w| + r^{-1}|w| \right\|_{L_t^2 L_\mu^2} \leq C \|f\|_E. \tag{38}$$

In the last inequality we have used the Hardy inequality.

Combining (37) and (38) we conclude the assertion. □

Remark The endpoint Strichartz estimates with $p = 2$ are not proved in Theorems 13 and 14. For these purposes, in place of the use of Christ-Kiselev lemma, we are necessary to acquire the estimates corresponding to (34) for the Schrödinger equation.

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Chapter 10
On an Optimal Control Problem for the Wave Equation in One Space Dimension Controlled by Third Type Boundary Data

Alexey Nikitin

Abstract In the present paper we study the boundary control by the third boundary condition on the left end of a string, the right end being fixed. An optimality criterion based on the minimization of an integral of a linear combination of the control itself and its antiderivative raised to an arbitrary power $p \geq 1$ is established. A method is developed permitting one to find a control satisfying this optimality criterion and write it out in the explicit form. The optimal control for $p > 1$ is proved. Thereby proposed optimality criterion uniquely determines the optimal solution of boundary control problem under consideration.

Mathematics Subject Classification 49K20

10.1 Statement of the Boundary Control Problem

In the present paper, we consider the boundary control problem for string vibrations governed by the wave equation

$$u_{xx}(x,t) - u_{tt}(x,t) = 0. \tag{1}$$

This control is realized at the end x = 0 by the third boundary condition

$$u_x(0,t) - h \cdot u(0,t) = \mu(t),$$

and the end $x = \ell$ is fixed. For an arbitrary time interval T multiple of 4ℓ, the considered control brings the string from an arbitrary initial state

$$\{u(x,0) = \varphi(x);\, u_t(x,0) = \psi(x)\} \tag{2}$$

to an arbitrary terminal state

$$\{u(x,T) = \widehat{\varphi}(x);\, u_t(x,T) = \widehat{\psi}(x)\}. \tag{3}$$

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The investigation is performed in terms of a generalized solution of the wave equation (1) in the class $\widehat{W}_p^1(Q_T)$, where Q_T is the rectangle $[0 \le x \le \ell] \times [0 \le t \le T]$. This class was introduced in [1] and is defined as the set of functions $u(x,t)$ that are continuous in the rectangle Q_T and have both generalized derivatives $u_x(x,t)$, and $u_t(x,t)$ in it, and each of these derivatives belongs to the class $L_p(Q_T)$, to the class $L_p[0,\ell]$ for all $t \in [0,T]$, and to the class $L_p[0,T]$ for all $x \in [0,\ell]$.

Necessary conditions for the solution $u(x,t)$ to belong to the class $\widehat{W}_p^1(Q_T)$ are the following:

The Inclusion Conditions

$$u(x,0) = \varphi(x) \in W_p^1[0,\ell], \qquad u_t(x,0) = \psi(x) \in L_p[0,\ell], \tag{4}$$

$$u(x,T) = \widehat{\varphi}(x) \in W_p^1[0,\ell], \qquad u_t(x,T) = \widehat{\psi}(x) \in L_p[0,\ell], \tag{5}$$

$$\mu(t) \in L_p[0,T]. \tag{6}$$

The Fixing Condition

$$\varphi(\ell) = 0, \qquad \widehat{\varphi}(\ell) = 0. \tag{7}$$

For the further statement of results, we consider the mixed problem for the wave equation (1) with the initial and boundary conditions

$$u_{xx}(x,t) - u_{tt}(x,t) = 0, \tag{8}$$

$$u(x,0) = \varphi(x), \qquad u_t(x,0) = \psi(x), \tag{9}$$

$$u_x(0,t) - h \cdot u(0,t) = \mu(t), \qquad u(\ell,t) = 0, \tag{10}$$

where the functions $\varphi(x)$, $\psi(x)$, $\mu(t)$ belong to the classes (4)–(6) and satisfy the fixing conditions (7).

Definition 1 A generalized solution of this mixed problem in the class $\widehat{W}_p^1(Q_T)$ is defined as a function $u(x,t) \in \widehat{W}_p^1(Q_T)$, that satisfies the integral identity

$$\int_0^\ell \int_0^T u(x,t)\big[\Phi_{tt}(x,t) - \Phi_{xx}(x,t)\big]dxdt + \int_0^T \mu(t)\Phi(0,t)dt$$
$$+ \int_0^\ell \big[\varphi(x)\Phi_t(x,0) - \psi(x)\Phi(x,0)\big]dx = 0, \tag{11}$$

for an arbitrary function $\Phi(x,t)$ in the class $C^2(Q_T)$ subjected to the conditions

$$\Phi_x(0,t) - h\Phi(0,t) \equiv 0, \qquad \Phi(\ell,t) \equiv 0 \quad \text{for } 0 \le t \le T$$

and

$$\Phi(x,T) \equiv 0, \qquad \Phi_t(x,T) \equiv 0 \quad \text{for } 0 \le x \le \ell.$$

The following assertion is a consequence of the results in [2]:

Proposition 1 *For any $T > 0$ the mixed problem has at most one generalized solution of the class $\widehat{W}_p^1(Q_T)$.*

Definition 2 A solution of the corresponding boundary control problem is defined as a function $\mu(t) \in L_p[0, T]$ for which a generalized solution $u(x, t) \in \widehat{W}_p^1(Q_T)$ of the mixed problem (8)–(10) satisfies the terminal condition (3).

If $T > 2\ell$, then this problem has infinitely many solutions. Therefore the task can be posed to find the optimality criterion which uniquely determines the `optimal solution` among them. In the present paper, we formulate an optimality criterion for the solution of the considered boundary control problem. This criterion is based on the minimization of the integral of a linear combination of the control itself and its antiderivative raised to an arbitrary power $p \geq 1$.

Let us consider this problem for the time interval T satisfying the condition

$$\mathbf{T} = 4\ell \cdot (\mathbf{n} + \mathbf{1}), \quad \text{where } n = 0, 1, 2, \ldots. \tag{12}$$

Remark 1 By using the approach in [3], one can consider the investigated problem for the case of arbitrary time intervals T, which are not necessarily multiples of 4ℓ.

For the statement of the optimization problem, we introduce the function:

$$\mathbf{H}_m(\tau) = \left\{e^{-h\tau} \cdot \left[\mathbf{L}^1_{2n-m+1}(2h\tau) + \mathbf{L}^1_{2n-m}(2h\tau)\right], m = \overline{0, 2n+1}\right\}, \tag{13}$$

where $\mathbf{L}^1_k(2h\tau)$—is a `Laguerre polynomial`, [4].

Now we introduce the function $\mathbf{H}(t, \tau)$ by the relation

$$\mathbf{H}(t, \tau) = \left\{\mathbf{H}_m(\tau) \text{ for } 2\ell m < t \leq 2\ell(m+1), m = \overline{0, 2n+1}\right\}. \tag{14}$$

We pose the problem of finding among all $\mu(t) \in L_p[0, T]$, a function minimizing the integral

$$\int_0^T \left|\mu(t) - h \cdot \int_0^t \mathbf{H}(t, t-\xi)\mu(\xi)d\xi\right|^p dt \tag{15}$$

under the constraints that follow from the validity of arbitrarily posed initial and terminal conditions.

In the present paper the idea to find required special optimality criterion was based on utilization of this integral which is modified (up to equivalence) "boundary energy integral" (see [3, 5, 6]). An analogical functional, which was minimized in previous papers and can be obtained from the one presented in this work exploited the Neumann condition (i.e. $h = 0$) instead of the third boundary condition.

We continue the functions $\varphi(x)$ and $\psi(x)$ in the initial conditions (2) and the functions $\widehat{\varphi}(x)$ and $\widehat{\psi}(x)$ in the terminal conditions (3) as odd functions around the point $x = \ell$ from the interval $[0, \ell]$ to the interval $[\ell, 2\ell]$. The fixing conditions (5) guarantee that the functions thus continued belong to the classes

$$\varphi(x), \widehat{\varphi}(x) \in W_p^1[0, 2\ell], \qquad \psi(x), \widehat{\psi}(x) \in L_p[0, 2\ell]. \tag{4*}$$

10.2 Preparation for the Optimization

Theorem 1 *There exists a solution $\mu = \mu(t)$ of the considered boundary control problem satisfying the given optimality criterion, and on each segment*

$$\big[2\ell m, 2\ell(m+1)\big] \quad (m = \overline{0, 2n+1}),$$

it can be represented by the formula

$$\mu(y) = \frac{(-1)^{m+1}\mathbf{D}(y-2\ell m)}{2n+2} + h\int_0^y \mathbf{R}_m(y-t)\frac{(-1)^{m+1}\mathbf{D}(t-2\ell m)}{2n+2}dt, \quad (16)$$

where $D(t)$ is a function represented in closed form and depending only on initial and terminal conditions of the problem,

$$\mathbf{R}_m(y-t) = \sum_{i=0}^{2n-m+2} h^{i-1}(y-t)^{i-1}\binom{2n-m+2}{i}\,{}_1\tilde{\mathbf{F}}_1\big(2n-m+1; i; h(y-t)\big), \quad (17)$$

and ${}_1\tilde{\mathbf{F}}_1(a; c; z)$—is a `degenerate Kummer hypergeometric function`, [7]. *If $p > 1$, then the above-mentioned optimal solution is unique.*

Proof Consider the function:

$$\widetilde{u}(x,t) = \begin{cases} \frac{1}{2}\cdot\left[\varphi(x+t)+\varphi(x-t)+\int_{x-t}^{x+t}\psi(\xi)d\xi\right] & \text{in } \triangle_1 \\ \frac{1}{2}\cdot\left[\varphi(x+t)+\varphi(0)+\int_{0}^{x+t}\psi(\xi)d\xi\right] & \text{in } \triangle_2, \\ 0 & \text{in } \triangle_3, \end{cases} \quad (18)$$

where $\triangle_1$—is the triangle bounded by segments of the lines $x-t=0$, $x-\ell=0$ and $t=0$; $\triangle_2$—is the triangle bounded by segments of the lines $x-t=0$, $x+t-2\ell=0$ and $x=0$; $\triangle_3$—is the quadrangle bounded by segments of the lines $x=0$, $x+t-2\ell=0$, $x-\ell=0$ and $t-T=0$.

Following [5], one can show that, for all $T > 2\ell$, the function (18) is the unique generalized solution of the mixed problem

$$\widetilde{u}_{xx}(x,t) - \widetilde{u}_{tt}(x,t) = 0,$$
$$\widetilde{u}(x,0) = \varphi(x), \qquad \widetilde{u}_t(x,0) = \psi(x),$$
$$\widetilde{u}_x(0,t) - h\cdot\widetilde{u}(0,t) = \widetilde{\mu}(t), \qquad \widetilde{u}(\ell,t) = 0,$$

in the class $\widehat{W}^1_p(Q_T)$, where

$$\widetilde{\mu}(t) = \frac{1}{2}\left\{\varphi'(t) + \psi(t) - h\left[\varphi(t) + \varphi(0) + \int_0^t \psi(\xi)d\xi\right]\right\}, \quad \text{if } 0 \le t \le 2\ell,$$
$$0, \quad \text{if } 2\ell < t \le T.$$

Let $u(x,t)$ be a generalized solution of the main problem (3), (8)–(10) in the class $\widehat{W}_p^1(Q_T)$, which is used for the minimization of the integral (15), and let $\widetilde{u}(x,t)$ be the constructed solution (18). Then the function

$$\widehat{u}(x,t) = u(x,t) - \widetilde{u}(x,t) \tag{19}$$

is a generalized solution of the mixed problem

$$\widehat{u}_{xx}(x,t) - \widehat{u}_{tt}(x,t) = 0, \tag{20}$$

$$\widehat{u}(x,0) = 0, \qquad \widehat{u}_t(x,0) = 0, \tag{21}$$

$$\widehat{u}_x(0,t) - h \cdot \widehat{u}(0,t) = \widehat{\mu}(t), \qquad \widehat{u}(\ell,t) = 0, \tag{22}$$

where

$$\widehat{\mu}(t) = \mu(t) - \widetilde{\mu}(t). \tag{23}$$

We use the following closed form [8] of the generalized solution of the mixed problem (20)–(22):

$$\begin{aligned}\widehat{u}(x,t) = &-\sum_{k=0}^{2n+1}(-1)^k \cdot \int_0^{t-x-2kl} \underline{\widehat{\mu}}(\xi)d\xi - \sum_{k=1}^{2n+2}(-1)^k \cdot \int_0^{t+x-2kl} \underline{\widehat{\mu}}(\xi)d\xi \\ &+ h \cdot \sum_{k=0}^{2n+1}(-1)^k \cdot \int_0^t e^{-h\tau}\mathbf{L}_k^1(2h\tau) \\ &\cdot \Bigg[\int_0^{t-2k\ell-x-\tau} \underline{\widehat{\mu}}(\xi)d\xi - \int_0^{t-2\ell(k+1)+x-\tau} \underline{\widehat{\mu}}(\xi)d\xi \\ &- \int_0^{t-2\ell(k+1)-x-\tau} \underline{\widehat{\mu}}(\xi)d\xi + \int_0^{t-2\ell(k+2)+x-\tau} \underline{\widehat{\mu}}(\xi)d\xi\Bigg]d\tau,\end{aligned} \tag{24}$$

where the symbol $\underline{\widehat{\mu}}(t)$ stands for the function that coincides with $\widehat{\mu}(t)$ for $t \ge 0$ and is zero for $t < 0$. Since the function $u(x,t)$ satisfies the terminal condition (3) and $\widetilde{u}(x,t)$ satisfies the zero terminal condition, it follows from relation (19) that the function $\widehat{u}(x,t)$ satisfies the terminal conditions

$$\widehat{u}(x,T) = \widehat{\varphi}(x); \qquad \widehat{u}_t(x,T) = \widehat{\psi}(x). \tag{25}$$

By using relation (24) and conditions (3) and (25), we establish constraints, which are necessary and sufficient for the function $\mu(t)$ to be a solution of the boundary control problem.

We introduce the notation

$$\boldsymbol{l}_k^1(2h\tau) = e^{-h\tau} \cdot \mathbf{L}_k^1(2h\tau), \qquad \widehat{\mu}_k(x) = \widehat{\mu}(x + 2k\ell),$$

and evaluate the derivatives of the function (24):

$$\begin{aligned}
\widehat{u}_x(x,t) = & \sum_{k=0}^{2n+1} (-1)^k \cdot \underline{\widehat{\mu}}_{-k}(t-x) - \sum_{k=1}^{2n+2} (-1)^k \cdot \underline{\widehat{\mu}}_{-k}(t+x) \\
& + h \cdot \sum_{k=0}^{2n+1} (-1)^k \cdot \int_0^t \boldsymbol{l}_k^1(2h\tau) \cdot \big[-\underline{\widehat{\mu}}_{-k}(t-x-\tau) \\
& - \underline{\widehat{\mu}}_{-(k+1)}(t+x-\tau) + \underline{\widehat{\mu}}_{-(k+1)}(t-x-\tau) + \underline{\widehat{\mu}}_{-(k+2)}(t+x-\tau) \big] d\tau, \\
\widehat{u}_t(x,t) = & -\sum_{k=0}^{2n+1} (-1)^k \cdot \underline{\widehat{\mu}}_{-k}(t-x) - \sum_{k=1}^{2n+2} (-1)^k \cdot \underline{\widehat{\mu}}_{-k}(t+x) \\
& + h \cdot \sum_{k=0}^{2n+1} (-1)^k \cdot \int_0^t \boldsymbol{l}_k^1(2h\tau) \cdot \big[\underline{\widehat{\mu}}_{-k}(t-x-\tau) \\
& - \underline{\widehat{\mu}}_{-(k+1)}(t+x-\tau) - \underline{\widehat{\mu}}_{-(k+1)}(t-x-\tau) + \underline{\widehat{\mu}}_{-(k+2)}(t+x-\tau) \big] d\tau.
\end{aligned}$$

By setting $t = \mathbf{T} = 4\ell(n+1)$ and by using the terminal conditions (25), we obtain

$$\begin{aligned}
\widehat{\varphi}'(x) = & \sum_{k=0}^{2n+1} (-1)^k \cdot \underline{\widehat{\mu}}_{2n-k+2}(-x) - \sum_{k=1}^{2n+2} (-1)^k \cdot \underline{\widehat{\mu}}_{2n-k+2}(x) \\
& + h \cdot \sum_{k=0}^{2n+1} (-1)^k \cdot \int_0^{4\ell(n+1)} \boldsymbol{l}_k^1(2h\tau) \cdot \big[-\underline{\widehat{\mu}}_{2n-k+2}(-x-\tau) \\
& - \underline{\widehat{\mu}}_{2n-k+1}(x-\tau) + \underline{\widehat{\mu}}_{2n-k+1}(-x-\tau) + \underline{\widehat{\mu}}_{2n-k}(x-\tau) \big] d\tau, \\
\widehat{\psi}(x) = & -\sum_{k=0}^{2n+1} (-1)^k \cdot \underline{\widehat{\mu}}_{2n-k+2}(-x) - \sum_{k=1}^{2n+2} (-1)^k \cdot \underline{\widehat{\mu}}_{2n-k+2}(x) \\
& + h \cdot \sum_{k=0}^{2n+1} (-1)^k \cdot \int_0^{4\ell(n+1)} \boldsymbol{l}_k^1(2h\tau) \cdot \big[\underline{\widehat{\mu}}_{2n-k+2}(-x-\tau) \\
& - \underline{\widehat{\mu}}_{2n-k+1}(x-\tau) - \underline{\widehat{\mu}}_{2n-k+1}(-x-\tau) + \underline{\widehat{\mu}}_{2n-k}(x-\tau) \big] d\tau.
\end{aligned}$$

The half-sum and half-difference of the last two relations provide constraints for all x in the interval $[0, \ell]$:

$$\frac{1}{2}\big[\widehat{\varphi}'(x)+\widehat{\psi}(x)\big] = -\sum_{k=1}^{2n+2}(-1)^k \underline{\widehat{\mu}}_{2n-k+2}(x) - h\sum_{k=0}^{2n+1}(-1)^k \times \int_0^T \boldsymbol{l}_k^1(2h\tau)\big[\underline{\widehat{\mu}}_{2n-k+1}(x-\tau) - \underline{\widehat{\mu}}_{2n-k}(x-\tau)\big]d\tau, \tag{26}$$

$$\frac{1}{2}\big[\widehat{\varphi}'(x)-\widehat{\psi}(x)\big] = \sum_{k=0}^{2n+1}(-1)^k \underline{\widehat{\mu}}_{2n-k+2}(-x) - h\sum_{k=0}^{2n+1}(-1)^k \times \int_0^T \boldsymbol{l}_k^1(2h\tau)\big[\underline{\widehat{\mu}}_{2n-k+2}(-x-\tau) - \underline{\widehat{\mu}}_{2n-k+1}(-x-\tau)\big]d\tau. \tag{27}$$

Relations (26) and (27) are equalities of elements of the class $L_p[0, \ell]$. In addition, note that if x is replaced by $2\ell - x$, then, by taking into account the odd extension of $\varphi(x)$ and $\psi(x)$ around the point $x = \ell$, one reduces relation (27) to (26) expressed for all x in the interval $[\ell, 2\ell]$. Consequently, relation (26) should be treated as an equality of elements of the class $L_p[0, 2\ell]$.

We perform the substitution $m = 2n - k + 2$ in the first sum and $m = 2n - k + 1$ in the second sum of relation (26):

$$\frac{1}{2}\big[\widehat{\varphi}'(x)+\widehat{\psi}(x)\big] = -\sum_{m=0}^{2n+1}(-1)^m \underline{\widehat{\mu}}_m(x) + h\cdot\sum_{m=0}^{2n+1}(-1)^m\cdot\int_0^T \boldsymbol{l}_{2n-m+1}^1(2h\tau) \cdot\big[\underline{\widehat{\mu}}_m(x-\tau) - \underline{\widehat{\mu}}_{m-1}(x-\tau)\big]d\tau. \tag{28}$$

We rewrite the second sum in relation (28) in the form:

$$h\cdot\sum_{m=0}^{2n+1}(-1)^m\cdot\int_0^T \boldsymbol{l}_{2n-m+1}^1(2h\tau)\cdot\big[\underline{\widehat{\mu}}_m(x-\tau) - \underline{\widehat{\mu}}_{m-1}(x-\tau)\big]d\tau$$

$$= h\cdot\sum_{m=0}^{2n+1}(-1)^m\cdot\int_0^T \boldsymbol{l}_{2n-m+1}^1(2h\tau)\cdot\underline{\widehat{\mu}}_m(x-\tau)d\tau$$

$$-h\cdot\sum_{m=-1}^{2n}(-1)^m\cdot\int_0^T \boldsymbol{l}_{2n-m}^1(2h\tau)\cdot\underline{\widehat{\mu}}_m(x-\tau)d\tau.$$

Since

$$\underline{\widehat{\mu}}_m(x-\tau) \equiv 0 \quad \text{for } m = -1,$$

and

$$\boldsymbol{l}^1_{2n-m}(2h\tau) \equiv 0 \quad \text{for } m = 2n+1,$$

in the last relation, so

$$h \cdot \sum_{m=0}^{2n+1} (-1)^m \cdot \int_0^T \boldsymbol{l}^1_{2n-m+1}(2h\tau) \cdot \big[\underline{\widehat{\mu}}_m(x-\tau) - \underline{\widehat{\mu}}_{m-1}(x-\tau)\big] d\tau$$
$$= h \cdot \sum_{m=0}^{2n+1} (-1)^m \cdot \int_0^T \big[\boldsymbol{l}^1_{2n-m+1}(2h\tau) + \boldsymbol{l}^1_{2n-m}(2h\tau)\big] \cdot \underline{\widehat{\mu}}_m(x-\tau) d\tau,$$

it follows that relation (28) can be represented in the form

$$\frac{1}{2}\big[\widehat{\varphi}'(x) + \widehat{\psi}(x)\big] = -\sum_{m=0}^{2n+1} (-1)^m \underline{\widehat{\mu}}_m(x) + h \sum_{m=0}^{2n+1} (-1)^m$$
$$\times \int_0^T \big[\boldsymbol{l}^1_{2n-m+1}(2h\tau) + \boldsymbol{l}^1_{2n-m}(2h\tau)\big] \underline{\widehat{\mu}}_m(x-\tau) d\tau. \quad (29)$$

Let us now derive a constraint in terms of $\mu(x)$. We use the relation

$$\widehat{\mu}_m(x) = \mu_m(x) - \widetilde{\mu}_m(x),$$

where

$$\widetilde{\mu}_m(x) = \frac{1}{2} \cdot \left\{ \varphi'(x) + \psi(x) - h \cdot \left[\varphi(x) + \varphi(0) + \int_0^x \psi(\xi) d\xi \right] \right\}$$
$$= \widetilde{\mathbf{A}}(x), \quad \text{for } m = 0,$$
$$0, \quad \text{for } m = \overline{1, 2n+1},$$

let us note that

$$\underline{\widehat{\mu}}_m(x) = \widehat{\mu}_m(x),$$

and

$$\underline{\widehat{\mu}}_m(x-\tau) = \widehat{\mu}_m(x-\tau), \quad 0 \le \tau \le 2\ell m + x,$$
$$0, \quad 2\ell m + x \le \tau \le T.$$

We transform the right-hand side of relation (29):

$$-\sum_{m=0}^{2n+1} (-1)^m \cdot \widehat{\mu}_m(x)$$
$$+ h \sum_{m=0}^{2n+1} (-1)^m \int_0^{2\ell m + x} \big[\boldsymbol{l}^1_{2n-m+1}(2h\tau) + \boldsymbol{l}^1_{2n-m}(2h\tau)\big] \cdot \widehat{\mu}_m(x-\tau) d\tau$$

$$= \sum_{m=0}^{2n+1} (-1)^m \cdot \mu_m(x) + \widetilde{\mathbf{A}}(x)$$

$$+ h \sum_{m=0}^{2n+1} (-1)^m \int_0^{2\ell m+x} \left[\boldsymbol{l}^1_{2n-m+1}(2h\tau) + \boldsymbol{l}^1_{2n-m}(2h\tau) \right] \cdot \mu_m(x-\tau) d\tau$$

$$- h \cdot \sum_{m=0}^{2n+1} (-1)^m \int_0^{2\ell m+x} \left[\boldsymbol{l}^1_{2n-m+1}(2h\tau) + \boldsymbol{l}^1_{2n-m}(2h\tau) \right] \cdot \widetilde{\mathbf{A}}(x-\tau) d\tau.$$

We set

$$\mathbf{D}(x) = \frac{1}{2} \left[\widehat{\varphi}'(x) + \widehat{\psi}(x) \right] - \widetilde{\mathbf{A}}(x)$$
$$+ h \sum_{m=0}^{2n+1} (-1)^m \int_0^{2\ell m+x} \left[\boldsymbol{l}^1_{2n-m+1}(2h\tau) + \boldsymbol{l}^1_{2n-m}(2h\tau) \right] \widetilde{\mathbf{A}}(x-\tau) d\tau. \tag{30}$$

Finally, the constraint (29) can be represented in the form

$$- \sum_{m=0}^{2n+1} (-1)^m \mu_m(x)$$
$$+ h \sum_{m=0}^{2n+1} (-1)^m \int_0^{2\ell m+x} \left[\boldsymbol{l}^1_{2n-m+1}(2h\tau) + \boldsymbol{l}^1_{2n-m}(2h\tau) \right] \cdot \mu_m(x-\tau) d\tau = \mathbf{D}(x),$$

or, by virtue of notation (13), in the form

$$\sum_{m=0}^{2n+1} (-1)^m \left\{ \mu_m(x) - h \int_0^{2\ell m+x} \mathbf{H}_m(\tau) \mu_m(x-\tau) d\tau \right\} = -\mathbf{D}(x). \tag{31}$$

Note that this relation is valid in the sense of the class $L_p[0, 2\ell]$.

We have thereby shown that if the function $\mu(t)$ is a solution of a boundary control problem, then it necessarily satisfies condition (31). Conversely, let $\mu(t)$ satisfy condition (31). Then, by performing considerations similar to the above-performed investigation, we obtain

$$\frac{1}{2} \left[\widehat{u}_x(x, T) + \widehat{u}_t(x, T) \right]$$
$$= - \sum_{m=0}^{2n+1} (-1)^m \mu_m(x) + \widetilde{\mathbf{A}}(x)$$

$$+h\sum_{m=0}^{2n+1}(-1)^m\int_0^{2\ell m+x}\left[\boldsymbol{l}^1_{2n-m+1}(2h\tau)+\boldsymbol{l}^1_{2n-m}(2h\tau)\right]\cdot\mu_m(x-\tau)d\tau$$

$$-h\sum_{m=0}^{2n+1}(-1)^m\int_0^{2\ell m+x}\left[\boldsymbol{l}^1_{2n-m+1}(2h\tau)+\boldsymbol{l}^1_{2n-m}(2h\tau)\right]\cdot\widetilde{\mathbf{A}}(x-\tau)d\tau,$$

where $\widehat{u}(x,T)$, $\widehat{u}_t(x,T)$, $\varphi(x)$, and $\psi(x)$ are extended from the interval $[0,\ell]$ to the interval $[\ell,2\ell]$ as odd functions around the point $x=\ell$. By using this relation, condition (30), and relation (31), we obtain the relation

$$\widehat{u}_x(x,T)+\widehat{u}_t(x,T)=\widehat{\varphi}'(x)+\widehat{\psi}(x)$$

which is valid almost everywhere on the interval $[0,2\ell]$. Since the functions $\widehat{u}_x(x,T)$, $\widehat{u}_t(x,T)$, $\widehat{\varphi}(x)$, and $\widehat{\psi}(x)$ are continued as odd functions around the point $x=\ell$, we have that the relations

$$\widehat{u}_x(x,T)+\widehat{u}_t(x,T)=\widehat{\varphi}'(x)+\widehat{\psi}(x),\qquad \widehat{u}_x(x,T)-\widehat{u}_t(x,T)=\widehat{\varphi}'(x)-\widehat{\psi}(x),$$

are valid almost everywhere on the interval $[0,\ell]$. Consequently, by virtue of (19) and the relations $\widetilde{u}(x,T)=0$, $\widetilde{u}_t(x,T)=0$, $\widehat{u}(\ell,T)=\widehat{\varphi}(\ell)=0$, the terminal conditions (3) are satisfied for the function $u(x,t)$. But the validity of the initial conditions (2) readily follows from (18), (19) and (24); i.e., the function $\mu(t)$ is a solution of the boundary control problem.

By performing the change of variables $\{\tau=t-\xi\}$ we reduce the integral (15) to be minimized to the form

$$\int_0^T\left|\mu(t)-h\cdot\int_0^t\mathbf{H}(t,\tau)\mu(t-\tau)d\tau\right|^p dt. \tag{32}$$

Note that this integral for the time interval, $\mathbf{T}$ given by relation (12), can be represented in the following form which is more convenient for the optimization:

$$\int_0^{2\ell}\sum_{m=0}^{2n+1}\left|(-1)^m\cdot\left\{\mu_m(x)-h\cdot\int_0^{2\ell m+x}\mathbf{H}_m(\tau)\mu_m(x-\tau)d\tau\right\}\right|^p dx, \tag{33}$$

where the variable t is replaced by x.

Therefore, the optimization problem can be reduced to finding the minimum of the integral (33) with the constraints given by (30), and (31).

10.3 The Optimization Process

Lemma 1 *Let N be some fixed positive integer, and let $A:(L_p[a,b])^N\to L_1[a,b]$ be the operator that takes each function*

$$F(x)=\big(f_1(x),f_2(x),\dots,f_N(x)\big)\in\big(L_p[a,b]\big)^N$$

to the sum

$$AF(x) = \sum_{i=1}^{N} \left| f_i(x) \right|^p \in L_1[a, b].$$

Let M be some subset of functions in $(L_p[a, b])^N$. If the pointwise minimum

$$\min_{F \in M} AF(x) = AF^0(x), \quad x \in [a, b],$$

is attained at a function $F^0(x) \in M$, then the minimum of the integral

$$\min_{F \in M} \int_a^b AF(x)dx = \int_a^b AF^0(x)dx$$

is attained at the same function $F^0(x)$.

The assertion of the lemma follows from the fact that the inequality

$$AF^0(x) \leq AF(x)$$

for all x in the interval $[a, b]$ is valid for an arbitrary function $F(x)$ in the set M; therefore,

$$\int_a^b AF^0(x)dx \leq \int_a^b AF(x)dx$$

for any function $F(x)$ from the set M.

For brevity of notation, we set

$$z_m = \mu_m(x) - h \cdot \int_0^{2\ell m + x} \mathbf{H}_m(\tau)\mu_m(x - \tau)d\tau, \quad m = \overline{0, 2n+1}. \tag{34}$$

By applying Lemma 1 to the functions z_m, we reduce the problem of finding the minimum of the integral (33) to finding the pointwise minimum of the sum

$$\sum_{m=0}^{2n+1} |z_m|^p \tag{35}$$

under the condition

$$\sum_{m=0}^{2n+1} z_m = -D(x). \tag{36}$$

The above-mentioned minimum can be found with the use of the Lagrange method. We form the Lagrange function

$$\mathrm{L}(x, \overline{\lambda}) = \sum_{m=0}^{2n+1} |z_m|^p + \overline{\lambda} \cdot \left[\sum_{m=0}^{2n+1} z_m + \mathbf{D}(x) \right].$$

Let $p > 1$. We set the derivative of the function $\mathrm{L}(x, \overline{\lambda})$ with respect to z_m equal to zero:

$$p \cdot |z_m|^{p-1} \cdot \operatorname{sgn}(z_m) + \lambda = 0, \quad m = \overline{0, 2n+1}. \tag{37}$$

Since $p > 1$ and $|z_m|^{p-1} \geq 0$, we have

$$\operatorname{sgn}(z_m) = \operatorname{sgn}(-\lambda), \quad m = \overline{0, 2n+1}. \tag{38}$$

We separately consider two cases, $\mathbf{D}(x) \leq 0$ and $\mathbf{D}(x) > 0$.

- If $\mathbf{D}(x) \leq 0$, then, by using relations (36) and (38) we obtain $z_m \geq 0$, $m = \overline{0, 2n+1}$, $\lambda \leq 0$. Consequently, z_m is equal to the same number $(-\frac{\lambda}{p})^{1/(p-1)}$ for all $m = \overline{0, 2n+1}$.
- If $\mathbf{D}(x) > 0$, then $z_m < 0$, $m = \overline{0, 2n+1}$, $\lambda > 0$. Therefore, z_m is equal to the same number $-(\frac{\lambda}{p})^{1/(p-1)}$ for all $m = \overline{0, 2n+1}$.

But then it follows from (36) that

$$z_m = -\frac{\mathbf{D}(x)}{2n+2} = z_m^0 \tag{39}$$

for all $m = \overline{0, 2n+1}$. The relations

$$\sum_{m=0}^{2n+1} |z_m| \geq \left| \sum_{m=0}^{2n+1} z_m \right| = \left| \mathbf{D}(x) \right| = \sum_{m=0}^{2n+1} \left| z_m^0 \right|$$

imply that, for $p = 1$, the same set of functions z_m^0 attains the pointwise minimum of the sum $\sum_{m=0}^{2n+1} |z_m|^p$.

Now we write out relation (39) in the original terms:

$$\mu_m(x) - h \int_0^{2\ell m + x} \mathbf{H}_m(\tau) \mu_m(x - \tau) d\tau = \frac{(-1)^{m+1} \mathbf{D}(x)}{2n+2},$$
$$\text{for } 0 \leq x \leq 2\ell;\ m = \overline{0, 2n+1}. \tag{40}$$

We denote the number $2\ell m + x$ for all $m = \overline{0, 2n+1}$ by y and perform the substitution $y - \tau = \xi$ in the integrand; then relation (40) acquires the form of the convolution-type Volterra integral equation of the second kind

$$\mu(y) - h \cdot \int_0^y \mathbf{H}_m(y - \xi) \mu(\xi) d\xi = \frac{(-1)^{m+1} \mathbf{D}(y - 2\ell m)}{2n+2},$$
$$\text{for } 2\ell m \leq y \leq 2\ell(m+1);\ m = \overline{0, 2n+1}, \tag{41}$$

where the equality is valid in the sense of the class $L_p[2\ell m, 2\ell(m+1)]$ and the kernel $\mathbf{H}_m(y - \xi)$ is given by (13).

Such equations are studied with the use of the Laplace transform

$$\mathcal{L}_p(\mathbf{F}) = \int_0^\infty e^{-pt}\mathbf{F}(t)dt,$$

which reduces a convolution into an ordinary product under some conditions related with its applicability.

10.4 The Integral Equation Solving

We denote the right side of (41) by $f(y)$ and find a solution to this equation, the function $\mu(t)$ from the class $L_p[0,T]$.

Applying the direct Laplace transform of (41) we obtain the following algebraic equation

$$\widetilde{\mu}(p) - h\widetilde{\mathbf{H}}_m(p)\widetilde{\mu}(p) = \widetilde{f}(p),$$

$$\widetilde{\mu}(p) = \frac{\widetilde{f}(p)}{1 - h\widetilde{\mathbf{H}}_m(p)} = \widetilde{f}(p) + \frac{h\widetilde{\mathbf{H}}_m(p)}{1 - h\widetilde{\mathbf{H}}_m(p)}\widetilde{f}(p).$$

To find the initial required solution of the integral equation we use the inverse Laplace transform. We obtain

$$\mu(y) = f(y) + h\int_0^y \mathbf{R}(y-t)f(t)dt, \quad 2\ell m \le y \le 2\ell(m+1), m = \overline{0, 2n+1},$$

where

$$\mathbf{R}(y-t) = \mathcal{L}_p^{-1}\left[\frac{\widetilde{\mathbf{H}}_m(p)}{1 - h\widetilde{\mathbf{H}}_m(p)}\right](y-t),$$

$$\mathbf{H}_m(\tau) = e^{-ht}\left[\sum_{k=0}^{2n-m+1}\binom{2n-m+2}{2n-m+1-k}\frac{(-2h\tau)^k}{k!} + \sum_{k=0}^{2n-m}\binom{2n-m+1}{2n-m-k}\frac{(-2h\tau)^k}{k!}\right],$$

$$\widetilde{\mathbf{H}}_m(p) = \left(\frac{p-h}{h+p}\right)^{-m}\left(2h\left(\frac{p-h}{h+p}\right)^m + 2p\left(\frac{p-h}{h+p}\right)^m + h\left(\frac{p-h}{h+p}\right)^{2n} - p\left(\frac{p-h}{h+p}\right)^{2n}\right)\Big/\big(2h(h+p)\big) - \frac{(h-p)^2(1-2h/(h+p))^{2n-m}}{2h(h+p)^2}, \tag{42}$$

$$\frac{\widetilde{\mathbf{H}}_m(p)}{1 - h\widetilde{\mathbf{H}}_m(p)} = -\frac{(h+p)^2((p-h)/(h+p))^{m-2n}}{h(h-p)p} - \frac{1}{h}.$$

For simplicity, taking account of the inverse Laplace transform of expression (43), it is represented in the form

$$\frac{\widetilde{\mathbf{H}}_m(p)}{1-h\widetilde{\mathbf{H}}_m(p)}$$
$$= -\frac{1}{h} + \sum_{i=0}^{2n-m+2} h^{-i-m+2n+1} p^{i-1}(p-h)^{m-2n-1}\binom{-m+2n+2}{i}. \quad (43)$$

The inverse Laplace transform of the function (43) is

$$\mathbf{R}(x) = \sum_{i=0}^{-m+2n+2} h^{i-1}x^{i-1}\binom{-m+2n+2}{i}\,{}_1\tilde{\mathbf{F}}_1(-m+2n+1;i;hx), \quad (44)$$

where ${}_1\tilde{\mathbf{F}}_1(a;c;z)$—*is a degenerate Kummer hypergeometric function*, [7],

$${}_1\tilde{\mathbf{F}}_1(a;c;z) = \boldsymbol{\Phi}(a;c;z) = \sum_{n=0}^{\infty}\frac{(a)_n}{(c)_n}\frac{z^n}{n!}.$$

Consequently, the solution of (41) can be represented as

$$\mu(y) = \frac{(-1)^{m+1}\mathbf{D}(y-2\ell m)}{2n+2} + h\int_0^y \mathbf{R}(y-t)\frac{(-1)^{m+1}\mathbf{D}(t-2\ell m)}{2n+2}dt,$$
$$2\ell m \le y \le 2\ell(m+1), m = \overline{0, 2n+1}, \quad (45)$$

where $\mathbf{R}(x)$ is expressed by the formula (44).

For example, when $T = 8\ell$, we have

$$\mathbf{R}(x) = \frac{16}{315}e^{hx}\big(2h^7x^7 + 49h^6x^6 + 462h^5x^5 + 2100h^4x^4 + 4830h^3x^3$$
$$+ 5355h^2x^2 + 2520hx + 315\big) + 1, \quad 0 < x < 2\ell,$$
$$\frac{16}{45}e^{hx}\big(h^6x^6 + 18h^5x^5 + 120h^4x^4 + 360h^3x^3$$
$$+ 495h^2x^2 + 270hx + 45\big) - 1, \quad 2\ell \le x < 4\ell,$$
$$\frac{4}{15}e^{hx}\big(4h^5x^5 + 50h^4x^4 + 220h^3x^3$$
$$+ 390h^2x^2 + 270hx + 45\big) + 1, \quad 4\ell \le x < 6\ell,$$
$$\frac{4}{3}e^{hx}\big(2h^4x^4 + 16h^3x^3 + 42h^2x^2 + 36hx + 9\big) - 1, \quad 6\ell \le x < 8\ell.$$

It is easy to see that for $h = 0$ the solution of (45) moves in the solution obtained in [6] in the study of boundary control problems with second boundary condition at the left end of the string when docked right for $T = 8\ell$.

10.5 The Uniqueness of the Optimal Control

Now we analyze the uniqueness of the optimal solution of the boundary control problem. We show that if $p > 1$, then only the set $\{z_m^0\}$ of functions of the class $L_p[0, 2\ell]$ minimizes the integral (15) under condition (36). Let $\{z_m\}$ be some set of functions satisfying the constraint (36) and

$$\int_0^{2\ell} \sum_{m=0}^{2n+1} |z_m|^p dx = \int_0^{2\ell} \sum_{m=0}^{2n+1} |z_m^0|^p dx.$$

By using the linearity of the integral we obtain

$$\int_0^{2\ell} \left[\sum_{m=0}^{2n+1} |z_m|^p - \sum_{m=0}^{2n+1} |z_m^0|^p \right] dx = 0.$$

At the same time, since the function set z_m^0 provides the pointwise minimum of the sum $\sum_{m=0}^{2n+1} |z_m|^p$, we have

$$\sum_{m=0}^{2n+1} |z_m|^p - \sum_{m=0}^{2n+1} |z_m^0|^p \geq 0.$$

This implies that

$$\sum_{m=0}^{2n+1} |z_m|^p = \sum_{m=0}^{2n+1} |z_m^0|^p$$

almost everywhere on $[0, 2\ell]$.

Suppose that there exists a function set $\{z_m^1\}$ such that

$$\sum_{m=0}^{2n+1} |z_m^1|^p = \sum_{m=0}^{2n+1} |z_m^0|^p = s(x)$$

for almost all x in the interval $[0, 2\ell]$. Then

$$\left[\sum_{m=0}^{2n+1} |z_m^1|^p \right]^{1/p} = \left[\sum_{m=0}^{2n+1} |z_m^0|^p \right]^{1/p} = s^{1/p}(x). \tag{46}$$

Since the set $\{z_m\} = \{\frac{1}{2} z_m^0 + \frac{1}{2} z_m^1\}$ satisfies condition (36), we have

$$\left[\sum_{m=0}^{2n+1} \left| \frac{1}{2} z_m^0 + \frac{1}{2} z_m^1 \right|^p \right]^{1/p} \geq s^{1/p}(x). \tag{47}$$

On the other hand, by using obvious inequalities, the `Minkowski inequality`, and relation (46), we obtain

$$\left[\sum_{m=0}^{2n+1}\left|\frac{1}{2}z_m^0+\frac{1}{2}z_m^1\right|^p\right]^{1/p}\le\frac{1}{2}\left[\sum_{m=0}^{2n+1}\left|z_m^0\right|^p\right]^{1/p}+\frac{1}{2}\left[\sum_{m=0}^{2n+1}\left|z_m^1\right|^p\right]^{1/p}=s^{1/p}(x). \tag{48}$$

The relations (47) and (48) are compatible if and only if the sign "≤" in the Minkowski inequality used in (48) becomes the sign "="; but it is known that, for $p>1$, this is possible only if

$$z_m^0=c(x)z_m^1,\quad m=\overline{0,2n+1}, c(x)>0.$$

By using the last relation and formula (46), we obtain $c(x)\equiv 1$, i.e., $z_m^1=z_m^0$ almost everywhere on $[0,2\ell]$, $m=\overline{0,2n+1}$. Consequently, to prove the desired uniqueness, it remains to note that the uniqueness of the solution of the integral equation (39) in the class of functions satisfying the inequality

$$\left|\mu(t)\right|\le Me^{(M_0|x|^\omega)}\quad (M>0, M_0>0, \omega>0)$$

was proved in [9] in a much more general case.

In conclusion, we note that if $p=1$, then the set of functions z_m minimizing the integral (15) and satisfying condition (36) is defined non uniquely. One can readily see that it can be any set $\{\widetilde{z}_m=-\mathbf{D}(x)\cdot\alpha_m(x)\}$, in which $\alpha_m(x)$ are functions, which are Lebesgue integrable on the interval $[0,2\pi]$ and such that

$$\alpha_m(x)\in L_p[0,2\ell],\qquad \alpha_m(x)\ge 0,$$

$$m=\overline{0,2n+1}\quad\text{and}\quad \sum_{m=0}^{2n+1}\alpha_m(x)=1,\quad \forall x\in[0,2\ell].$$

This completes the proof. □

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Chapter 11
Critical Exponent for the Semilinear Wave Equation with Time or Space Dependent Damping

Kenji Nishihara

Abstract Since the damped wave equation has the diffusion phenomenon, the critical exponent is expected to be the same as that for the corresponding diffusive equation with semilinear term. Therefore, we first remember the basic facts on the diffusion phenomenon. Then, from this point of view, we can conjecture the critical exponent for the damped wave equation and state several results. Finally, the small data global existence of solutions is shown in the supercritical exponent, while no global existence for some data is done in the critical and subcritical exponents. The latter part will be applied to the semilinear damped wave equation with quadratically decaying potential.

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11.1 Introduction

We mainly consider the Cauchy problem for the semilinear wave equation with time-dependent damping:

$$\begin{cases} u_{tt} - \Delta u + b(t)u_t = |u|^\rho, & (t,x) \in \mathbf{R}_+ \times \mathbf{R}^N \\ (u,u_t)(0,x) = (u_0,u_1)(x), & x \in \mathbf{R}^N, \end{cases} \tag{P}$$

where $\rho > 1$ and $b(t) = b_0(t+1)^{-\beta}$ with $-1 < \beta < 1$ and $b_0 = 1$ (WOLG). When

$$\rho < \frac{N+2}{[N-2]_+} := \begin{cases} \infty & N = 1,2, \\ \frac{N+2}{N-2} & N \geq 3, \end{cases}$$

and $(u_0,u_1) \in H^1 \times L^2$ are compactly supported, there exists a unique weak solution $u \in C([0,T];H^1) \cap C^1([0,T];L^2)$ for some $T > 0$, which has the compact support by the finite propagation property. Our concern is with large time behavior

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of the solution. The main aim in this note is to show that the critical exponent is still

$$\rho_F(N) := 1 + \frac{2}{N} \quad \text{(Fujita exponent)} \tag{1}$$

even for time-dependent damping. That is, if $\rho > \rho_F(N)$, then there exists a time-global solution $u(t, x)$ to (P) for any small data, while, if $\rho \leq \rho_F(N)$, then the solution does not exist time-globally for suitable data. When $b(t) \equiv 1$, our problem (P) is reduced to

$$\begin{cases} u_{tt} - \Delta u + u_t = |u|^\rho, & (t, x) \in \mathbf{R}_+ \times \mathbf{R}^N \\ (u, u_t)(0, x) = (u_0, u_1)(x), & x \in \mathbf{R}^N, \end{cases} \tag{DW}$$

which has the diffusion phenomenon. That is, the solution behaves as $t \to \infty$ like the solution to the corresponding diffusive equation

$$\begin{cases} -\Delta \phi + \phi_t = |\phi|^\rho, & (t, x) \in \mathbf{R}_+ \times \mathbf{R}^N \\ \phi(0, x) = \phi_0(x), & x \in \mathbf{R}^N. \end{cases} \tag{H}$$

For (H) with $\phi_0(x) \geq 0$, because of the maximum principle, the equation is equivalent to

$$\phi_t - \Delta \phi = |\phi|^{\rho - 1} \phi, \tag{2}$$

which is called the Fujita equation, named after his pioneering work [3]. As is known well, there is a critical exponent, the Fujita exponent, given by (1). When $\rho > \rho_F(N)$, there exists a unique and time-global solution $\phi(t, x)$ for small data $\phi_0(x)$, while, when $\rho \leq \rho_F(N)$, the time-local solution $\phi(t, x)$ blows up within a finite time for any small data $\phi_0(x) \geq 0$. By the diffusion phenomenon the solution $u(t, x)$ to (DW) is expected to behave like $\phi(t, x)$. In fact, (DW) has been investigated by many authors [5–9, 15–19, 24, 26–28, 34, 39] etc. See also the references therein. In the results, the Fujita exponent $\rho_F(N)$ is critical for (DW), too. More detailed discussions are referred to the survey paper [30]. Based on those results, we investigate (P) and determine the critical exponent.

In the supercritical exponent case the global existence of solution for small data will be shown by the weighted energy method. The proof is generally rather complicated, but the procedure is standard in some sense once we obtain the suitable weight. The original development in this direction was done in Todorova and Yordanov [34]. The proof of blow-up of solution is sometimes difficult. In case of the space-dependent damping $+b(x)u_t$, instead of $+b(t)u_t$ in (P), the blow-up was shown by Ikehata, Todorova and Yordanov [11] by applying the test function method, developed by Qi S. Zhang [39] for the constant coefficient damping. To our problem (P) we apply the same method together with the additional idea. However, in case of both space- and time-dependent damping, the blow-up is not yet proved. Therefore, we will focus to derive the blow-up results. Also, we apply the idea to the semilinear damped wave equation with quadratically decaying potential

$$u_{tt} + u_t - \Delta u + V(x)u = |u|^\rho, \tag{3}$$

where V is radial and

$$V(x) =: V\big(|x|\big) \sim \omega |x|^{-2} \quad (\omega > 0) \quad \text{as } |x| \to \infty. \tag{4}$$

The corresponding parabolic problem

$$\begin{cases} \phi_t - \Delta\phi + V(x)\phi = |\phi|^{\rho-1}\phi, & (t,x) \in \mathbf{R}_+ \times \mathbf{R}^N \\ \phi(0,x) = \phi_0(x) \ge 0, & x \in \mathbf{R}^N, \end{cases} \tag{5}$$

has been investigated by Ishige [13]. Qi S. Zhang [40] emphasized that the potential has an influence on the critical exponent and obtained the critical exponent depending on the potential. Ishige [13] covered the case of quadratically decaying potential and obtained the critical exponent

$$\rho_c(N,\omega) = 1 + \frac{2}{N + \alpha(\omega)}, \tag{6}$$

where

$$\alpha(\omega) = \frac{-(N-2) + \sqrt{(N-2)^2 + 4\omega}}{2} > 0, \tag{7}$$

which is the positive root of

$$\alpha^2 + (N-2)\alpha - \omega = 0. \tag{8}$$

By the diffusion phenomenon we expect that $\rho_c(N,\omega)$ is still critical for (3). Since their proofs are based on the maximum principle for the parabolic equation, it will be worth to obtain the critical exponent for (3).

Our plan of this note is as follows. In Sect. 11.2 we give the model showing the diffusion phenomenon and analyze the solution of linear damped wave equation. Similar discussions are done in [30]. In Sect. 11.3 we consider the semilinear problem (P) due to the analysis in the preceding section, and state the main theorems. In Sect. 11.4 we treat the weighted energy method in the supercritical exponent case. In the final section the proof of blow-up is given and the method will be applied to (3) with (4).

11.2 Model Equation and Diffusion Phenomenon

We first show the model equation showing the diffusion phenomenon.

Model Equation of Heat Conductive Equation (Li [18]) Consider an infinitely long wire. By $q(t,x)$ denote the heat flow at time t and position x of the wire. Let the heat be conducted only in the x-direction. When $\phi(t,x)$ is the temperature at (t,x) and its specific heat is normalized to one, the change of temperature in

$[a,b]$ is $\frac{d}{dt}\int_a^b \phi(t,x)dx$. The total flow is $q(t,a)-q(t,b)=-\int_a^b q_x(t,x)dx$ by the inflow and outflow at $x=a,b$, and hence

$$\frac{d}{dt}\int_a^b \phi(t,x)dx = -\int_a^b q_x(t,x)dx, \quad \text{that is, } \phi_t + q_x = 0. \tag{9}$$

The heat flow q is proportional to the change of temperature (the coefficient is minus) by *Fourier's law*, that is,

$$q(t,x) = -\kappa\phi_x(t,x) \quad (\kappa > 0\text{: coefficient of heat conductivity}). \tag{10}$$

Substituting (9) to (10), we have the simplest one-dimensional *linear heat equation*

$$\phi_t - \kappa\phi_{xx} = 0. \tag{11}$$

Equation (10) is the usual Fourier's law. But, if we assume *Fourier's law with time delay*, that is,

$$q(t+\tau,x) = -\kappa\phi_x(t,x) \quad (0<\tau\ll 1), \tag{12}$$

then we get

$$q(t,x) + \tau q_t(t,x) = -\kappa\phi_x(t,x), \tag{13}$$

after Taylor's expansion of $q(t+\tau,x)$ at t and neglecting the higher order terms because of $0<\tau\ll 1$. Differentiating (13) with respect to x and using (9), we arrive at the 1-D *linear damped wave equation*

$$\tau\phi_{tt} + \phi_t - \kappa\phi_{xx} = 0. \tag{14}$$

When $\tau\to 0+$, it is easily conjectured by the derivation that the solution of (14) approaches to the solution of (11). Note that the propagation speed of (14) is proportional to $1/\sqrt{\tau}$. The scale transformation in (14) means that the coefficient of ϕ_{tt} is normalized to be one and $\tau\to 0+$ corresponds to $t\to+\infty$, which implies the diffusion phenomenon.

There are nonlinear models, too, one of which is the compressible flow through porous media. The 1-D model is represented in the Lagrangian mass coordinate by

$$\begin{cases} v_t - u_x = 0, & (t,x)\in\mathbf{R}_+\times\mathbf{R}^1 \\ u_t + p_x = -\alpha u & (\alpha>0\text{: constant}) \end{cases} \tag{15}$$

Here, v (>0) is the specific volume ($=1/\rho$, ρ: density), u the velocity, and p the pressure with the barotropic relation $p=p(v)$. The typical example is $p(v)=v^{-\gamma}$ ($\gamma\ge 1$), in which $\gamma=1$ means the isothermal flow and $\gamma>1$ does the isentropic one. The solution $(v,u)(t,x)$ had been conjectured to behave like the solution $(\bar{v},\bar{u})(t,x)$ to

$$\begin{cases} \bar{v}_t - \bar{u}_x = 0, \\ p(\bar{v})_x = -\alpha\bar{u}. \end{cases} \tag{16}$$

In fact, Hsiao and Liu [8] showed under suitable conditions that the solution v of (15) behaves like $\bar{v}$ of (16). Since $\bar{v}$ in (16) satisfies quasilinear parabolic equation and v in (15) does the quasilinear damped wave equation, their result means the diffusion phenomenon. In this model there are many developments [4, 27] etc. (Here we only cite them because our interest is in the second order damped wave equation.)

We now analyze the solution of the Cauchy problem for linear damped wave equation of second order

$$\begin{cases} u_{tt} - \Delta u + u_t = 0, & (t,x) \in \mathbf{R}_+ \times \mathbf{R}^N, \\ (u, u_t)(0,x) = (u_0, u_1)(x), & x \in \mathbf{R}^N. \end{cases} \tag{LDW}$$

Fortunately, we know the explicit formula of solution ([1]). By $v = S_N(t)g$ denote the solution to

$$\begin{cases} v_{tt} - \Delta v + v_t = 0, & (t,x) \in \mathbf{R}_+ \times \mathbf{R}^N, \\ (u, u_t)(0,x) = (0, g)(x), & x \in \mathbf{R}^N, \end{cases} \tag{17}$$

then the solution u to (*LDW*) is given by

$$u(t,x) = S_N(t)(u_0 + u_1) + \partial_t\big(S_N(t)u_0\big). \tag{18}$$

Hence, it is necessary to analyze both $S_N(t)(g)$ and $\partial_t(S_N(t)g)$. When $N = 1, 2, 3$,

$$\big(S_1(t)g\big)(x) = \frac{e^{-t/2}}{2} \int_{|z|\le t} I_0\left(\frac{1}{2}\sqrt{t^2 - |z|^2}\right) g(x+z)dz \tag{19$_1$}$$

$$\big(S_2(t)g\big)(x) = \frac{e^{-t/2}}{2\pi} \int_{|z|\le t} \frac{\cosh(1/2)\sqrt{t^2 - |z|^2}}{\sqrt{t^2 - |z|^2}} g(x+z)dz \tag{19$_2$}$$

$$\big(S_3(t)g\big)(x) = \frac{e^{-t/2}}{4\pi t} \partial_t \int_{|z|\le t} I_0\left(\frac{1}{2}\sqrt{t^2 - |z|^2}\right) g(x+z)dz. \tag{19$_3$}$$

Here $I_\nu(y)$ is the modified Bessel function with the series form

$$I_\nu(y) = \sum_{m=0}^{\infty} \frac{1}{m!\Gamma(m+\nu+1)} \left(\frac{y}{2}\right)^{2m+\nu} \quad (\Gamma\text{: Gamma function}).$$

Note that $S_N(t)g$ is obtained by the method of descent for the explicit formula of the wave equation without damping for the dimension $N+1$. Here we only analyze $S_3(t)g$ (for $S_N(t)g$, $N = 1, 2$ see [10, 23]). The properties of the modified Bessel function are the followings.

Lemma 1 *The modified Bessel function* I_ν $(\nu \in \mathbf{N}_0)$ *satisfies*

$$I_0(0) = 1, \qquad I_1(y)/y|_{y=0} = 1/2, \qquad \left(I_0(y) - \frac{2}{y} I_1(y)\right)/y^2\bigg|_{y=0} = 1/8,$$
$$I_0'(y) = I_1(y), \qquad I_1'(y) = I_0(y) - I_1(y)/y$$

and, moreover, the following expansion formula as $y \to \infty$:

$$\begin{aligned} I_\nu(y) = \frac{e^y}{\sqrt{2\pi y}}\bigg(&1 - \frac{(\nu-1/2)(\nu+1/2)}{2y} \\ &+ \frac{(\nu-1/2)(\nu-3/2)(\nu+3/2)(\nu+1/2)}{2!2^2 y^2} \\ &- \cdots + (-1)^k \frac{(\nu-1/2)\cdots(\nu-(k-1/2))(\nu+(k-1/2))\cdots(\nu+1/2)}{k!2^k y^k} \\ &+ O\big(y^{-k-1}\big)\bigg). \end{aligned}$$

By Lemma 1 the terms $S_3(t)g$ and $\partial_t(S_3(t)g)$ are, respectively,

$$\begin{aligned} S_3(t)g &= e^{-t/2} \cdot \frac{t}{4\pi} \int_{S^2} g(x + t\omega) d\omega \\ &\quad + \frac{e^{-t/2}}{8\pi} \int_{|z| \le t} I_1\left(\frac{\sqrt{t^2 - |z|^2}}{2}\right) \frac{g(x+z)dz}{\sqrt{t^2 - |z|^2}} \\ &=: e^{-t/2} W(t)g + J_0(t)g \quad \big(W(t)g\text{: Kirchhoff formula}\big) \end{aligned}$$

and

$$\begin{aligned} \partial_t\big(S(t)g\big) &= e^{-t/2} \cdot \left\{\left(-\frac{1}{2} + \frac{t}{8}\right) W(t)g + \partial_t\big(W(t)g\big)\right\} \\ &\quad + \int_{|z| \le t} \partial_t \left[\frac{e^{-t/2} I_1(\sqrt{t^2 - |z|^2}/2)}{8\pi\sqrt{t^2 - |z|^2}}\right] g(x+z)dz \\ &=: e^{-t/2} \tilde{W}(t)g + J_1(t)g. \end{aligned}$$

Hence, by (18) the solution $u(t,x)$ to (*LDW*) is decomposed as

$$\begin{aligned} u(t,x) &= \big[S(t)(u_0 + u_1)\big](x) + \partial_t\big[\big(S(t)u_0\big)\big](x) \\ &= e^{-t/2}\big\{W(t)(u_0 + u_1) + \tilde{W}(t)u_0\big\} + J_0(t)(u_0 + u_1) + J_1(t)u_0. \end{aligned}$$

By Lemma 1 the remainder terms $J_i(t)g$ are estimated as follows.

Lemma 2 ([28], L^p-L^q estimate in $N = 3$) *For p, q with $1 \le q \le p \le \infty$, it holds that*

$$\big\| J_0(t)g \big\|_{L^p} \le C\|g\|_{L^q}(t+1)^{-(3/2)(1/q-1/p)}, \quad t \ge 0$$
$$\big\| \big(J_0(t) - e^{t\Delta}\big)g \big\|_{L^p} \le C\|g\|_{L^q} t^{-(3/2)(1/q-1/p)-1}, \quad t > 0$$
$$\big\| J_1(t)g \big\|_{L^p} \le C\|g\|_{L^q}(t+1)^{-(3/2)(1/q-1/p)-1}, \quad t \ge 0$$

where $e^{t\Delta}g = \int_{\mathbf{R}^N} G(t, x-y)g(y)dy$ with the Gauss kernel

$$G(t,x) = (4\pi t)^{-N/2} e^{-|x|^2/(4t)}.$$

Therefore, the decomposition of solution $u(t,x)$ to (*LDW*) is symbolically seen as

$$u(t,x) = \underbrace{e^{-t/2}\big\{W(t)(u_0+u_1) + \tilde{W}(t)u_0\big\}}_{\text{wave part decaying fast}} + \underbrace{J_0(t)(u_0+u_1) + J_1(t)u_0}_{\text{diffusive part}}. \tag{20}$$

Thus the solution u to (*LDW*) behaves like the solution ϕ to the linear heat equation with data $\phi_0(x) = (u_0 + u_1)(x)$, which implies the diffusion phenomenon. The decomposition also implies that $u(t,x)$ has the finite propagation property and may have the singularity, but, $u(t,x)$ has not the smoothing effect nor the maximum principle. These observations suggest the methods to treat the damped wave equations, which are different from the parabolic equations.

11.3 Semilinear Damped Wave Equation

Based on the decomposition (20), we proceed to study the semilinear problems

$$\begin{cases} u_{tt} - \Delta u + u_t = f(u), & (t,x) \in \mathbf{R}_+ \times \mathbf{R}^N \\ (u, u_t)(0,x) = (u_0, u_1)(x), & x \in \mathbf{R}^N, \end{cases} \tag{21}$$

and

$$\begin{cases} u_{tt} - \Delta u + b(t)u_t = f(u), & (t,x) \in \mathbf{R}_+ \times \mathbf{R}^N \\ (u, u_t)(0,x) = (u_0, u_1)(x), & x \in \mathbf{R}^N, \end{cases} \tag{22}$$

compared with the corresponding parabolic problems. Here we assume

$$(u_0, u_1) \in H^1 \times L^2 \quad \text{with compact supports} \tag{23}$$

and

$$f(u) = -|u|^{\rho-1}u \quad \text{or} \quad f(u) = +|u|^{\rho-1}u, |u|^\rho \quad \left(1 < \rho < \frac{N+2}{[N-2]_+}\right). \tag{24}$$

Then we have a time-local solution u to (21) or (22). Our interest is in the large-time behavior. Depending on the semilinear term $f(u)$ the behavior is much different from each other. We have several known results depending on the semilinear absorbing term $f(u) = -|u|^{\rho-1}u$ or semilinear source term $f(u) = +|u|^{\rho-1}u$, $|u|^\rho$ for (21). Denoting $f \sim g$ as $t \to \infty$ when $\frac{f-g}{g} \to 0$ as $t \to \infty$, we can sum them up, roughly speaking, as follows.

Absorbing Semilinear Term If $f(u) = -|u|^{\rho-1}u$, then there exists a time-global solution $u(t)$ to (21) for any large data (u_0, u_1), which satisfies

(i) when $\rho > \rho_F(N) = 1 + \frac{2}{N}$, $u(t) \sim M_0 G(t,x)$ as $t \to \infty$, where $M_0 = \int_{\mathbf{R}^N} (u_0 + u_1)(x)dx - \int_0^\infty \int_{\mathbf{R}^N} |u|^{\rho-1}u(t,x)dxdt$,
(ii) when $\rho = \rho_F(N)$, $\|u(t)\|_{L^2} = O(t^{-N/4}(\log t)^{-N/2})$ as $t \to \infty$,
(iii) when $1 < \rho < \rho_F(N)$, $\|u(t)\|_{L^2} = O(t^{-1/(\rho-1)+N/4})$ as $t \to \infty$.

Here we note that there exists a similarity solution $w_0(t,x)$ of the form

$$w_0(t,x) = (t+1)^{-1/(\rho-1)} f_0\big(|x|/\sqrt{t+1}\big)$$

to the corresponding semilinear heat equation $-\Delta\phi + \phi_t + |\phi|^{\rho-1}\phi = 0$, where f_0 is a solution of

$$-f_0'' - \left(\frac{r}{2} + \frac{N-1}{r}\right) f_0' + |f_0|^{\rho-1} f_0 = \frac{1}{\rho-1} f_0, \qquad \lim_{r\to\infty} r^{2/(\rho-1)} f_0(r) = 0.$$

The L^2-norm of w_0 is the same as $\|u(t)\|_{L^2}$ in (iii).

Source Semilinear Term If $f(u) = +|u|^{\rho-1}u$, $|u|^\rho$, then the followings hold:

(i) when $\rho > \rho_F(N)$, there exists a time-global solution $u(t)$ for small data (u_0, u_1), and $u(t) \sim M_0 G(t,x)$ as $t \to \infty$ with $M_0 = \int_{\mathbf{R}^N} (u_0 + u_1)(x)dx + \int_0^\infty \int_{\mathbf{R}^N} f(u)(t,x)dxdt$.
(ii, iii) when $\rho \le \rho_F(N)$, $f(u) = |u|^\rho$ ($f(u) = +|u|^{\rho-1}u$ in some cases), there is no existence of time-global solution $u(t)$ for some data (u_0, u_1).

Thus the critical exponent is the Fujita exponent $\rho_F(N)$ in both cases.

Next we consider the time-dependent damping problem (22). For the linear equation

$$v_{tt} - \Delta v + b(t)v_t = 0 \quad \big(b(t) = (t+1)^{-\beta}\big), \qquad (u,u_t)(0,x) = (v_0, v_1)(x), \quad (25)$$

Wirth ([36, 37]) showed, by the Fourier transformation, that when $\beta > 1$, the damping is non-effective and the solution of (25) has the wave property. When $-1 < \beta < 1$, the damping is effective and the decay rate is the same as that of solutions of the corresponding diffusive equation (see also Yamazaki [38] for $0 \le \beta < 1$). The remaining case $\beta < -1$ is classified as the over-damping.

Here we consider the corresponding parabolic problem when $-1 < \beta < 1$ from another point of view. The linear equation is

$$-\Delta\phi + b(t)\phi_t = 0 \quad \text{or} \quad \phi_t = \frac{1}{b(t)}\Delta\phi, \quad \text{with } \phi(0,x) = \phi_0(x).$$

The explicit formula of solution is

$$\phi(t,x) = \int_{\mathbf{R}^N} G_B(t,x-y)\phi_0(y)dy := \int_{\mathbf{R}^N} \big(4\pi B(t)\big)^{-N/2} e^{-|x-y|^2/(4B(t))}\phi_0(y)dy$$

with $B(t) = \int_0^t \frac{d\tau}{b(\tau)} \sim t^{1+\beta}$. Hence, for $-1 < \beta < 1$ we have the L^p-L^q estimate

$$\big\|\phi(t)\big\|_{L^p} \le C\|\phi_0\|_{L^q} t^{-((1+\beta)N/2)(1/q-1/p)} \quad (1 \le q \le p \le \infty),$$

in particular,

$$\big\|\phi(t)\big\|_{L^2} = O\big(t^{-(N/4)(1+\beta)}\big). \tag{26}$$

On the other hand, the corresponding nonlinear equation is

$$b(t)\phi_t - \Delta\phi + |\phi|^{\rho-1}\phi = 0.$$

For this equation we have the self-similar solution of the form

$$w_0(t,x) = (c+ct)^{-(1+\beta)/(\rho-1)} f\left(\frac{|x|}{(c+ct)^{(1+\beta)/2}}\right) \quad \text{if } \rho < 1 + \frac{2}{N}$$

with $c^{1+\beta}(1+\beta) = 1$ ([33]), which satisfies the decay rate

$$\big\|w_0(t,\cdot)\big\|_{L^2} = O\big(t^{-(1/(\rho-1)-N/4)(1+\beta)}\big). \tag{27}$$

Comparing the decay rates of (26) and (27), we can expect that $\rho_F(N)$ is still critical in the case of effective time-dependent damping.

In the case of absorbing semilinear term we have the following theorem.

Theorem 1 *When* $f(u) = -|u|^{\rho-1}u$, $1 < \rho < \frac{N+2}{[N-2]_+}$, *the time-global solution* u *to* (22) *with* $-1 < \beta < 1$ *satisfies*

(i) $\|u(t,\cdot)\|_{L^2} = O(t^{-(1/(\rho-1)-N/4)(1+\beta)})$ *provided that* $1 < \rho < \rho_F(N)$ *(Nishihara and Zhai* [33]*), and*
(ii) $\|u(t,\cdot)\|_{L^2} = O(t^{-(N/4)(1+\beta)+\varepsilon})$ $(0 < \varepsilon \ll 1)$ *provided that* $\rho_F(N) \le \rho < \frac{N+2}{[N-2]_+}$ *(Nishihara* [31]*).*

Moreover, when $N = 1$ *with* $(u_0, u_1) \in H^2 \times H^1$ *additionally,*

$$\big\|u(t,\cdot) - \theta_0 G_B(t,\cdot)\big\|_{L^2} = o\big(t^{-(1/4)(1+\beta)}\big) \quad \text{as } t \to \infty \tag{28}$$

holds (*Nishihara* [32]), *where*

$$\theta_0 = \int_{\mathbf{R}^1} \big(u_1 + (1-\beta)u_0\big)(x)dx \\ + \int_0^\infty \left[\frac{\beta(1-\beta)}{(\tau+1)^{(2-\beta)}} \int_{\mathbf{R}^1} u dx - (\tau+1)^\beta \int_{\mathbf{R}^1} |u|^{\rho-1} u dx \right] d\tau.$$

From the viewpoint of the diffusion phenomenon, the decay rate (i) in Theorem 1 is optimal in the subcritical exponent, and (ii) is almost optimal in the supercritical one. When $N = 1$, by (28) we can conclude that the Fujita exponent $\rho_F(N)$ is completely critical. When $N \geq 2$, we believe the decay rates are (almost) optimal even for the damped wave equation, but these are not proved and so we cannot say that $\rho_F(N)$ is critical. In the critical exponent, we will have the slightly faster decay rate than $G_B(t,x)$ like (ii) in the absorbing semilinear term, which is remained open.

Finally we consider the source semilinear problem (22) with $-1 < \beta < 1$. Our main theorems are the followings.

Theorem 2 (Global existence in the supercritical exponent) *Assume* $|f(u)| = O(|u|^\rho)$ *in* (22). *If*

$$I_0^2 := \int_{\mathbf{R}^N} e^{(1+\beta)|x|^2/(2(2+\delta))} \big(|u_1|^2 + |\nabla u_0|^2 + |u_0|^{\rho+1}\big) dx$$

is suitably small for some small $\delta > 0$, *then there exists a time-global solution* $u \in C([0,\infty); H^1) \cap C^1([0,\infty); L^2)$ *to* (22), *which satisfies*

$$\|u(t)\|_{L^2} \leq C_\delta I_0 (t+1)^{-(N/4)(1+\beta)+\varepsilon/2} \quad \textit{for } \varepsilon = \varepsilon(\delta) \to 0 \ (\delta \to 0)$$

provided that $\rho_F(N) < \rho < \frac{N+2}{[N-2]_+}$.

Theorem 3 (Blow-up in critical and subcritical exponents) *Suppose that* $f(u) = |u|^\rho$ *in* (22) *and the data* (u_0, u_1) *satisfy*

$$\int_{\mathbf{R}^N} \big(u_1(x) + \hat{b}_1 u_0(x)\big) dx > 0, \quad \hat{b}_1^{-1} = \int_0^\infty e^{-\int_0^t (\tau+1)^{-\beta} d\tau} dt.$$

Then the global solution $u \in C([0,\infty); H^1) \cap C^1([0,\infty); L^2)$ *to* (22) *does not exist provided that* $1 < \rho \leq \rho_F(N)$.

The proof of Theorem 2 is done by the energy method with suitable weight, which is stated in the next section together with the reason why such kind of weight is chosen. The blow-up result in Theorem 3 will be proved in the final section by the test function method together with an additional idea. A similar idea can be applied to the semilinear damped wave equation (3) with quadratically decaying potential (4). For the semi-linear problem we also refer the recent paper [2] by D'Abbicco, Lucente and Reissig and the references therein.

11.4 Weighted Energy Method in the Supercritical Exponent

Our problem is the Cauchy problem

$$\begin{cases} u_{tt} - \Delta u + b(t)u_t = f(u), & (t,x) \in \mathbf{R}_+ \times \mathbf{R}^N \\ (u, u_t)(0,x) = (u_0, u_1)(x), & x \in \mathbf{R}^N, \end{cases} \tag{22}$$

with

$$\begin{aligned} &b(t) = (t+1)^{-\beta} \quad (-1 < \beta < 1) \quad \text{and} \\ &\big|f(u)\big| = O\big(|u|^{\rho}\big) \quad \left(\rho_F(N) < \rho < \frac{N+2}{[N-2]_+}\right). \end{aligned} \tag{29}$$

The weighted energy method for the damped wave equation is now well-known, which was originally developed in Todorova and Yordanov [34]. But, here we show the basic energy estimate for the simplest problem, and then apply it to our problem. Because this basic treatment implies how to apply the method to more complicated problems. The simplest problem is

$$(u_s)_t - \Delta u_s = 0, \quad u_s(0,x) = u_0(x). \tag{30}$$

By the diffusion phenomenon u_{tt} in (22) decays fast. Since $\int_0^\infty \int_{\mathbf{R}^N} |f(u)| dx dt < \infty$ in the supercritical exponent, $f(u)$ is small as $t \to \infty$. So, we have the simplest problem (30), taking $\beta = 0$. For the solution u_s to (30) we easily have the L^p-L^q estimate

$$\big\|u_s(t)\big\|_{L^p} \le C t^{-(N/2)(1/q - 1/p)} \|u_0\|_{L^q} \quad (1 \le q \le p \le \infty),$$

applying the Hausdorff and Young inequality, in particular,

$$\big\|u_s(t)\big\|_{L^2} \le C t^{-N/4} \|u_0\|_{L^2}. \tag{31}$$

We want to obtain (31) by the weighted energy method, assuming that

$$u_s(t,x) \text{ has a compact support.} \tag{32}$$

The assumption (32) is reasonable because the solution to (22) is compactly supported by the finite propagation property.

We now multiply (30) by $2e^{2\psi}u_s$ with $\psi = \frac{a|x|^2}{t+1}$ to get

$$\begin{aligned} &\big(e^{2\psi}u_s^2\big)_t - 2\nabla \cdot \big(e^{2\psi} u_s \nabla u_s\big) \\ &\quad + 2\big[e^{2\psi}(-\psi_t)u_s^2 + e^{2\psi}\nabla\psi \cdot 2u_s \nabla u_s + e^{2\psi}|\nabla u_s|^2\big] = 0. \end{aligned} \tag{33}$$

If we change $e^{2\psi} 2\nabla\psi \cdot u_s \nabla u_s$ as

$$e^{2\psi}\nabla\psi \cdot 2u_s\nabla u_s = \nabla \cdot \big(e^{2\psi}u_s^2 \nabla\psi\big) - e^{2\psi}\big(2|\nabla\psi|^2 + \Delta\psi\big)u_s^2,$$

then the last term becomes bad. Hence, we change the term after using an additional idea as

$$\begin{aligned} &e^{2\psi}\nabla\psi\cdot 2u_s\nabla u_s\\ &\quad = 4e^{2\psi}\nabla\psi\cdot u_s\nabla u_s - e^{2\psi}\nabla\psi\cdot 2u_s\nabla u_s\\ &\quad = 4e^{2\psi}\nabla\psi\cdot u_s\nabla u_s - \nabla\cdot\left(e^{2\psi}u_s^2\nabla\psi\right) + e^{2\psi}2|\nabla\psi|^2u_s^2 + e^{2\psi}(\Delta\psi)u_s^2. \end{aligned}$$

Then, (33) becomes

$$\begin{aligned} &\left(e^{2\psi}u_s^2\right)_t - 2\nabla\cdot e^{2\psi}\left(u_s\nabla u_s + u_s^2\nabla\psi\right)\\ &\quad + 2e^{2\psi}\left[\left(-\psi_t + 2|\nabla\psi|^2\right)u_s^2 + 4u_s\nabla\psi\cdot\nabla u_s + |\nabla u_s|^2\right]\\ &\quad + e^{2\psi}(2\Delta\psi)u_s^2 = 0. \end{aligned} \tag{34}$$

Here we note that, since $2e^{2\psi}(u_s\nabla u_s + u_s^2\nabla\psi) = \nabla(e^{2\psi}u_s^2)$, (34) is re-written as

$$\begin{aligned} &\left(e^{2\psi}u_s^2\right)_t - \Delta\left(e^{2\psi}u_s^2\right) + 2e^{2\psi}\left[\left(-\psi_t + 2|\nabla\psi|^2\right)u_s^2 + 4u_s\nabla\psi\cdot\nabla u_s + |\nabla u_s|^2\right]\\ &\quad + e^{2\psi}(2\Delta\psi)u_s^2 = 0. \end{aligned} \tag{34$'$}$$

By the definition of ψ,

$$-\psi_t = \frac{a|x|^2}{(t+1)^2} = \frac{1}{4a}|\nabla\psi|^2, \qquad \Delta\psi = \frac{2aN}{t+1}. \tag{35}$$

By taking $a = \frac{1}{8}$, the discriminant of the second term in (34) is equal to zero. Hence, substituting (35) with $a = \frac{1}{8}$ to (34) and integrating the resultant equation over $\mathbf{R}^N$, we have

$$\frac{d}{dt}\int_{\mathbf{R}^N} e^{2\psi}u_s^2dx + 2\int_{\mathbf{R}^N} e^{2\psi}|2u_s\nabla\psi + \nabla u_s|^2dx + \frac{N/2}{t+1}\int_{\mathbf{R}^N} e^{2\psi}u_s^2dx = 0. \tag{36}$$

Moreover, multiplying (36) by $(t+1)^{N/2}$ and integrating the equation over $[0,t]$, we reach to

$$(t+1)^{N/2}\int_{\mathbf{R}^N} e^{2\psi(t,x)}u_s^2(t,x)dx \le \int_{\mathbf{R}^N} e^{2\psi(0,x)}u_0^2(x)dx$$

or

$$\int_{\mathbf{R}^N} e^{|x|^2/(4(t+1))}u_s^2(t,x)dx \le (t+1)^{-N/2}\int_{\mathbf{R}^N} e^{|x|^2/4}u_0^2(x)dx, \tag{37}$$

which yields the desired estimate (31).

Note that, if we take $a = \frac{1}{8} - \eta$ $(0 < \eta \ll 1)$ then the discriminant of the second term in (34) is negative and, instead of (36), we have

$$\frac{d}{dt}\int_{\mathbf{R}^N} e^{2\psi} u_s^2 dx + \delta \int_{\mathbf{R}^N} e^{2\psi} |\nabla u_s|^2 dx + \frac{N/2 - \delta'}{t+1}\int_{\mathbf{R}^N} e^{2\psi} u_s^2 dx \le 0 \qquad (36')$$

for $0 < \delta, \delta' \ll 1$, that is, the decay rate is a little bit less, but a useful term on ∇u_s comes out.

For the problem

$$(t+1)^{-\beta}(u_\beta)_t - \Delta u_\beta = 0, \quad u_\beta(0, x) = u_0(x), \qquad (38)$$

we multiply (38) by $2(t+1)^\beta e^{2\psi} u_\beta$ to get

$$\begin{aligned}
&\left(e^{2\psi} u_\beta^2\right)_t - 2\nabla \cdot (t+1)^\beta e^{2\psi}\left(u_\beta \nabla u_\beta + u_\beta^2 \nabla \psi\right) \\
&\quad 2(t+1)^\beta e^{2\psi}\left[\left(\frac{-\psi_t}{(t+1)^\beta} + 2|\nabla\psi|^2\right)u_\beta^2 + 4u_\beta \nabla\psi \cdot \nabla u_\beta + |\nabla u_\beta|^2\right] \\
&\quad + (t+1)^\beta e^{2\psi}(2\Delta\psi)u_\beta^2 \\
&= 0. \qquad (39)
\end{aligned}$$

If we take

$$\psi(t, x) = \frac{a|x|^2}{(t+1)^{1+\beta}}, \qquad (40)$$

then

$$\frac{-\psi_t}{(t+1)^\beta} = \frac{a(1+\beta)|x|^2}{(t+1)^{2+2\beta}} = \frac{1+\beta}{4a}|\nabla\psi|^2, \qquad \Delta\psi = \frac{2aN}{(t+1)^{1+\beta}}, \qquad (41)$$

and hence, by taking $a = \frac{1+\beta}{8}$, (39) becomes

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbf{R}^N} e^{2\psi} u_\beta^2 dx + 2(t+1)^\beta \int_{\mathbf{R}^N} e^{2\psi} |2u_\beta \nabla\psi + \nabla u_\beta|^2 dx \\
&\quad + \frac{(1+\beta)N/2}{t+1}\int_{\mathbf{R}^N} e^{2\psi} u_\beta^2 dx = 0, \qquad (42)
\end{aligned}$$

and

$$\int_{\mathbf{R}^N} e^{(1+\beta)|x|^2/(4(t+1)^{1+\beta})} u_\beta^2(t, x) dx \le (t+1)^{-(1+\beta)N/2} \int_{\mathbf{R}^N} e^{(1+\beta)|x|^2/4} u_0^2(x) dx, \qquad (43)$$

which is almost the same estimate as (37).

Now, let us return back to our problem (22). If we multiply (22) by $2e^{2\psi}u$, then we have the additional terms

$$2e^{2\psi}uu_{tt} = 2\left\{\left(e^{2\psi}uu_t\right)_t + e^{2\psi}\left(2(-\psi_t)uu_t - u_t^2\right)\right\}.$$

Especially, the final term is working bad. To cover this, the same as (36′), we choose ψ as

$$\psi(t,x) = \left(\frac{1+\beta}{4} - \eta\right)\frac{|x|^2}{(t+1)^{1+\beta}} \quad (0 < \eta \ll 1)$$

or

$$\psi(t,x) = \frac{(1+\beta)|x|^2}{2(2+\delta)(t+1)^{1+\beta}} \quad (0 < \delta \ll 1).$$

For details, see [22].

Finally, we note that, for space-dependent damping $+\langle x\rangle^{-\alpha}u_t$ $(0 \le \alpha < 1)$, ψ is chosen as $\psi(t,x) = \frac{a|x|^{2-\alpha}}{t+1}$ $(a > 0)$ ([29]) , and $\psi(t,x) = \frac{a|x|^{2-\alpha}}{(t+1)^{1+\beta}}$ for both time and space-dependent damping $+\langle x\rangle^{-\alpha}(t+1)^{-\beta}u_t$ ([20, 21, 35]). See also [17] for space-time dependent damping.

11.5 Proof of Blow-up and Its Application

To assert the necessity of the additional idea to prove Theorem 3, we observe the test function method for the simpler equation with $\beta = 0$

$$u_{tt} - \Delta u + u_t = |u|^\rho, \quad (u, u_t)(0,x) = (u_0, u_1)(x). \tag{44}$$

Let u be a global non-trivial solution to (44). We derive a contradiction. With ρ' denoting the dual number of ρ, i.e. $\frac{1}{\rho} + \frac{1}{\rho'} = 1$, we put

$$I_R := \int_{Q_R} |u|^\rho \cdot (\psi_R)^{\rho'}(t,x)dxdt = \int_{Q_R} (u_{tt} - \Delta u + u_t)\cdot(\psi_R)^{\rho'}dxdt, \tag{45}$$

where $Q_R = [0, R^2] \times B_R(0)$, $B_R(0) = \{|x| \le R\}$ and the test function

$$\begin{aligned}
&\psi_R(t,x) = \eta_R(t)\phi_R(r) = \eta\left(\frac{t}{R^2}\right)\phi\left(\frac{r}{R}\right), \quad r = |x| \\
&0 \le \eta \le 1, \qquad \eta(t) = \begin{cases} 1 & t \in [0, \frac{1}{4}] \\ 0 & t \in [1, \infty) \end{cases}, \qquad |\eta'(t)|, |\eta''(t)| \le C, \\
&\text{with} \quad 0 \le \phi \le 1, \qquad \phi(r) = \begin{cases} 1 & r \in [0, \frac{1}{2}] \\ 0 & r \in [1, \infty) \end{cases}, \qquad |\phi'(r)|, |\phi''(r)| \le C, \\
&(\eta')^2/\eta \le C \quad (0 \le t \le 1), \qquad |\nabla\phi|^2/|\phi| \le C \quad (0 \le r \le 1).
\end{aligned} \tag{46}$$

Then, after integration by parts we obtain, for example,

$$\begin{aligned}
&\int_{Q_R} u_t(\psi_R)^{\rho'} dxdt \\
&= -\int_{B_R} u_0 dx - \int_{\hat{Q}_{R,t}} u\cdot \rho'(\psi_R)^{\rho'-1}\cdot \frac{1}{R^2}\cdot \eta'\left(\frac{t}{R^2}\right)\psi\left(\frac{|x|}{R}\right)dxdt \\
&\le -\int_{B_R} u_0 dx \\
&\quad + \left(\int_{\hat{Q}_{R,t}} |u|^\rho (\psi_R)^{\rho'} dxdt\right)^{1/\rho}\left(\int_{\hat{Q}_{R,t}}\left\{\eta'\left(\frac{t}{R^2}\right)\phi\left(\frac{|x|}{R}\right)\right\}^{\rho'} dxdt\right)^{1/\rho'}\frac{C}{R^2} \\
&\le -\int_{B_R} u_0 dx + C(\hat{I}_{R,t})^{1/\rho} R^{(2+N)(1/\rho')-2}, \quad B_R := B_R(0).
\end{aligned}$$

Here we have used the Hölder inequality with $\rho' = (\rho'-1)\rho$ and

$$\hat{I}_{R,t} := \int_{\hat{Q}_{R,t}} |u|^\rho (\psi_R)^{\rho'} dxdt = \int_{R^2/4}^{R^2}\int_{B_R} |u|^\rho (\psi_R)^{\rho'} dxdt.$$

By similar ways to treat the other terms in I_R we have

$$\begin{aligned}
I_R &\le -\int_{B_R}(u_0+u_1)dx + C(\hat{I}_{R,t}+\hat{I}_{R,|x|})^{1/\rho} R^{(N+2)/\rho'-2} \\
&\le C(I_R)^{1/\rho} R^{(N+2)(1-1/\rho)-2} \quad \text{if } \int_{\mathbf{R}^N}(u_0+u_1)dx > 0,
\end{aligned} \tag{47}$$

where $\hat{I}_{R,|x|} = \int_0^t \int_{B_R\setminus B_{R/2}} |u|^\rho (\psi_R)^{\rho'} dxdt$. Here $\frac{N+2}{\rho'} - 2 < 0$ is equivalent to $\rho < 1 + \frac{2}{N} = \rho_F(N)$. Hence, if $\rho < \rho_F(N)$, then $(I_R)^{1-1/\rho} \le CR^{(N+2)(1-1/\rho)-2}$ and $I_R \to 0$ as $R \to \infty$, which contradicts to the non-triviality of u. If $\rho = \rho_F(N)$, then $I_R \le C$, that is, $\int_0^\infty \int_{\mathbf{R}^N} |u|^\rho dxdt < \infty$ by taking $R = \infty$. Hence (44) becomes

$$I_R \le -\int_{B_R}(u_0+u_1)dx + C(\hat{I}_{R,t}+\hat{I}_{R,|x|})^{1/\rho}.$$

Since both $\hat{I}_{R,t}$ and $\hat{I}_{R,|x|}$ tend to zero as $R \to \infty$, we again reach to the contradiction.

By this observation we know it is a key point that the left hand side of (44) is in divergence form. However, our equation in (P) is not divergent and some idea is necessary. To overcome this, we multiply (P) by some non-negative function $g(t)$ to get

$$\big(g(t)u\big)_{tt} - \Delta\big(g(t)u\big) - \big(g'(t)u\big)_t + \big(-g'(t)+b(t)g(t)\big)u_t = g(t)|u|^\rho.$$

Hence we choose $g(t)$ by the ordinary differential equation

$$-g'(t)+b(t)g(t)=1, \quad g(0)=1/\hat{b}_1, \quad \hat{b}_1=\left(\int_0^\infty e^{-\int_0^t b(s)ds}dt\right)^{-1},$$

that is, explicitly, $g(t)=e^{\int_0^t b(s)ds}(\int_0^\infty e^{-\int_0^\tau b(s)ds}d\tau-\int_0^t e^{-\int_0^\tau b(s)ds}d\tau)$. Thus, we have the divergence form

$$\big(g(t)u\big)_{tt}-\Delta\big(g(t)u\big)-\big(g'(t)u\big)_t+u_t=g(t)|u|^\rho, \tag{48}$$

and, as above, set $I_R=\int_{Q_R} g(t)|u|^\rho\cdot(\psi_R)^{\rho'}dxdt$ with $Q_R=[0,R^{2/(1+\beta)}]\times B_R(0)$ and $\psi_R(t,x)=\eta_R(t)\cdot\phi_R(r)=\eta(\frac{t}{R^{2/(1+\beta)}})\cdot\phi(\frac{|x|}{R})$.

Again, assuming that u is a non-trivial global solution, we can derive the contradiction if $\rho\le\rho_F(N)$. We omit the details here. □

If we consider the same problem for the space-dependent damping

$$\begin{cases} u_{tt}-\Delta u+b(x)u_t=|u|^\rho, & (t,x)\in\mathbf{R}_+\times\mathbf{R}^N\\ (u,u_t)(0,x)=(u_0,u_1)(x), & x\in\mathbf{R}^N\end{cases} \tag{49}$$

with $b(x)=\langle x\rangle^{-\alpha}$ $(0\le\alpha<1)$, $\langle x\rangle=\sqrt{1+|x|^2}$, then the left-hand side is already in divergence form. Therefore, the original test function method is applicable and the critical exponent

$$\rho_c(N,\alpha)=1+\frac{2}{N-\alpha} \tag{50}$$

is completely determined in [11]. Note that, if $\alpha>1$, then the solution has the wave structure ([25]). In the critical case $\alpha=1$ see the recent result [12].

For the problem with both space- and time-dependent damping.

$$\begin{cases} u_{tt}-\Delta u+b(t,x)u_t=|u|^\rho, & (t,x)\in\mathbf{R}_+\times\mathbf{R}^N\\ (u,u_t)(0,x)=(u_0,u_1)(x), & x\in\mathbf{R}^N\end{cases} \tag{51}$$

with $b(t,x)=\langle x\rangle^{-\alpha}(t+1)^{-\beta}$ $(\alpha>0,\ \beta>0,\ 0\le\alpha+\beta<1)$, the test function method with the idea used above does not seem to be applicable and the blow-up result remained open. Several estimates from above for the solution to (51) are obtained in [17, 20, 21, 35]. Therefore, the critical exponent $\rho_c(N,\alpha,\beta)$ is determined if we prove the blow-up result.

To aim to prove the blow-up for (51), as an étude, we consider the space-dependent problem

$$\begin{cases} u_{tt}+u_t-\Delta u+V(x)u=|u|^\rho, & (t,x)\in\mathbf{R}_+\times\mathbf{R}^N\\ (u,u_t)(0,x)=(u_0,u_1)(x), & x\in\mathbf{R}^N.\end{cases} \tag{52}$$

The left-hand side of (52) is not in divergence form and so a similar idea is necessary. Combination of time and space dependent cases may hint to treat (51).

Following Ishige and Kabeya [14], we assume that the potential V has to satisfy

$$\begin{aligned} &\text{(i)} \quad V \text{ is radial} \quad \text{and} \quad V(x) =: V\big(|x|\big) \in C^1\big(\mathbf{R}^N\big), \\ &\text{(ii)} \quad V(r) \geq 0 \text{ on } [0,\infty), \quad r = |x|, \\ &\text{(iii)} \quad \sup_{r\geq 1} r^{2+\theta}\left|V(r) - \frac{\omega}{r^2}\right| < \infty, \quad \theta > 0, \\ &\text{(iv)} \quad \sup_{r\geq 1}\left|r^3\frac{dV}{dr}(r)\right| < \infty. \end{aligned} \tag{V}$$

Similar to (48), we multiply (52) by a suitable function $g(x) \geq 0$ to get

$$\begin{aligned} &(gu)_{tt} + (gu)_t - \Delta(gu) + 2\nabla\cdot(u\nabla g) \\ &\quad + \big(-\Delta g + V(x)g\big)u = g|u|^\rho. \end{aligned} \tag{53}$$

Hence, if it is possible to choose g as

$$-\Delta g + V(x)g = 0, \tag{54}$$

then the left-hand side of (53) becomes the divergence form

$$(gu)_{tt} + (gu)_t - \Delta(gu) + 2\nabla\cdot(u\nabla g) = g|u|^\rho. \tag{55}$$

Fortunately, it is shown in [14] that there exists a unique and positive $g(x) =: g(r)$, $r = |x|$ such that

$$g'' + \frac{N-1}{r}g' - V(r)g = 0, \quad r \in \mathbf{R}^N_+ \ (N \geq 2) \tag{g}$$

with

$$\begin{aligned} &\lim_{r\to 0}\sup\big|g(r)\big| < \infty, \qquad \lim_{r\to\infty} r^{-\alpha(\omega)}g(r) = 1, \\ &\text{and} \quad \lim_{r\to\infty} r^{-\alpha(\omega)-1}\big|g'(r)\big| = \text{const} > 0, \end{aligned} \tag{56}$$

where $\alpha(\omega)$ was defined in (7)–(8). Therefore, there exists $g(r) \geq 0$ satisfying (54) with (56) including the case $N = 1$. Note that (8) is roughly derived as follows. Since $V(r) \sim \omega r^{-2}$, (g) is approximated by

$$r^2 g'' + (N-1)rg' - \omega g = 0,$$

which is a Cauchy differential equation. By the change of variable $g(r) = g(e^s)$,

$$\frac{d^2}{ds^2}g + (N-2)\frac{d}{ds}g - \omega g = 0,$$

whose characteristic equation is (8).

Thus, following the procedure of test function method, we define

$$I_R = \int_{Q_R} g(x)|u|^\rho (\psi_R)^{\rho'}(t,x)dxdt, \quad R \gg 1.$$

The notations such as ψ_R, Q_R etc. are the same as in (46). Hence,

$$\begin{aligned} I_R &= \int_{Q_R} \big[(gu)_{tt} + (gu)_t - \Delta(gu) + 2\nabla\cdot(u\nabla g)\big](\psi_R)^{\rho'}dxdt \\ &=: J_1 + \cdots + J_4. \end{aligned}$$

Then

$$\begin{aligned} J_4 &= -2\int_{\hat{Q}_{R,x}} u\nabla g\cdot\rho'(\psi_R)^{\rho'-1}(\nabla\psi_R)\cdot\frac{1}{R}dxdt \\ &\le C\int_{\hat{Q}_{R,x}} g^{1/\rho}|u|(\psi_R)^{\rho'-1}\cdot g^{1/\rho'}\frac{|\nabla g|}{g}|\nabla\psi_R|\cdot\frac{1}{R}dxdt \\ &\le C\Big(\int_{\hat{Q}_{R,x}} g|u|^\rho(\psi_R)^{\rho'}dxdt\Big)^{1/\rho}\Big(\int_{\hat{Q}_{R,x}} r^{-\rho'}g(r)dxdt\Big)^{1/\rho'}\frac{1}{R} \\ &\le C(\hat{I}_{R,x})^{1/\rho}R^{-2+(\alpha(\omega)+N+2)/\rho'}, \end{aligned}$$

and, similarly,

$$J_3 \le C(\hat{I}_{R,x})^{1/\rho}R^{-2+(\alpha(\omega)+N+2)/\rho'}.$$

Here we have used (56). After integration by parts we conclude

$$\begin{aligned} &J_1 + J_2 \\ &= \int_{Q_R}\frac{\partial}{\partial t}\Big[gu(\psi_R)^{\rho'} + gu_t(\psi_R)^{\rho'} - gu\cdot\rho'(\psi_R)^{\rho'-1}(\psi_{Rt})\cdot\frac{1}{R^2}\Big]dxdt \\ &\quad + \rho'\int_{\hat{Q}_{R,t}} gu\Big[-(\psi_R)^{\rho'-1}(\psi_{Rt})\frac{1}{R^2} \\ &\quad + \big\{(\rho'-1)(\psi_R)^{\rho'-2}(\psi_{Rt})^2 + (\psi_R)^{\rho'-1}(\psi_{Rtt})\big\}\frac{1}{R^4}\Big]dxdt \\ &\le -\int_{B_R(0)} g(x)(u_0+u_1)(x)(\phi_R)^{\rho'}dx \\ &\quad + C\Big(\int_{\hat{Q}_{R,t}} g|u|^\rho(\psi_R)^{\rho'}dxdt\Big)^{1/\rho}\Big(\int_{\hat{Q}_{R,t}} g(r)dxdt\Big)^{1/\rho'}\frac{1}{R^2} \\ &\le -\int_{B_R(0)} g(x)(u_0+u_1)(x)(\phi_R)^{\rho'}dx + C(\hat{I}_{R,t})^{1/\rho}R^{-2+(\alpha(\omega)+N+2)/\rho'}. \end{aligned}$$

Combining all, we obtain

$$I_R \le -\int_{B_R(0)} g(x)(u_0+u_1)(x)(\phi_R)^{\rho'}dx + C(\hat{I}_{R,t}+\hat{I}_{R,x})^{1/\rho} R^{-2+(\alpha(\omega)+N+2)/\rho'}. \tag{57}$$

Here, $-2+\frac{\alpha(\omega)+N+2}{\rho'} \le 0$ is equivalent to $\rho \le 1+\frac{2}{N+\alpha(\omega)} = \rho_c(N,\omega)$. Hence, when $\rho < \rho_c(N,\omega)$, taking $R=\infty$ in (57), we have

$$I_\infty \le -\int_{B_R(0)} g(x)(u_0+u_1)(x)(\phi_R)^{\rho'}dx, \tag{58}$$

which contradicts provided that

$$(u_0+u_1)(x) \ge 0, \quad \int_{\mathbf{R}^N}(u_0+u_1)(x)dx > 0. \tag{59}$$

When $\rho=\rho_c(N,\omega)$ and (59) is assumed, (57) yields $I_R \le C(I_R)^{1/\rho}$ and $I_R \le C$ for any R. Hence

$$\int_0^\infty \int_{\mathbf{R}^N} g(x)|u|^\rho(t,x)dxdt < \infty$$

and both $\hat{I}_{R,t}$ and $\hat{I}_{R,x}$ tends to zero as $R\to\infty$. Taking $R=\infty$ in (57) again, we have (58), which contradicts to (59).

Thus we have obtained the blow-up result for (52) when $\rho \le \rho_c(N,\omega)$. We have just showed our ideas for both time-dependent and space-dependent cases, and we expect that those ideas may be applicable to the blow-up problem for (51).

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Chapter 12
A Note on a Class of Conservative, Well-Posed Linear Control Systems

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Abstract We discuss a class of linear control problems in a Hilbert space setting. The aim is to show that these control problems fit in a particular class of evolutionary equations such that the discussion of well-posedness becomes easily accessible. Furthermore, we study the notion of conservativity. For this purpose we require additional regularity properties of the solution operator in order to allow point-wise evaluations of the solution. We exemplify our findings by a system with unbounded control and observation operators.

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12.1 Introduction

Abstract linear control systems are commonly described by a system of equations of the form

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad y(t) = Cx(t) + Du(t), \quad t \in \mathbb{R}_{>0},$$

with appropriate linear operators A, B, C and D and $\dot{x}$ denoting the time derivative of x in Newton's notation, linking the time development of state x, control u and observation y. The first equation is called *state differential equation* and the second one *observation equation*. The system is formally completed by an initial condition prescribing $x(0+) = x^{(0)}$ for the state trajectory x. As a matter of convenience we will consider this system on the whole real time-line $\mathbb{R}$ in which case the initial data $x^{(0)}$ turns into a Dirac-δ-source at time 0. Writing ∂_0 for time differentiation on the

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full time-line this yields

$$\partial_0 x = Ax + Bu + \delta \otimes x^{(0)}, \qquad y = Cx + Du \quad \text{on } \mathbb{R}.$$

We may formally re-write this into a single block operator matrix equation as

$$\left(\partial_0 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -C & 1 \end{pmatrix} + \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}\right) \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & B \\ 0 & D \end{pmatrix} \begin{pmatrix} \delta \otimes x^{(0)} \\ u \end{pmatrix}, \tag{1}$$

which brings our linear control system into the realm of a problem class discussed in [8, 9]. In a suitable setting ∂_0 can be established as a normal operator with continuous inverse so that for continuous linear operators (A, B, C, D) the solution theory is little more than matrix algebra. If (A, B, C, D) contains unbounded linear operators matters are more complicated. If only A is unbounded but such that $\partial_0 + A$ is invertible the solution theory can be largely salvaged. A common instrument here is to express $(\partial_0 + A)^{-1}$ in terms of a semi-group generated by A. Matters become exceedingly complicated if also other operators in the list (A, B, C, D) are also permitted to be unbounded (see [4, 6, 7] for a survey, also [14]). The answer of questions concerning for example well-posedness along this line of reasoning may be quite involved. The classical approach to well-posedness is the concept of so-called admissible control and observation operators, using the theory of strongly continuous semigroups, see for instance [1, 2, 11–13, 15] and [3] for a survey.

Here we want to give a more elementary approach to this issue, by changing the perspective to the above type of space-time operators, which in effect by-passes C_0-semi-groups as a solution tool and at the same time enlarges the class of accessible control problems considerably. On the other hand, we use elementary C_0-semigroup theory as a tool for discussing regularity issues.

We shall consider systems of the general form

$$(\partial_0 M_0 + M_1 + \mathcal{A}) \begin{pmatrix} x \\ y \end{pmatrix} = J \begin{pmatrix} f \\ u \end{pmatrix},$$

$M_0 : X \oplus Y \to X \oplus Y$, $M_1 : X \oplus Y \to X \oplus Y$ continuous linear operators, $\mathcal{A} : D(\mathcal{A}) \subseteq X \oplus Y \to X \oplus Y$ a closed and densely defined operator. Mostly we shall assume that $J : F \oplus U \mapsto X \oplus Y$ is such that

$$J = \begin{pmatrix} E & B \\ 0 & D \end{pmatrix}$$

with $B : U \to X$, $D : U \to Y$, $E : F \to X$ continuous linear operators. Here X, Y, F, U are Hilbert spaces referred to as state, observation, data and control spaces, respectively.

There is little harm in assuming $X = F$ and $U = Y$ and we shall do so.

As the space to model time-dependence we consider the weighted L^2-space $H_{\varrho,0}(\mathbb{R})$, $\varrho \in \mathbb{R}_{>0}$, generated by the completion of $\mathring{C}_\infty(\mathbb{R})$ with respect to the inner product $\langle\cdot|\cdot\rangle_{\varrho,0}$

$$(\varphi, \psi) \mapsto \int_{\mathbb{R}} \varphi(t)^* \psi(t) \exp(-2\varrho t) dt.$$

The associated norm will be denoted by $|\cdot|_{\varrho,0}$. The time-derivative ∂_0 can be lifted canonically to corresponding Hilbert-space-valued generalized functions making ∂_0 a normal operator in the resulting Hilbert space $H_{\varrho,0}(\mathbb{R}, H)$, where H is an arbitrary Hilbert space. Thus the linear control system under consideration is a quaternary relation of the form

$$\mathcal{C}_{M_0,M_1,\mathcal{A},J} = \left\{ (x, y, f, u) \middle| (\partial_0 M_0 + M_1 + \mathcal{A}) \begin{pmatrix} x \\ y \end{pmatrix} = J \begin{pmatrix} f \\ u \end{pmatrix} \right\}$$

in spaces derived from this consideration. We say $\mathcal{C}_{M_0,M_1,\mathcal{A},J}$ is *well-posed*, if $\mathcal{C}_{M_0,M_1,\mathcal{A},J}$ considered as the associated binary relation

$$\left\{ \big((x, y), (f, u)\big) \middle| (\partial_0 M_0 + M_1 + \mathcal{A}) \begin{pmatrix} x \\ y \end{pmatrix} = J \begin{pmatrix} f \\ u \end{pmatrix} \right\}$$

induces for all sufficiently large $\varrho \in \mathbb{R}_{>0}$ a continuous linear mapping in a suitable Hilbert space setting linking a solution (x, y) with any given (f, u). Of course we would want the solution operator $(\partial_0 M_0 + M_1 + \mathcal{A})^{-1} J$ also to be causal in the intuitive sense. If there is no danger of confusion and the coefficient operators M_0, M_1, $\mathcal{A}$, J are clear from the context, we simply write $\mathcal{C}$ for $\mathcal{C}_{M_0,M_1,\mathcal{A},J}$.

Another extract of our current linear control system $\mathcal{C}$ is frequently of particular interest. It is the so-called transfer relation T_f which is given for a fixed f by

$$T_f := \left\{ (u, y) \middle| \bigvee_x (\partial_0 M_0 + M_1 + \mathcal{A}) \begin{pmatrix} x \\ y \end{pmatrix} = J \begin{pmatrix} f \\ u \end{pmatrix} \right\}.$$

For a well-posed linear control system this is just reading off the second block component of the solution and yields that T_f is a mapping, the transfer mapping. Frequently, one prefers to consider the unitarily equivalent operator

$$\mathcal{L}_\varrho T_f \mathcal{L}_\varrho^*,$$

where $\mathcal{L}_\varrho$ is the unitary Fourier-Laplace transformation (see Sect. 12.2), as the transfer mapping.

We also address a question approached in [15], namely conservativity of a linear control system. In [15] this notion was defined by means of a certain energy balance equality, that should be fulfilled by state, observation and control.

By considering abstract control system in the above sense we shall show that for reasonable state differential equations it is always possible to construct an observation equation, which leads to a conservative linear control system. Moreover, although in [15] unbounded control and observation operators were considered, we shall see that in the generalized form such system are reduced to the bounded operator case (with $\mathcal{A}$ being the only unbounded linear operator involved).

12.2 Setting

The particular time-derivative defined as a normal and invertible operator in the exponentially weighted space $H_{\varrho,0}(\mathbb{R}) := L_2(\mathbb{R}, \exp(-2\varrho x)\mathrm{d}x)$ (for some $\varrho \in \mathbb{R}_{>0}$) is given in various articles of the authors of this paper. The core issues are discussed in [5]. We state the basic facts as follows. Let $\varrho \in \mathbb{R}_{>0}$. We define ∂_0 as the closure of the operator $\mathring{C}_\infty(\mathbb{R}) \subseteq H_{\varrho,0}(\mathbb{R}) \to H_{\varrho,0}(\mathbb{R}) : f \mapsto f'$, where $\mathring{C}_\infty(\mathbb{R})$ denotes the space of infinitely often differentiable functions with compact support. It can be shown that $\partial_0^{-1} \in L(H_{\varrho,0}(\mathbb{R}), H_{\varrho,0}(\mathbb{R}))$ and $\|\partial_0^{-1}\| \le 1/\varrho$.

It is well-known that there is an explicit spectral representation as a multiplication operator of the one-dimensional derivative on the real line, which is given by the unitary *Fourier transformation* $\mathcal{F} : L_2(\mathbb{R}) \to L_2(\mathbb{R})$. An analogous representation can be found for ∂_0: Denote by m the multiplication-by-argument-operator in $L_2(\mathbb{R})$ with natural domain and $\exp(-\varrho m) : H_{\varrho,0}(\mathbb{R}) \to L_2(\mathbb{R}) : f \mapsto \exp(-\varrho(\cdot))f(\cdot)$. Then we have the following unitary representation of ∂_0:

$$\mathcal{L}_\varrho^*(im + \varrho)\mathcal{L}_\varrho = \partial_0$$

with the unitary *Fourier-Laplace transformation* $\mathcal{L}_\varrho := \mathcal{F}\exp(-\varrho m)$. This formula can canonically be lifted to the Hilbert-space-valued case. Moreover, the latter unitary representation results in a functional calculus for ∂_0^{-1}. More precisely, let $r > \frac{1}{2\varrho}$ and H be a Hilbert space. Let $M : B(r,r) \to L(H)$ be an element of the Hardy space $\mathcal{H}^\infty(B(r,r), L(H))$ of bounded and analytic functions defined on the open ball $B(r,r) \subseteq \mathbb{C}$ with values in $L(H)$, the set of continuous linear operators within H. Define

$$M\big(\partial_0^{-1}\big) := \mathcal{L}_\varrho^* M\left(\frac{1}{im+\varrho}\right)\mathcal{L}_\varrho,$$

where $M(\frac{1}{im+\varrho})\phi(t) := M(\frac{1}{it+\varrho})\phi(t)$ for all $\phi \in \mathring{C}_\infty(\mathbb{R}, H)$ and $t \in \mathbb{R}$. It is easy to see that $M(\partial_0^{-1}) \in L(H_{\varrho,0}(\mathbb{R}, H))$ and $\partial_0^{-1}M(\partial_0^{-1}) = M(\partial_0^{-1})\partial_0^{-1}$. As it was already mentioned in [5], for $h > 0$ the time-shift τ_{-h} defined as $\tau_{-h} f := f(\cdot - h)$ or the convolution with a $L_1(\mathbb{R})$-function supported on the positive reals yield analytic and bounded functions of ∂_0^{-1} in the above sense.

In the following we shall also make use of the concept of Sobolev lattices, which are related to abstract distribution spaces associated with particular (unbounded) operators in a Hilbert space. The whole set-up is described in [10]. We sketch it as follows. Let C, D be densely defined, closed, linear operators in a Hilbert space H. Furthermore, assume that $0 \in \varrho(C) \cap \varrho(D)$ and $C^{-1}D^{-1} = D^{-1}C^{-1}$. For $k, n \in \mathbb{Z}$ the Hilbert space $H_{k,n}(C,D)$ is defined as the completion of $D(C^{|k|}) \cap D(D^{|n|})$ with respect to the (well-defined) inner product $(\phi, \psi) \mapsto \langle C^k D^n \phi, C^k D^n \psi\rangle$. The family $(H_{k,n}(C,D))_{(k,n)\in\mathbb{Z}^2}$ is called *Sobolev lattice associated with* (C, D). One can show that for $k_1, n_1 \in \mathbb{Z}$ with $k_1 \le k$ and $n_1 \le n$ we have dense and continuous embeddings

$$H_{k,n}(C,D) \hookrightarrow H_{k_1,n_1}(C,D).$$

The latter relation justifies the term “lattice”. Indeed, $(H_{k,n}(C, D))_{(k,n)\in\mathbb{Z}^2}$ is a lattice with respect to the order relation $\hookrightarrow$, which is isomorphic to $\mathbb{Z}^2$ endowed with component-wise order.

Moreover, by continuous extension, we have unitary operators

$$C^{k_2} D^{n_2} : H_{k,n}(C, D) \to H_{k-k_2,n-n_2}(C, D)$$

for all $n_2, k_2 \in \mathbb{Z}$. It should be mentioned that any continuous linear operator $B : H \to H$, which commutes with $C^{-1} \in L(H)$, has a unique continuous extension (restriction) to $H_{k,0}(C, D)$. We shall use the construction of Sobolev lattices in the aforementioned situation of linear control systems. For the special case that D is the identity on H, we will write $H_k(C) := H_{k,n}(C, D)$. Moreover, given a densely defined, closed linear operator $\mathcal{A} : D(\mathcal{A}) \subseteq \mathcal{H} \to \mathcal{H}$ with non-empty resolvent set $\varrho(\mathcal{A})$. Then, for $\varrho \in \mathbb{R}_{>0}$ and $\lambda \in \varrho(\mathcal{A})$, we define

$$H_{\varrho,k}\big(\mathbb{R}, \mathcal{H}_n(\mathcal{A} - \lambda)\big) := H_{k,n}(\partial_0, \mathcal{A} - \lambda).$$

If it is clear from the context, which operator $\mathcal{A}$ is under consideration, we shall also write $H_{\varrho,k}(\mathbb{R}, \mathcal{H}_n)$ for short. Clearly, the latter set does not depend on the particular choice of $\lambda \in \varrho(\mathcal{A})$. As another short-hand notation we also define

$$H_{\varrho,\infty}(\mathbb{R}, \mathcal{H}_n) := \bigcap_{k\in\mathbb{N}} H_{\varrho,k}(\mathbb{R}, \mathcal{H}_n).$$

12.3 Solution Theory for Abstract Linear Control Systems

We summarize the core issues of the solution theory used in this paper. In the whole section, we make the following assumptions. Let X and Y be Hilbert spaces and define $\mathcal{H} := X \oplus Y$. Moreover, let $M_0 : \mathcal{H} \to \mathcal{H}$, $M_1 : \mathcal{H} \to \mathcal{H}$, $J : \mathcal{H} \to \mathcal{H}$ be continuous linear operators and let $\mathcal{A} : D(\mathcal{A}) \subseteq \mathcal{H} \to \mathcal{H}$ be a closed linear operator. We assume that

- M_0 is selfadjoint, non-negative and strictly positive on its range, whereas
- $\mathfrak{Re}\, M_1 : \mathcal{H} \to \mathcal{H}$ is strictly positive on the null space of M_0.

To simplify matters, we shall also assume that

- $\mathcal{A}$ is skew-selfadjoint in $\mathcal{H}$, which is a standard case for most problems.

We will use the extension of these operators to the Hilbert space of $\mathcal{H}$-valued $H_{\varrho,0}(\mathbb{R})$ functions. From the 3 aforementioned properties, it is easy to see that the following lemma holds. For a set $S \subseteq \mathbb{R}$, we denote by $\chi_S(m_0)$ the truncation operator, mapping a function $f : \mathbb{R} \to \mathcal{H}$ to the truncated one: $\chi_S(m_0) f := (t \mapsto \chi_S(t) f(t))$.

Lemma 1 *There is a constant $\beta_0 \in \mathbb{R}_{>0}$ such that for all $\xi \in D(\mathcal{A}) \cap D(\partial_0)$ and all sufficiently large $\varrho \in \mathbb{R}_{>0}$*

$$\mathfrak{Re}\big\langle \chi_{\mathbb{R}_{<0}}(m_0)\xi \,|\, (\partial_0 M_0 + M_1 + \mathcal{A})\xi \big\rangle_{\varrho,0,0} \geq \beta_0 \big\langle \chi_{\mathbb{R}_{<0}}(m_0)\xi \,|\, \xi \big\rangle_{\varrho,0,0}. \tag{++a}$$

It follows

$$\mathfrak{Re}\big\langle \xi \,|\, (\partial_0 M_0 + M_1 + \mathcal{A})\xi \big\rangle_{\varrho,0,0} \geq \beta_0 \langle \xi \,|\, \xi \rangle_{\varrho,0,0}. \tag{++b}$$

The proof can be found in Chap. 7 in [10]. It is remarkable that the core of the proof of the solution theory only relies on the positive definiteness as stated in Lemma 1 and the explicit spectral representation of ∂_0.

Theorem 1 *For every sufficiently large $\varrho \in \mathbb{R}_{>0}$ and every $\left(\begin{smallmatrix} f \\ u \end{smallmatrix}\right) \in H_{\varrho,0}(\mathbb{R}, X \oplus Y)$ there is a unique solution $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \in H_{\varrho,0}(\mathbb{R}, X \oplus Y)$ of the problem*

$$\overline{(\partial_0 M_0 + M_1 + \mathcal{A})} \begin{pmatrix} x \\ y \end{pmatrix} = J \begin{pmatrix} f \\ u \end{pmatrix}.$$

Moreover, the solution depends continuously on the data in $H_{\varrho,0}(\mathbb{R}, X \oplus Y)$ and the solution operator $(\partial_0 M_0 + M_1 + \mathcal{A})^{-1} J$ is causal in the sense that

$$\begin{aligned} &\chi_{\mathbb{R}_{<a}}(m_0)\overline{(\partial_0 M_0 + M_1 + \mathcal{A})^{-1}} J \\ &\quad = \chi_{\mathbb{R}_{<a}}(m_0)\overline{(\partial_0 M_0 + M_1 + \mathcal{A})^{-1}} J \chi_{\mathbb{R}_{<a}}(m_0) \end{aligned}$$

for all $a \in \mathbb{R}$.

Remark 1 The assumptions on the operators $\mathcal{A}$, M_1 and M_0 are sharp in the sense that we can easily construct ill-posed systems, if one of the assumptions fails. For instance consider the system

$$\left(\partial_0 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 0 & -C^* \\ C & 0 \end{pmatrix}\right) \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix},$$

where C is an unbounded, closed and densely defined linear operator. Now $\mathfrak{Re}\, M_1 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ is not strictly positive definite on the kernel of $M_0 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. Substituting the second equation $v = Cu$ into the first yields

$$\big(\partial_0 - C^* C\big) u = f,$$

which is an abstract heat equation with time reversed and well-known to be ill-posed as a forward causal equation. Even in the ode case, i.e. for $C = 0$, taking now $M_1 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and considering the resulting system

$$\left(\partial_0 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}$$

would yield

$$u = g,$$
$$\partial_0 u + v = f,$$

which can only have a solution $u, v \in H_{\varrho,0}(\mathbb{R}, H)$ if $g = u = \partial_0^{-1}(f - v) \in H_{\varrho,1}(\mathbb{R}, H)$ and not for general data $f, g \in H_{\varrho,0}(\mathbb{R}, H)$.

Using the Sobolev lattice $(H_{\varrho,k}(\mathbb{R}, \mathcal{H}_n))_{(k,n)\in\mathbb{Z}^2}$, we shall extend the operators ∂_0, M_0, M_1, $\mathcal{A}$ to $H_{\varrho,-\infty}(\mathbb{R}, \mathcal{H}_{-1}) := \bigcup_{k\in\mathbb{Z}} H_{\varrho,k}(\mathbb{R}, \mathcal{H}_{-1})$. This has the effect that we do not need to write the closure bar anymore. However, this has the consequence that, whereas the equation

$$(\partial_0 M_0 + M_1 + \mathcal{A}) \begin{pmatrix} x \\ y \end{pmatrix} = J \begin{pmatrix} f \\ u \end{pmatrix}$$

holds in $H_{\varrho,0}(\mathbb{R}, X \oplus Y)$, the equation

$$\partial_0 M_0 \begin{pmatrix} x \\ y \end{pmatrix} + M_1 \begin{pmatrix} x \\ y \end{pmatrix} + \mathcal{A} \begin{pmatrix} x \\ y \end{pmatrix} = J \begin{pmatrix} f \\ u \end{pmatrix}$$

only holds in $H_{\varrho,-1}(\mathbb{R}, \mathcal{H}_{-1})$. This line of reasoning also yields that $\begin{pmatrix} x \\ y \end{pmatrix} \in H_{\varrho,-1}(\mathbb{R}, \mathcal{H}_1)$. We will use this observation in the forthcoming sections. To incorporate non-vanishing initial data we record the following corollary, where we use the continuous extension of the solution operator—a particular bounded and analytic function of ∂_0^{-1} (cf. Sect. 12.2)—to the space $H_{\varrho,-1}(\mathbb{R}, \mathcal{H})$.

Corollary 1 *For every sufficiently large $\varrho \in \mathbb{R}_{>0}$ and every $\begin{pmatrix} f \\ u \end{pmatrix} \in H_{\varrho,0}(\mathbb{R}, X \oplus Y)$ and $\begin{pmatrix} x^{(0)} \\ y^{(0)} \end{pmatrix} \in M_0[X \oplus Y]$ there is a unique solution $\begin{pmatrix} x \\ y \end{pmatrix} \in H_{\varrho,-1}(\mathbb{R}, X \oplus Y)$ of the problem*

$$(\partial_0 M_0 + M_1 + \mathcal{A}) \begin{pmatrix} x \\ y \end{pmatrix} = J \begin{pmatrix} f \\ u \end{pmatrix} + \delta \otimes M_0 \begin{pmatrix} x^{(0)} \\ y^{(0)} \end{pmatrix}. \tag{2}$$

The solution depends continuously on the data in $H_{\varrho,-1}(\mathbb{R}, X \oplus Y)$.

Proof The existence result follows by applying the previous theorem to

$$(\partial_0 M_0 + M_1 + \mathcal{A}) \begin{pmatrix} \xi \\ \eta \end{pmatrix} = J \begin{pmatrix} \partial_0^{-1} f \\ \partial_0^{-1} u \end{pmatrix} + \chi_{\mathbb{R}_{>0}} \otimes M_0 \begin{pmatrix} x^{(0)} \\ y^{(0)} \end{pmatrix}$$

and then differentiating and letting

$$\begin{pmatrix} x \\ y \end{pmatrix} := \partial_0 \begin{pmatrix} \xi \\ \eta \end{pmatrix}.$$

The uniqueness and continuous dependence part follows conversely by applying ∂_0^{-1} to (2) and using the uniqueness and continuous dependence result of Theorem 1. □

12.4 Regularity

In this section we discuss regularity issues. The method is based on "see-saw"-type arguments and relies on the Sobolev lattice associated with $(\partial_0, \mathcal{A}+1)$, i.e.,

$$\big(H_{\varrho,s}(\mathbb{R}, \mathcal{H}_k)\big)_{(s,k)\in\mathbb{Z}^2}.$$

Our main focus will be initial value problems. We need the following definition.

Definition 1 Let $\mathcal{C}_{M_0,M_1,\mathcal{A},J}$ be a well-posed[1] linear control system. If

$$P_0\big((\partial_0 M_0 + M_1 + \mathcal{A})^{-1}\delta \otimes M_0 - \chi_{\mathbb{R}_{>0}} \otimes P_0\big)[\mathcal{U}] \subseteq H_{\varrho,1}(\mathbb{R}, \mathcal{H})$$

for some subspace $\mathcal{U} \subseteq D(\mathcal{A})$, which is dense in $\mathcal{H}$, then we call $\mathcal{C}_{M_0,M_1,\mathcal{A},J}$ a *globally regularizing* linear control system. If for all $T \in \mathbb{R}$ we have

$$\begin{aligned}
&\chi_{\mathbb{R}_{<T}}(m_0) P_0\big((\partial_0 M_0 + M_1 + \mathcal{A})^{-1}\delta \otimes M_0 - \chi_{\mathbb{R}_{>0}} \otimes P_0\big)[\mathcal{U}]\\
&\quad \subseteq \chi_{\mathbb{R}_{<T}}(m_0)\big[H_{\varrho,1}(\mathbb{R}, \mathcal{H})\big]
\end{aligned}$$

we call $\mathcal{C}_{M_0,M_1,\mathcal{A},J}$ a *locally regularizing* linear control system. Here $P_0 := \pi_0^* \pi_0$, where π_0 denotes the orthogonal projector onto $M_0[\mathcal{H}]$.

Obviously, the regularizing property is independent of J. For locally regularizing linear control systems we have according to the Sobolev embedding property (cf. Lemma 3.1.59 in [10]) point-wise evaluation as a continuous operation and we can define, what it means for such a system to be conservative. In the forthcoming sections, we deal with a system studied in [15]. This system may be rewritten into a first order system such that the above theory becomes applicable. Moreover, it can be shown that the respective system is a special case of the system occurring in the next theorem, for which the notion of conservativity can be established.

Theorem 2 *Let $\mathcal{C}_{M_0,M_1,\mathcal{A},J}$ be a linear control system with*

$$M_0 = \begin{pmatrix} M_{00} & 0 \\ 0 & 0 \end{pmatrix}, \qquad M_1 = \begin{pmatrix} M_{11} & 0 \\ \alpha R^{-1}\pi_1 & \alpha \end{pmatrix},$$
$$\mathcal{A} = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}, \qquad J = \begin{pmatrix} 0 & 2\,\mathfrak{Re}(M_{11})\pi_1^* R \\ 0 & \alpha \end{pmatrix},$$

[1] In this case

$$\begin{aligned}
\big((\partial_0 M_0 + M_1 + \mathcal{A})^{-1}\delta \otimes M_0\big) : \mathcal{H} &\to H_{\varrho,-1}(\mathbb{R}, \mathcal{H}).\\
z &\mapsto (\partial_0 M_0 + M_1 + \mathcal{A})^{-1}\delta \otimes M_0 z
\end{aligned}$$

is a continuous linear operator.

where $M_{00} \in L(X)$ *is selfadjoint and strictly positive definite on* $M_{00}[X]$, $M_{11} \in L(X)$ *with* $(\mathfrak{Re}\, M_{11}) \geq 0$ *and* $\mathfrak{Re}\, M_{11}$ *is strictly positive definite on* $[\{0\}]M_{00}$, $U_1 := (\mathfrak{Re}\, M_{11})[X]$, $R : U_1 \to U_1$ *is a continuous linear bijection,* $\pi_1 : X \to U_1$ *is the orthogonal projector, A is a skew-selfadjoint operator on X and* $\alpha \in \mathbb{R} \setminus \{0\}$, *is such that*

$$4\left\| \left(\sqrt{(\mathfrak{Re}\, M_{11})|_{[\{0\}]M_{00}}}\right)^{-1} \right\|^{-2} \left\| R^{-1} \right\|^{-2} > \alpha > 0.$$

Then $\mathcal{C}_{M_0,M_1,\mathcal{A},J}$ *is well-posed. Let* $U_0 := M_{00}[X]$ *and* $\pi_0 : X \to U_0$, $P_0 := \pi_0^* \pi_0$ *the corresponding orthogonal projections. Assume in addition that* $\mathcal{C}_{M_0,M_1,\mathcal{A},J}$ *is locally regularizing. Then* $\mathcal{C}_{M_0,M_1,\mathcal{A},J}$ *is* conservative *in the sense of* [15], *i.e., the solution* $\binom{x}{y}$ *of*

$$(\partial_0 M_0 + M_1 + \mathcal{A}) \begin{pmatrix} x \\ y \end{pmatrix} = J \begin{pmatrix} 0 \\ u \end{pmatrix} + \delta \otimes \begin{pmatrix} M_{00} x^{(0)} \\ 0 \end{pmatrix}$$

for the initial data $x^{(0)}$ *and control u gives rise to mappings*

$$\Sigma_T : \begin{pmatrix} \sqrt{M_{00}} & 0 \\ 0 & \sqrt{2\, \mathfrak{Re}\, M_{11}} R \end{pmatrix} \begin{pmatrix} P_0 x^{(0)} \\ \chi_{\mathbb{R}_{<T}}(m_0) u \end{pmatrix}$$
$$\mapsto \begin{pmatrix} \sqrt{M_{00}} & 0 \\ 0 & \sqrt{2\, \mathfrak{Re}\, M_{11}} R \end{pmatrix} \begin{pmatrix} P_0 x(T) \\ \chi_{\mathbb{R}_{<T}}(m_0) y \end{pmatrix},$$

which are densely defined isometries on $U_0 \oplus L^2(\mathbb{R}_{>0}, U_1)$ *for all* $T \in \mathbb{R}_{\geq 0}$.

Remark 2 In the setting of the theorem above, the state space is given by $\mathcal{H} = X \oplus U_1$. Furthermore we shall note here that for the definition of conservativity the parameter $\alpha \in \mathbb{R} \setminus \{0\}$ is irrelevant. However, it is used to adjust for the assumptions of our above solution theory.

Proof of Theorem 2 At first we show well-posedness of $\mathcal{C}_{M_0,M_1,\mathcal{A},J}$. We need to consider the positive definiteness of

$$\mathfrak{Re}\, M_1 = \begin{pmatrix} \mathfrak{Re}\, M_{11} & \frac{1}{2}\alpha \pi_1^* (R^{-1})^* \\ \frac{1}{2}\alpha R^{-1} \pi_1 & \alpha \end{pmatrix}$$

on $[\{0\}]M_0 = [\{0\}]M_{00} \oplus U_1$. Let $z \oplus y \in [\{0\}]M_{00} \oplus U_1$. For $\varepsilon > 0$, we compute

$$\begin{aligned} \left\langle \mathfrak{Re}\, M_1 (z \oplus y) | z \oplus y \right\rangle &= \langle z | \mathfrak{Re}\, M_{11} z \rangle + \left\langle z | \alpha \pi_1^* \left(R^{-1}\right)^* y \right\rangle + \alpha \langle y | y \rangle \\ &\geq \langle \sqrt{\mathfrak{Re}\, M_{11}} z | \sqrt{\mathfrak{Re}\, M_{11}} z \rangle - \frac{1}{2\varepsilon} |z|^2 \\ &\quad - \frac{\varepsilon}{2} \alpha^2 \left| \pi_1^* \left(R^{-1}\right)^* y \right|^2 + \alpha \langle y | y \rangle \end{aligned}$$

$$\geq \left(1-\frac{1}{2\varepsilon}\left\|\left(\sqrt{(\mathfrak{Re}\,M_{11})|_{[\{0\}]M_{00}}}\right)^{-1}\right\|^{2}\right)|\sqrt{\mathfrak{Re}\,M_{11}}z|^{2}$$
$$+\alpha\left(1-\frac{\varepsilon}{2}\alpha\left\|R^{-1}\right\|^{2}\right)|y|^{2}.$$

From the first term of the right-hand side of the latter inequality it follows that ε has to be chosen such that

$$\frac{1}{\varepsilon}<2\left\|\left(\sqrt{(\mathfrak{Re}\,M_{11})|_{[\{0\}]M_{00}}}\right)^{-1}\right\|^{-2}$$

holds. From the second term, we read off that

$$1-\frac{\varepsilon}{2}\alpha\left\|R^{-1}\right\|^{2}>0$$

should hold. Thus, we want ε to satisfy in addition

$$\frac{1}{\varepsilon}>\frac{\alpha}{2}\left\|R^{-1}\right\|^{2}.$$

The condition on α ensures that the interval

$$\left]\frac{\alpha}{2}\left\|R^{-1}\right\|^{2},2\left\|\left(\sqrt{(\mathfrak{Re}\,M_{11})|_{[\{0\}]M_{00}}}\right)^{-1}\right\|^{-2}\right[$$

is not empty. Employing Theorem 1, we conclude that the abstract linear control system $\mathcal{C}$ is well-posed. Assume now that $\mathcal{C}$ is locally regularizing. Due to the block structure of the operator matrices M_0, M_1, $\mathcal{A}$ and J there exists a subspace $\mathcal{U}\subseteq D(A)$, dense in X, such that for $x^{(0)}\in\mathcal{U}$, we have

$$\chi_{\mathbb{R}_{<T}}(m_0)\begin{pmatrix}P_0&0\\0&0\end{pmatrix}\left((\partial_0M_0+M_1+\mathcal{A})^{-1}\delta\otimes M_0-\chi_{\mathbb{R}_{>0}}\otimes\begin{pmatrix}P_0&0\\0&0\end{pmatrix}\right)\begin{pmatrix}x^{(0)}\\0\end{pmatrix}$$
$$\in\chi_{\mathbb{R}_{<T}}(m_0)\left[H_{\varrho,1}(\mathbb{R},\mathcal{H})\right]$$

for all $T\in\mathbb{R}$. Let $x^{(0)}\in\mathcal{U}$ and $u\in H_{\varrho,1}(\mathbb{R}_{\geq0},U_1)$. Our general solution theory yields the unique existence of $(x,y)\in H_{\varrho,-1}(\mathbb{R},X\oplus U_1)$ of the problem

$$\left(\partial_0\begin{pmatrix}M_{00}&0\\0&0\end{pmatrix}+\begin{pmatrix}M_{11}&0\\\alpha R^{-1}\pi_1&\alpha\end{pmatrix}+\begin{pmatrix}A&0\\0&0\end{pmatrix}\right)\begin{pmatrix}x\\y\end{pmatrix}$$
$$=\begin{pmatrix}0&2\,\mathfrak{Re}(M_{11})\pi_1^{*}R\\0&\alpha\end{pmatrix}\begin{pmatrix}0\\u\end{pmatrix}+\begin{pmatrix}\delta\otimes M_{00}x^{(0)}\\0\end{pmatrix},$$

where $\operatorname{supp}x\subseteq\mathbb{R}_{\geq0}$ and $\operatorname{supp}y\subseteq\mathbb{R}_{\geq0}$ due to the causality of the solution operator. This leads to

$$\begin{aligned}
&\begin{pmatrix} P_0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} - \chi_{\mathbb{R}_{>0}} \otimes \begin{pmatrix} P_0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x^{(0)} \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} P_0 & 0 \\ 0 & 0 \end{pmatrix} (\partial_0 M_0 + M_1 + \mathcal{A})^{-1} \begin{pmatrix} 0 & 2\,\mathfrak{Re}(M_{11})\pi_1^* R \\ 0 & \alpha \end{pmatrix}\begin{pmatrix} 0 \\ u \end{pmatrix} \\
&\quad + \begin{pmatrix} P_0 & 0 \\ 0 & 0 \end{pmatrix} (\partial_0 M_0 + M_1 + \mathcal{A})^{-1} \begin{pmatrix} \delta \otimes M_{00} x^{(0)} \\ 0 \end{pmatrix} \\
&\quad - \chi_{\mathbb{R}_{>0}} \otimes \begin{pmatrix} P_0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x^{(0)} \\ 0 \end{pmatrix}.
\end{aligned}$$

Since $(\partial_0 M_0 + M_1 + \mathcal{A})^{-1}$ leaves $H_{\varrho,1}(\mathbb{R}, \mathcal{H})$ invariant, we read off that

$$\begin{aligned}
&\chi_{\mathbb{R}_{<T}}(m_0)\left(\begin{pmatrix} P_0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} - \chi_{\mathbb{R}_{>0}} \otimes \begin{pmatrix} P_0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x^{(0)} \\ 0 \end{pmatrix}\right) \\
&\quad \in \chi_{\mathbb{R}_{<T}}(m_0)\big[H_{\varrho,1}(\mathbb{R}, \mathcal{H})\big]
\end{aligned}$$

holds for all $T \in \mathbb{R}$. We fix $T \in \mathbb{R}_{>0}$ for the rest of the proof. Let $\varphi \in C_\infty(\mathbb{R})$ be such that $\varphi = 1$ on $\mathbb{R}_{<T+1}$ and $\varphi = 0$ on $\mathbb{R}_{>T+2}$. Using the Sobolev lattice associated with $(\partial_0, \mathcal{A} + 1)$ we get that

$$\begin{aligned}
\begin{pmatrix} \varphi(m_0) & 0 \\ 0 & 1 \end{pmatrix} (\partial_0 M_0 + M_1 + \mathcal{A}) \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} \varphi(m_0) & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 2\,\mathfrak{Re}(M_{11})\pi_1^* R \\ 0 & \alpha \end{pmatrix}\begin{pmatrix} 0 \\ u \end{pmatrix} \\
&\quad + \begin{pmatrix} \varphi(0)\delta \otimes M_{00} x^{(0)} \\ 0 \end{pmatrix},
\end{aligned}$$

which implies

$$\begin{aligned}
&(\partial_0 M_0 + M_1 + \mathcal{A}) \begin{pmatrix} \varphi(m_0) & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \qquad (3) \\
&= \begin{pmatrix} \varphi(m_0) & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 2\,\mathfrak{Re}(M_{11})\pi_1^* R \\ 0 & \alpha \end{pmatrix}\begin{pmatrix} 0 \\ u \end{pmatrix} \\
&\quad + \begin{pmatrix} \varphi(0)\delta \otimes M_{00} x^{(0)} \\ 0 \end{pmatrix} + \begin{pmatrix} M_{00}\varphi'(m_0) & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \\
&= \begin{pmatrix} \varphi(m_0) 2\,\mathfrak{Re}(M_{11})\pi_1^* R u \\ \alpha u \end{pmatrix} + \begin{pmatrix} \delta \otimes M_{00} x^{(0)} \\ 0 \end{pmatrix} \\
&\quad + \begin{pmatrix} M_{00}\varphi'(m_0) x \\ 0 \end{pmatrix}. \qquad (4)
\end{aligned}$$

Define $x_\varphi := \varphi(m_0)x$. Employing the local regularizing property and using that $M_{00}\varphi'(m_0)x \in H_{\varrho,1}(\mathbb{R}, X)$, we deduce that $P_0 x_\varphi - \chi_{\mathbb{R}_{>0}} \otimes P_0 x^{(0)} \in H_{\varrho,1}(\mathbb{R}, X)$.

Moreover, from

$$\begin{aligned}(\partial_0 M_0 + M_1 + \mathcal{A})\begin{pmatrix} x_\varphi \\ y \end{pmatrix} &= \left(\partial_0 M_0 \begin{pmatrix} x_\varphi \\ y \end{pmatrix} + M_1 \begin{pmatrix} x_\varphi \\ y \end{pmatrix} + \mathcal{A}\begin{pmatrix} x_\varphi \\ y \end{pmatrix}\right) \\ &= \begin{pmatrix} \varphi(m_0) 2\,\mathfrak{Re}(M_{11})\pi_1^* R u \\ \alpha u \end{pmatrix} + \begin{pmatrix} \delta \otimes M_{00} x^{(0)} \\ 0 \end{pmatrix} + \begin{pmatrix} M_{00}\varphi'(m_0) x \\ 0 \end{pmatrix}\end{aligned}$$

it follows that

$$\begin{aligned}&\partial_0 M_0 \left(\begin{pmatrix} x_\varphi \\ y \end{pmatrix} - \chi_{\mathbb{R}_{>0}} \otimes \begin{pmatrix} x^{(0)} \\ 0 \end{pmatrix}\right) + M_1 \left(\begin{pmatrix} x_\varphi \\ y \end{pmatrix} - \chi_{\mathbb{R}_{>0}} \otimes \begin{pmatrix} x^{(0)} \\ 0 \end{pmatrix}\right) \\ &\quad + \mathcal{A}\left(\begin{pmatrix} x_\varphi \\ y \end{pmatrix} - \chi_{\mathbb{R}_{>0}} \otimes \begin{pmatrix} x^{(0)} \\ 0 \end{pmatrix}\right) \qquad (5) \\ &= \begin{pmatrix} \varphi(m_0) 2\,\mathfrak{Re}(M_{11})\pi_1^* R u \\ \alpha u \end{pmatrix} - \begin{pmatrix} \chi_{\mathbb{R}_{>0}} \otimes A x^{(0)} \\ 0 \end{pmatrix} \\ &\quad - \chi_{\mathbb{R}_{>0}} \otimes M_1 \begin{pmatrix} x^{(0)} \\ 0 \end{pmatrix} + \begin{pmatrix} M_{00}\varphi'(m_0) x \\ 0 \end{pmatrix}, \qquad (6)\end{aligned}$$

where equality holds in $H_{\varrho,-1}(\mathbb{R}, \mathcal{H}_{-1})$. However, since the right-hand of the latter equation lies in $H_{\varrho,0}(\mathbb{R}, \mathcal{H}_0)$ we get $x_\varphi - \chi_{\mathbb{R}_{>0}} \otimes x^{(0)} \in H_{\varrho,0}(\mathbb{R}, X)$ and since $P_0 x_\varphi - \chi_{\mathbb{R}_{>0}} \otimes P_0 x^{(0)} \in H_{\varrho,1}(\mathbb{R}, X)$, we deduce that $\begin{pmatrix} x_\varphi \\ y \end{pmatrix} - \chi_{\mathbb{R}_{>0}} \otimes \begin{pmatrix} x^{(0)} \\ 0 \end{pmatrix} \in H_{\varrho,0}(\mathbb{R}, \mathcal{H}_1)$. In particular, this yields $x_\varphi \in H_{\varrho,0}(\mathbb{R}, H_1(A+1))$. We read off the first row equation of (5):

$$\begin{aligned}&\partial_0 M_{00}\big(x_\varphi - \chi_{\mathbb{R}_{>0}} \otimes x^{(0)}\big) + M_{11}\big(x_\varphi - \chi_{\mathbb{R}_{>0}} \otimes x^{(0)}\big) + A\big(x_\varphi - \chi_{\mathbb{R}_{>0}} \otimes x^{(0)}\big) \\ &\quad = 2\,\mathfrak{Re}(M_{11})\pi_1^* R \varphi(m_0) u - M_{11}\big(\chi_{\mathbb{R}_{>0}} \otimes x^{(0)}\big) - \chi_{\mathbb{R}_{>0}} \otimes A x^{(0)} + M_{00}\varphi'(m_0) x.\end{aligned}$$

Thus, we get that

$$\begin{aligned}&\partial_0 M_{00}\big(x_\varphi - \chi_{\mathbb{R}_{>0}} \otimes x^{(0)}\big) + M_{11} x_\varphi + A x_\varphi \\ &\quad = 2\,\mathfrak{Re}(M_{11})\pi_1^* R \varphi(m_0) u + M_{00}\varphi'(m_0) x,\end{aligned}$$

with equality in $H_0(A+1)$ pointwise almost everywhere. Multiplying by $\langle \cdot | x_\varphi \rangle_X$, taking real-parts and using $\mathfrak{Re}\langle A x_\varphi(s) | x_\varphi(s)\rangle = 0$ for almost every $s \in]0, T[$, we deduce that for almost every $t \in]0, T[$ it holds

$$\begin{aligned}&\mathfrak{Re}\big\langle \partial_0 M_{00}\big(x_\varphi - \chi_{\mathbb{R}_{>0}} \otimes x^{(0)}\big)(t) | x_\varphi(t)\big\rangle + \big\langle \mathfrak{Re}\, M_{11} x_\varphi(t) | x_\varphi(t)\big\rangle \\ &\quad = \mathfrak{Re}\big\langle 2\,\mathfrak{Re}(M_{11})\pi_1^* R u(t) | x_\varphi(t)\big\rangle.\end{aligned}$$

We let $\mu_{00} := \pi_0 M_{00} \pi_0^*$, $\mu_{11} := \pi_1(\mathfrak{Re}(M_{11}))\pi_1^*$. Thus, for almost every $t \in]0, T[$ it holds

$$\mathfrak{Re}\big\langle \partial_0\big(\mu_{00}\big(\pi_0 x_\varphi - \chi_{\mathbb{R}_{>0}} \otimes \pi_0 x^{(0)}\big)\big)(t)|\pi_0 x_\varphi(t)\big\rangle + \big\langle \mu_{11}\pi_1 x_\varphi(t)|\pi_1 x_\varphi(t)\big\rangle$$
$$= \mathfrak{Re}\big\langle 2\mu_{11} Ru(t)|\pi_1 x_\varphi(t)\big\rangle.$$

Hence, we conclude that for almost every $t \in]0, T[$:

$$\frac{1}{2}\big(s \mapsto \big\langle \mu_{00}\big(\pi_0 x_\varphi - \chi_{\mathbb{R}_{>0}} \otimes \pi_0 x^{(0)}\big)(s)|\big(\pi_0 x_\varphi - \chi_{\mathbb{R}_{>0}} \otimes \pi_0 x^{(0)}\big)(s)\big\rangle\big)'(t)$$
$$= -\mathfrak{Re}\big\langle \partial_0\big(\mu_{00}\big(\pi_0 x_\varphi - \chi_{\mathbb{R}_{>0}} \otimes \pi_0 x^{(0)}\big)\big)(t)|\chi_{\mathbb{R}_{>0}} \otimes \pi_0 x^{(0)}(t)\big\rangle$$
$$- \big\langle \mu_{11}\pi_1 x_\varphi(t)|\pi_1 x_\varphi(t)\big\rangle + \mathfrak{Re}\big\langle 2\mu_{11} Ru(t)|\pi_1 x_\varphi(t)\big\rangle. \tag{7}$$

The second row equation of (5) gives

$$\alpha R^{-1}\pi_1 x_\varphi + \alpha y = \alpha u.$$

Hence,

$$\pi_1 x_\varphi = R(u - y).$$

Since $x_\varphi(t) = x(t)$ for all $t \in]0, T[$, the latter equation put into (7) gives

$$\frac{1}{2}\big(s \mapsto \big\langle \mu_{00}\big(\pi_0 x - \chi_{\mathbb{R}_{>0}} \otimes \pi_0 x^{(0)}\big)(s)|\big(\pi_0 x - \chi_{\mathbb{R}_{>0}} \otimes \pi_0 x^{(0)}\big)(s)\big\rangle\big)'(t)$$
$$= -\mathfrak{Re}\big\langle \partial_0\big(\mu_{00}\big(\pi_0 x - \chi_{\mathbb{R}_{>0}} \otimes \pi_0 x^{(0)}\big)\big)(t)|\chi_{\mathbb{R}_{>0}} \otimes \pi_0 x^{(0)}(t)\big\rangle$$
$$- \big\langle \mu_{11} R(u-y)(t)|R(u-y)(t)\big\rangle + \mathfrak{Re}\big\langle 2\mu_{11} Ru(t)|R(u-y)(t)\big\rangle$$
$$= -\mathfrak{Re}\big\langle \partial_0\big(\mu_{00}\big(\pi_0 x - \chi_{\mathbb{R}_{>0}} \otimes \pi_0 x^{(0)}\big)\big)(t)|\chi_{\mathbb{R}_{>0}} \otimes \pi_0 x^{(0)}(t)\big\rangle$$
$$- \big\langle \mu_{11} Ry(t)|Ry(t)\big\rangle + \big\langle \mu_{11} Ru(t)|Ru(t)\big\rangle.$$

We integrate the latter equation over $]0, T[$. We conclude that

$$\frac{1}{2}\big\langle \mu_{00}\big(\pi_0 x - \chi_{\mathbb{R}_{>0}} \otimes \pi_0 x^{(0)}\big)(T)|\big(\pi_0 x - \chi_{\mathbb{R}_{>0}} \otimes \pi_0 x^{(0)}\big)(T)\big\rangle$$
$$= -\mathfrak{Re}\big\langle \mu_{00}\big(\pi_0 x - \chi_{\mathbb{R}_{>0}} \otimes \pi_0 x^{(0)}\big)(T)|\pi_0 x^{(0)}\big\rangle$$
$$- \int_0^T \big\langle \mu_{11} Ry(t)|Ry(t)\big\rangle dt + \int_0^T \big\langle \mu_{11} Ru(t)|Ru(t)\big\rangle dt.$$

Thus, we get that

$$\frac{1}{2}\big\langle \mu_{00}\pi_0 x(T)|\pi_0 x(T)\big\rangle + \int_0^T \big\langle \mu_{11} Ry(t)|Ry(t)\big\rangle dt$$
$$= \frac{1}{2}\big\langle \mu_{00}\pi_0 x^{(0)}|\pi_0 x^{(0)}\big\rangle + \int_0^T \big\langle \mu_{11} Ru(t)|Ru(t)\big\rangle dt.$$

This shows the conservativity of $\mathcal{C}$. □

Example 1 The heat equation yields a conservative, linear control system. With $G : D(G) \subseteq H_0 \to H_1$ closed and densely defined we consider the heat equation in the abstract form

$$\left(\partial_0 + G^*G\right)\theta = -G^*u,$$

which is equivalent to

$$\left(\partial_0 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & G^* \\ -G & 0 \end{pmatrix}\right) \begin{pmatrix} \theta \\ q \end{pmatrix} = \begin{pmatrix} 0 \\ u \end{pmatrix}.$$

We have $X = H_0 \oplus H_1$ and $Y = U = H_1$. Following the above construction we use

$$2q + y = u$$

with $R = \frac{1}{2}$ as observation equation. So, we get

$$\begin{pmatrix} \left(\partial_0 \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) + \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right) + \left(\begin{smallmatrix} 0 & G^* \\ -G & 0 \end{smallmatrix}\right)\right) & \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right) \\ (0\ 2) & 1 \end{pmatrix} \begin{pmatrix} \left(\begin{smallmatrix} \theta \\ q \end{smallmatrix}\right) \\ y \end{pmatrix} = \begin{pmatrix} \left(\begin{smallmatrix} 0 \\ u \end{smallmatrix}\right) \\ u \end{pmatrix}.$$

For $\alpha \neq 0$ we have equivalently

$$\begin{pmatrix} \left(\partial_0 \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) + \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right) + \left(\begin{smallmatrix} 0 & G^* \\ -G & 0 \end{smallmatrix}\right)\right) & \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right) \\ (0\ 2\alpha) & \alpha \end{pmatrix} \begin{pmatrix} \left(\begin{smallmatrix} \theta \\ q \end{smallmatrix}\right) \\ y \end{pmatrix} = \begin{pmatrix} \left(\begin{smallmatrix} 0 \\ u \end{smallmatrix}\right) \\ \alpha u \end{pmatrix},$$

where we choose α suitably to make

$$\mathfrak{Re} \begin{pmatrix} 1 & 0 \\ 2\alpha & \alpha \end{pmatrix} = \begin{pmatrix} 1 & \alpha \\ \alpha & \alpha \end{pmatrix}$$

strictly positive on $H_1 \oplus H_1$. This is the case if

$$0 < \alpha < 1.$$

This makes the example system

$$\begin{aligned} &\begin{pmatrix} \left(\partial_0 \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) + \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right) + \left(\begin{smallmatrix} 0 & G^* \\ -G & 0 \end{smallmatrix}\right)\right) & \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right) \\ (0\ 1) & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \left(\begin{smallmatrix} \theta \\ q \end{smallmatrix}\right) \\ y \end{pmatrix} \\ &= \begin{pmatrix} \left(\begin{smallmatrix} 0 \\ u \end{smallmatrix}\right) \\ \frac{1}{2}u \end{pmatrix} + \delta \otimes \begin{pmatrix} \left(\begin{smallmatrix} \theta^{(0)} \\ 0 \end{smallmatrix}\right) \\ 0 \end{pmatrix} \end{aligned} \tag{8}$$

a well-posed and at least formally conservative system. It remains to establish the required regularity. To this end put $u = 0$ and let $\theta^{(0)} \in D(G^*G)$ in (8). We compute

$$\begin{aligned}\theta &= \left(\partial_0 + G^*G\right)^{-1}\delta\otimes\theta^{(0)}\\ \theta - \chi_{\mathbb{R}_{>0}}\otimes\theta^{(0)} &= \left(\left(\partial_0 + G^*G\right)^{-1}\delta\otimes\theta^{(0)} - \chi_{\mathbb{R}_{>0}}\otimes\theta^{(0)}\right)\\ &= -\left(\partial_0 + G^*G\right)^{-1}\left(\chi_{\mathbb{R}_{>0}}\otimes G^*G\theta^{(0)}\right).\end{aligned}$$

This shows that the system is globally regularizing. Indeed, for $\phi := \chi_{\mathbb{R}_{>0}} \otimes G^*G\theta^{(0)}$ we estimate

$$\begin{aligned}\left|\left(\partial_0 + G^*G\right)^{-1}\phi\right|^2_{\varrho,1,0} &= \left|\partial_0\left(\partial_0 + G^*G\right)^{-1}\phi\right|^2_{\varrho,0,0}\\ &= \left|\partial_0\left(\partial_0 + |G|^2\right)^{-1}\phi\right|^2_{\varrho,0,0}\\ &= \left|\phi - |G|^2\left(\partial_0 + |G|^2\right)^{-1}\phi\right|^2_{\varrho,0,0}\\ &\leq 2|\phi|^2_{\varrho,0,0}.\end{aligned}$$

Thus, $\theta - \chi_{\mathbb{R}_{>0}} \otimes \theta^{(0)} \in H_{\varrho,1}(\mathbb{R}, H_0)$.

12.5 The Tucsnak-Weiss System

12.5.1 A First Order Formulation

Tucsnak and Weiss suggested the following particular system class, [15], describing a class of linear wave phenomena. In this reference, it is assumed that $H := X = F$, $Y = U$, $E = 1$ and $D = 1$. Let $A_0 : D(A_0) \subseteq H \to H$ be a selfadjoint positive operator. The observation operator C is an unbounded, closed linear operator

$$C : H_1(\sqrt{A_0} + \mathrm{i}) \subseteq H_0(\sqrt{A_0} + \mathrm{i}) \to U.$$

Then

$$\begin{aligned}C_0 : H_1(\sqrt{A_0} + \mathrm{i}) &\to U\\ x &\mapsto Cx\end{aligned}$$

is a continuous linear operator, according to the Closed Graph Theorem. The control operator B is now given as the dual operator $C_0^\diamond$ of C_0, where U and U^* as well as $H_1(\sqrt{A_0} + \mathrm{i})^*$ and $H_{-1}(\sqrt{A_0} + \mathrm{i})$ are identified so that we have $C_0^\diamond : U \to H_{-1}(\sqrt{A_0} + \mathrm{i})$. It is

$$C^* \subseteq C_0^\diamond.$$

The system considered in [15] is formally

$$\partial_0^2 z + A_0 z + \frac{1}{2} C_0^\diamond C_0 \partial_0 z = C_0^\diamond u,$$
$$C\partial_0 z + y = u$$

(C observation operator, $C_0^\diamond$ control operator) on $\mathbb{R}_{>0}$ for a given function $u \in H_{\varrho,0}(\mathbb{R}, U)$. We shall instead consider the first order system

$$\left(\partial_0 \begin{pmatrix} 1 & (0\,0) & 0 \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 1\,0 \\ 0\,0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ 0 & (0\,0) & 0 \end{pmatrix} + \begin{pmatrix} 0 & (0\,0) & 0 \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0\,0 \\ 0\,1 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ 0 & (0\,\sqrt{2}) & 1 \end{pmatrix}\right.$$
$$\left. + \begin{pmatrix} 0 & \mathbb{DIV} & 0 \\ \mathbb{GRAD} & \begin{pmatrix} 0\,0 \\ 0\,0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ 0 & (0\,0) & 0 \end{pmatrix}\right) \begin{pmatrix} v \\ \begin{pmatrix} \zeta \\ w \end{pmatrix} \\ y \end{pmatrix}$$
$$= \begin{pmatrix} 0 \\ \begin{pmatrix} 0 \\ -\sqrt{2}u \end{pmatrix} \\ -u \end{pmatrix} + \delta \otimes \begin{pmatrix} z^{(1)} \\ \begin{pmatrix} \sqrt{A_0} z^{(0)} \\ 0 \end{pmatrix} \\ 0 \end{pmatrix} \tag{9}$$

with

$$\mathbb{GRAD} := \begin{pmatrix} -\sqrt{A_0} \\ -\frac{1}{\sqrt{2}} C \end{pmatrix} : H_1(\sqrt{A_0} + \mathrm{i}) \subseteq H_0(\sqrt{A_0} + \mathrm{i}) \to H_0(\sqrt{A_0} + \mathrm{i}) \oplus U$$

and $\mathbb{DIV} := -(\mathbb{GRAD})^*$. Thus the whole systems acts in the space

$$H_{\varrho,0}\big(\mathbb{R}, H_0(\sqrt{A_0} + \mathrm{i}) \oplus \big(H_0(\sqrt{A_0} + \mathrm{i}) \oplus U\big) \oplus U\big).$$

Here

$$z^{(0)} \in H_1(\sqrt{A_0} + \mathrm{i}), \qquad z^{(1)} \in H_0(\sqrt{A_0} + \mathrm{i})$$

are the implementation of the initial data. Our first observation is that this system is a linear control system in a simple case:

- $$\mathcal{A} := \begin{pmatrix} 0 & \mathbb{DIV} & 0 \\ \mathbb{GRAD} & \begin{pmatrix} 0\,0 \\ 0\,0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ 0 & (0\,0) & 0 \end{pmatrix}$$

  is skew-selfadjoint,
- M_0 is the orthogonal projector onto $H_0(\sqrt{A_0} + \mathrm{i}) \oplus (H_0(\sqrt{A_0} + \mathrm{i}) \oplus \{0\}) \oplus \{0\}$,
- $$\mathfrak{Re}\, M_1 = \begin{pmatrix} 0 & (0\,0) & 0 \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0\,0 \\ 0\,1 \end{pmatrix} & \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix} \\ 0 & (0\,\frac{1}{\sqrt{2}}) & 1 \end{pmatrix}$$

  is strictly positive on the null space $\{0\} \oplus (\{0\} \oplus U) \oplus U$ of M_0.

Thus, well-posedness in the above sense is clear. We will show that this system is the appropriate interpretation of the original system. As a first step we compute the adjoint of $\mathbb{GRAD}$ explicitly.

Lemma 2 *Assume* $0 \in \varrho(A_0)$. *Then*

$$\mathbb{DIV} \subseteq \begin{pmatrix} \sqrt{A_0} & \frac{1}{\sqrt{2}} C_0^\diamond \end{pmatrix} : H_0(\sqrt{A_0}+\mathrm{i}) \oplus U \to H_{-1}(\sqrt{A_0}+\mathrm{i})$$
$$\begin{pmatrix} \zeta \\ w \end{pmatrix} \mapsto \sqrt{A_0}\zeta + \frac{1}{\sqrt{2}} C_0^\diamond w$$

and

$$D(\mathbb{DIV}) = \left\{ \begin{pmatrix} \zeta \\ w \end{pmatrix} \in H_0(\sqrt{A_0}+\mathrm{i}) \oplus U \,\middle|\, \begin{pmatrix} \sqrt{A_0} & \frac{1}{\sqrt{2}} C_0^\diamond \end{pmatrix} \begin{pmatrix} \zeta \\ w \end{pmatrix} \in H_0(\sqrt{A_0}+\mathrm{i}) \right\}.$$

Proof We consider

$$\widetilde{\mathbb{DIV}} : D(\widetilde{\mathbb{DIV}}) \subseteq H_0(\sqrt{A_0}+\mathrm{i}) \oplus U \to H_0(\sqrt{A_0}+\mathrm{i})$$
$$\begin{pmatrix} \zeta \\ w \end{pmatrix} \mapsto \begin{pmatrix} \sqrt{A_0} & \frac{1}{\sqrt{2}} C_0^\diamond \end{pmatrix} \begin{pmatrix} \zeta \\ w \end{pmatrix}$$

with $D(\widetilde{\mathbb{DIV}})$ being the set

$$\left\{ \begin{pmatrix} \zeta \\ w \end{pmatrix} \in H_0(\sqrt{A_0}+\mathrm{i}) \oplus U \,\middle|\, \begin{pmatrix} \sqrt{A_0} & \frac{1}{\sqrt{2}} C_0^\diamond \end{pmatrix} \begin{pmatrix} \zeta \\ w \end{pmatrix} \in H_0(\sqrt{A_0}+\mathrm{i}) \right\}.$$

We want to show that

$$\widetilde{\mathbb{DIV}} = \mathbb{DIV}$$

and we shall do so by showing that

$$\widetilde{\mathbb{DIV}}^* = -\mathbb{GRAD}.$$

Clearly,

$$\begin{pmatrix} \sqrt{A_0} & \frac{1}{\sqrt{2}} C^* \end{pmatrix} : H_1(\sqrt{A_0}+\mathrm{i}) \oplus D\big(C^*\big) \subseteq H_0(\sqrt{A_0}+\mathrm{i}) \oplus U \to H_0(\sqrt{A_0}+\mathrm{i})$$
$$\subseteq \widetilde{\mathbb{DIV}} \subseteq \begin{pmatrix} \sqrt{A_0} & \frac{1}{\sqrt{2}} C_0^\diamond \end{pmatrix} : H_0(\sqrt{A_0}+\mathrm{i}) \oplus U \to H_{-1}(\sqrt{A_0}+\mathrm{i})$$

and hence $\widetilde{\mathbb{DIV}}$ is densely defined. So let $v \in D(\widetilde{\mathbb{DIV}}^*)$. Then for some $\begin{pmatrix} f \\ g \end{pmatrix} \in H_0(\sqrt{A_0}+\mathrm{i}) \oplus U$ we have

$$\bigwedge_{\begin{pmatrix} \zeta \\ w \end{pmatrix} \in D(\widetilde{\mathbb{DIV}})} \left\langle \widetilde{\mathbb{DIV}} \begin{pmatrix} \zeta \\ w \end{pmatrix} \middle| v \right\rangle_{H_0(\sqrt{A_0}+\mathrm{i})} = \left\langle \begin{pmatrix} \zeta \\ w \end{pmatrix} \middle| \begin{pmatrix} f \\ g \end{pmatrix} \right\rangle_{H_0(\sqrt{A_0}+\mathrm{i}) \oplus U}.$$

It follows by testing with elements in $H_1(\sqrt{A_0}+\mathrm{i})\oplus\{0\}\subseteq D(\widetilde{\mathbb{DIV}})$ that

$$\bigwedge_{\zeta\in H_1(\sqrt{A_0}+\mathrm{i})}\langle\sqrt{A_0}\zeta\,|\,v\rangle_{H_0(\sqrt{A_0}+\mathrm{i})}=\langle\zeta\,|\,f\rangle_{H_0(\sqrt{A_0}+\mathrm{i})},$$

which implies

$$v\in D(\sqrt{A_0})$$

and

$$\sqrt{A_0}v=f.$$

Let now $w\in U$ be arbitrary. Then with $\zeta=-\frac{1}{\sqrt{2}}\sqrt{A_0}^{-1}C_0^{\diamond}w$ we get[2]

$$\begin{aligned}0&=\left\langle\widetilde{\mathbb{DIV}}\begin{pmatrix}\zeta\\ w\end{pmatrix}\middle|v\right\rangle_{H_0(\sqrt{A_0}+\mathrm{i})}\\&=\left\langle\begin{pmatrix}-\frac{1}{\sqrt{2}}\sqrt{A_0}^{-1}C_0^{\diamond}w\\ w\end{pmatrix}\middle|\begin{pmatrix}\sqrt{A_0}v\\ g\end{pmatrix}\right\rangle_{H_0(\sqrt{A_0}+\mathrm{i})\oplus U}\\&=\left\langle-\frac{1}{\sqrt{2}}\sqrt{A_0}^{-1}C_0^{\diamond}w\middle|\sqrt{A_0}v\right\rangle_{H_0(\sqrt{A_0}+\mathrm{i})}+\langle w|g\rangle_U\\&=\left\langle-\frac{1}{\sqrt{2}}C_0^{\diamond}w\middle|v\right\rangle_{H_0(\sqrt{A_0}+\mathrm{i})}+\langle w|g\rangle_U\\&=\left\langle-\frac{1}{\sqrt{2}}w\middle|C_0v\right\rangle_U+\langle w|g\rangle_U.\end{aligned}$$

This implies

$$\frac{1}{\sqrt{2}}Cv=g$$

and thus, we have

$$\widetilde{\mathbb{DIV}}^*v=\begin{pmatrix}\sqrt{A_0}v\\ \frac{1}{\sqrt{2}}Cv\end{pmatrix}=-\mathbb{GRAD}v,$$

i.e.

$$\widetilde{\mathbb{DIV}}^*\subseteq-\mathbb{GRAD}.$$

[2] Note that in the fourth equality $\langle\cdot|\cdot\rangle_{H_0(\sqrt{A_0}+\mathrm{i})}$ is used not as the inner product in $H_0(\sqrt{A_0}+\mathrm{i})$ but as its continuous extension to the duality pairing between $H_{-1}(\sqrt{A_0}+\mathrm{i})$ and $H_1(\sqrt{A_0}+\mathrm{i})$. This will be utilized throughout without explicit mention.

Moreover, let now $v \in D(\mathbb{GRAD})$. Then for all $\left(\begin{smallmatrix}\zeta\\ w\end{smallmatrix}\right) \in D(\widetilde{\mathbb{DIV}})$

$$
\begin{aligned}
&\left\langle \widetilde{\mathbb{DIV}} \begin{pmatrix}\zeta\\ w\end{pmatrix} \middle| v \right\rangle_{H_0(\sqrt{A_0}+\mathrm{i})} + \left\langle \begin{pmatrix}\zeta\\ w\end{pmatrix} \middle| \mathbb{GRAD} v \right\rangle_{H_0(\sqrt{A_0}+\mathrm{i})\oplus U} \\
&= \left\langle \sqrt{A_0}\zeta + \frac{1}{\sqrt{2}} C_0^{\diamond} w \middle| v \right\rangle_{H_0(\sqrt{A_0}+\mathrm{i})} - \left\langle \begin{pmatrix}\zeta\\ w\end{pmatrix} \middle| \begin{pmatrix}\sqrt{A_0}v\\ \frac{1}{\sqrt{2}}Cv\end{pmatrix} \right\rangle_{H_0(\sqrt{A_0}+\mathrm{i})\oplus U} \\
&= \left\langle \sqrt{A_0}\zeta + \frac{1}{\sqrt{2}} C_0^{\diamond} w \middle| v \right\rangle_{H_0(\sqrt{A_0}+\mathrm{i})} \\
&\quad - \left\langle \frac{1}{\sqrt{2}} C_0^{\diamond} w \middle| v \right\rangle_{H_0(\sqrt{A_0}+\mathrm{i})} - \langle \zeta | \sqrt{A_0} v\rangle_{H_0(\sqrt{A_0}+\mathrm{i})} \\
&= \left\langle \sqrt{A_0}\zeta + \frac{1}{\sqrt{2}} C_0^{\diamond} w \middle| v \right\rangle_{H_0(\sqrt{A_0}+\mathrm{i})} - \left\langle \sqrt{A_0}\zeta + \frac{1}{\sqrt{2}} C_0^{\diamond} w \middle| v \right\rangle_{H_0(\sqrt{A_0}+\mathrm{i})} = 0,
\end{aligned}
$$

from which we see that

$$-\mathbb{GRAD} \subseteq \widetilde{\mathbb{DIV}}^*.$$

Thus, we have shown that

$$\widetilde{\mathbb{DIV}} = -\mathbb{GRAD}^* = \mathbb{DIV}. \qquad \square$$

Noting that the solution

$$\begin{pmatrix} v \\ \left(\begin{smallmatrix}\zeta\\ w\end{smallmatrix}\right) \\ y \end{pmatrix}$$

of (9) is in $H_{\varrho,-1}(\mathbb{R}, \mathcal{H}_0) \cap H_{\varrho,-2}(\mathbb{R}, \mathcal{H}_1)$, by the results of Sect. 12.3, we can read (9) line by line under the assumption that $0 \in \varrho(A_0)$ and we obtain

$$
\begin{aligned}
\partial_0 v + \sqrt{A_0}\zeta + \frac{1}{\sqrt{2}} C_0^{\diamond} w &= \delta \otimes z^{(1)} \\
\partial_0 \zeta - \sqrt{A_0} v &= \delta \otimes \sqrt{A_0} z^{(0)} \\
w - \frac{1}{\sqrt{2}} C v &= -\sqrt{2} u \\
\sqrt{2} w + y &= -u.
\end{aligned}
$$

Since $v, \zeta \in H_{\varrho,-1}(\mathbb{R}, H_0(\sqrt{A_0}+\mathrm{i}))$ and $y, w \in H_{\varrho,-1}(\mathbb{R}, U)$, we see that the first equation holds in $H_{\varrho,-2}(\mathbb{R}, H_{-1}(\sqrt{A_0}+\mathrm{i}))$. Since also $v \in H_{\varrho,-2}(\mathbb{R}, H_1(\sqrt{A_0}+\mathrm{i}))$ and $z^{(0)} \in H_1(\sqrt{A_0}+\mathrm{i})$, the second equation holds in $H_{\varrho,-2}(\mathbb{R}, H_0(\sqrt{A_0}+\mathrm{i}))$ and the third one in $H_{\varrho,-2}(\mathbb{R},U)$. If we let $z := \sqrt{A_0}^{-1}\zeta \in H_{\varrho,-1}(\mathbb{R}, H_1(\sqrt{A_0}+\mathrm{i})) \cap H_{\varrho,0}(\mathbb{R}, H_{-1}(\sqrt{A_0}+\mathrm{i}))$ then $\partial_0 z = v + \delta\otimes z^{(0)}$ and

$$\begin{aligned}\partial_0^2 z + A_0 z + \frac{1}{\sqrt{2}}C_0^\diamond w &= \delta\otimes z^{(1)} + \partial_0\delta\otimes z^{(0)}\\ w - \frac{1}{\sqrt{2}}Cv &= -\sqrt{2}u\\ \sqrt{2}w + y &= -u.\end{aligned}$$

Thus, eliminating w we get

$$\begin{aligned}\partial_0^2 z + A_0 z + \frac{1}{2}C_0^\diamond\big(C\big(\partial_0 z - \delta\otimes z^{(0)}\big) - 2u\big) &= \delta\otimes z^{(1)} + \partial_0\delta\otimes z^{(0)}\\ y &= u - C\big(\partial_0 z - \delta\otimes z^{(0)}\big),\end{aligned}$$

or

$$\begin{aligned}\partial_0^2 z + A_0 z + \frac{1}{2}C_0^\diamond C\partial_0 z &= C_0^\diamond u + \delta\otimes z^{(1)} + \partial_0\delta\otimes z^{(0)} + \frac{1}{2}\delta\otimes C_0^\diamond C z^{(0)}\\ y &= u - C\partial_0 z + \delta\otimes C z^{(0)},\end{aligned}$$

which is on $\mathbb{R}_{>0}$ formally the equation we started out with. Here the first equality holds in $H_{\varrho,-2}(\mathbb{R}, H_{-1}(\sqrt{A_0}+\mathrm{i}))$ and the second one in $H_{\varrho,-2}(\mathbb{R},U)$.

12.5.2 The Tucsnak-Weiss System as a Conservative Linear Control System

In this section we want to prove that the system considered in the previous part is conservative as it was formulated in Theorem 2 under appropriate assumptions on the initial values $z^{(0)}, z^{(1)}$. In order to formulate pointwise evaluations of the solution, we have to inspect regularity properties for the system. Since the regularization property does not depend on u we may set $u = 0$. By assuming $0 \in \varrho(A_0)$ we arrive at the equations

$$\begin{aligned}\partial_0 v + \sqrt{A_0}\zeta + \frac{1}{\sqrt{2}}C_0^\diamond w &= \delta\otimes z^{(1)}\\ \partial_0\zeta - \sqrt{A_0}v &= \delta\otimes\sqrt{A_0}z^{(0)}\\ w - \frac{1}{\sqrt{2}}Cv &= 0\\ \sqrt{2}w + y &= 0\end{aligned}$$

and re-assemble them in a different way. As was already pointed out, the first equation holds in the space $H_{\varrho,-2}(\mathbb{R}, H_{-1}(\sqrt{A_0}+\mathrm{i}))$ and the second one in $H_{\varrho,-2}(\mathbb{R}, H_0(\sqrt{A_0}+\mathrm{i}))$, while both the third and fourth one hold in $H_{\varrho,-2}(\mathbb{R}, U)$. Using the third equation to eliminate w in the first one, we get the following system

$$\partial_0 v + \sqrt{A_0}\zeta + \frac{1}{2}C_0^\diamond C_0 v = \delta \otimes z^{(1)},$$
$$\partial_0 \zeta - \sqrt{A_0} v = \delta \otimes \sqrt{A_0} z^{(0)}.$$

Rewriting this in an operator-matrix form we get

$$\partial_0 \begin{pmatrix} \zeta \\ v \end{pmatrix} + \begin{pmatrix} 0 & -\sqrt{A_0} \\ \sqrt{A_0} & \frac{1}{2}C_0^\diamond C_0 \end{pmatrix} \begin{pmatrix} \zeta \\ v \end{pmatrix} = \delta \otimes \begin{pmatrix} \sqrt{A_0} z^{(0)} \\ z^{(1)} \end{pmatrix} \tag{10}$$

as an equation in $H_{\varrho,-2}(\mathbb{R}, H_0(\sqrt{A_0}+\mathrm{i}) \oplus H_{-1}(\sqrt{A_0}+\mathrm{i}))$. We define the following linear operator

$$A : D(A) \subseteq H_0(\sqrt{A_0}+\mathrm{i})^2 \to H_0(\sqrt{A_0}+\mathrm{i})^2,$$

where the domain of A, $D(A)$, is the set

$$\left\{ (\zeta, v) \in H_0(\sqrt{A_0}+\mathrm{i})^2 \middle| v \in H_1(\sqrt{A_0}+\mathrm{i}), \sqrt{A_0}\zeta + \frac{1}{2}C_0^\diamond C_0 v \in H_0(\sqrt{A_0}+\mathrm{i}) \right\}$$

and

$$A \begin{pmatrix} \zeta \\ v \end{pmatrix} := \begin{pmatrix} 0 & -\sqrt{A_0} \\ \sqrt{A_0} & \frac{1}{2}C_0^\diamond C_0 \end{pmatrix} \begin{pmatrix} \zeta \\ v \end{pmatrix}.$$

The density of the domain of A follows by arguing analogously to the proof of Lemma 2.

Lemma 3 *The operator A is closed and continuously invertible. Furthermore the following holds*

$$\mathfrak{Re}\left\langle \begin{pmatrix} \zeta \\ v \end{pmatrix} \middle| A \begin{pmatrix} \zeta \\ v \end{pmatrix} \right\rangle_{H_0(A)} = \frac{1}{2}\left\langle \begin{pmatrix} 0 & C_0 \end{pmatrix} \begin{pmatrix} \zeta \\ v \end{pmatrix} \middle| \begin{pmatrix} 0 & C_0 \end{pmatrix} \begin{pmatrix} \zeta \\ v \end{pmatrix} \right\rangle_U \geq 0$$

and

$$\mathfrak{Re}\left\langle \begin{pmatrix} r \\ s \end{pmatrix} \middle| A^* \begin{pmatrix} r \\ s \end{pmatrix} \right\rangle_{H_0(A)} = \frac{1}{2}\left\langle \begin{pmatrix} 0 & C_0 \end{pmatrix} \begin{pmatrix} r \\ s \end{pmatrix} \middle| \begin{pmatrix} 0 & C_0 \end{pmatrix} \begin{pmatrix} r \\ s \end{pmatrix} \right\rangle_U \geq 0$$

for all $\begin{pmatrix} \zeta \\ v \end{pmatrix} \in D(A)$, $\begin{pmatrix} r \\ s \end{pmatrix} \in D(A^*)$.

Proof The operator A is a restriction of the bounded linear operator

$$\begin{pmatrix} 0 & -\sqrt{A_0} \\ \sqrt{A_0} & \frac{1}{2}C_0^\diamond C_0 \end{pmatrix} : H_0(\sqrt{A_0}+\mathrm{i}) \oplus H_1(\sqrt{A_0}+\mathrm{i}) \to H_0(\sqrt{A_0}+\mathrm{i}) \oplus H_{-1}(\sqrt{A_0}+\mathrm{i}).$$

An easy computation shows that its inverse is given by

$$\begin{pmatrix} \frac{1}{2}A_0^{-1/2}C_0^{\diamond}C_0A_0^{-1/2} & A_0^{-1/2} \\ -A_0^{-1/2} & 0 \end{pmatrix} : H_0(\sqrt{A_0}+\mathrm{i}) \oplus H_{-1}(\sqrt{A_0}+\mathrm{i})$$
$$\to H_0(\sqrt{A_0}+\mathrm{i}) \oplus H_1(\sqrt{A_0}+\mathrm{i}),$$

which is again bounded. If we consider the restriction

$$H_0(\sqrt{A_0}+\mathrm{i})^2 \to H_0(\sqrt{A_0}+\mathrm{i})^2$$
$$\begin{pmatrix} r \\ s \end{pmatrix} \mapsto \begin{pmatrix} \frac{1}{2}A_0^{-1/2}C_0^{\diamond}C_0A_0^{-1/2} & A_0^{-1/2} \\ -A_0^{-1/2} & 0 \end{pmatrix}\begin{pmatrix} r \\ s \end{pmatrix},$$

we again obtain a bounded linear operator, whose range is a subset of $D(A)$. Hence it is the inverse of A and thus A^{-1} is a bounded linear operator, which shows that A is closed with $0 \in \varrho(A)$. For $z, v \in H_0(\sqrt{A_0}+\mathrm{i})$ we compute

$$\begin{aligned} \langle A_0^{-1/2}C_0^{\diamond}C_0A_0^{-1/2}z|v\rangle_{H_0(\sqrt{A_0}+\mathrm{i})} &= \langle C_0^{\diamond}C_0A_0^{-1/2}z|A_0^{-1/2}v\rangle_{H_0(\sqrt{A_0}+\mathrm{i})} \\ &= \langle C_0A_0^{-1/2}z|C_0A_0^{-1/2}v\rangle_U \\ &= \langle A_0^{-1/2}z|C_0^{\diamond}C_0A_0^{-1/2}v\rangle_{H_0(\sqrt{A_0}+\mathrm{i})} \\ &= \langle z|A_0^{-1/2}C_0^{\diamond}C_0A_0^{-1/2}v\rangle_{H_0(\sqrt{A_0}+\mathrm{i})}, \end{aligned}$$

proving that $A_0^{-1/2}C_0^{\diamond}C_0A_0^{-1/2}$ is self-adjoint. Thus, we obtain

$$\begin{aligned} (A^*)^{-1} &= (A^{-1})^* \\ &= \begin{pmatrix} \frac{1}{2}A_0^{-1/2}C_0^{\diamond}C_0A_0^{-1/2} & -A_0^{-1/2} \\ A_0^{-1/2} & 0 \end{pmatrix} : H_0(\sqrt{A_0}+\mathrm{i})^2 \\ &\to H_0(\sqrt{A_0}+\mathrm{i})^2 \end{aligned}$$

and so the operator $(A^*)^{-1}$ is a restriction of the operator

$$\begin{pmatrix} \frac{1}{2}A_0^{-1/2}C_0^{\diamond}C_0A_0^{-1/2} & -A_0^{-1/2} \\ A_0^{-1/2} & 0 \end{pmatrix} : H_0(\sqrt{A_0}+\mathrm{i}) \oplus H_{-1}(\sqrt{A_0}+\mathrm{i})$$
$$\to H_0(\sqrt{A_0}+\mathrm{i}) \oplus H_1(\sqrt{A_0}+\mathrm{i}).$$

Using this, we get that

$$\begin{aligned}
A^* &\subseteq \begin{pmatrix} \frac{1}{2}A_0^{-1/2}C_0^\diamond C_0 A_0^{-1/2} & -A_0^{-1/2} \\ A_0^{-1/2} & 0 \end{pmatrix}^{-1} \\
&= \begin{pmatrix} 0 & A_0^{-1/2} \\ -A_0^{-1/2} & \frac{1}{2}C_0^\diamond C_0 \end{pmatrix} : H_0(\sqrt{A_0}+\mathrm{i}) \oplus H_1(\sqrt{A_0}+\mathrm{i}) \\
&\to H_0(\sqrt{A_0}+\mathrm{i}) \oplus H_{-1}(\sqrt{A_0}+\mathrm{i}).
\end{aligned}$$

Now we are able to show the two asserted equalities. For $(\zeta, v) \in D(A)$ we have

$$\begin{aligned}
&\mathfrak{Re}\left\langle \begin{pmatrix} \zeta \\ v \end{pmatrix} \middle| A \begin{pmatrix} \zeta \\ v \end{pmatrix} \right\rangle_{H_0(\sqrt{A_0}+\mathrm{i})^2} \\
&= \mathfrak{Re}\left\langle \begin{pmatrix} \zeta \\ v \end{pmatrix} \middle| \begin{pmatrix} 0 & -\sqrt{A_0} \\ \sqrt{A_0} & \frac{1}{2}C_0^\diamond C_0 \end{pmatrix} \begin{pmatrix} \zeta \\ v \end{pmatrix} \right\rangle_{H_0(\sqrt{A_0}+\mathrm{i})^2} \\
&= \mathfrak{Re}\left\langle v \middle| \sqrt{A_0}\zeta + \frac{1}{2}\big(C_0^\diamond C_0\big)v \right\rangle_{H_0(\sqrt{A_0}+\mathrm{i})} + \mathfrak{Re}\langle \zeta | \sqrt{A_0} v\rangle_{H_0(\sqrt{A_0}+\mathrm{i})} \\
&= \mathfrak{Re}\left\langle v \middle| \frac{1}{2}\big(C_0^\diamond C_0\big)v \right\rangle_{H_0(\sqrt{A_0}+\mathrm{i})} \\
&= \frac{1}{2}\mathfrak{Re}\langle C_0 v | C_0 v\rangle_U \\
&= \frac{1}{2}\left\langle \begin{pmatrix} 0 & C_0 \end{pmatrix} \begin{pmatrix} \zeta \\ v \end{pmatrix} \middle| \begin{pmatrix} 0 & C_0 \end{pmatrix} \begin{pmatrix} \zeta \\ v \end{pmatrix} \right\rangle_U .
\end{aligned}$$

Analogously we get for $(r, s) \in D(A^*)$

$$\begin{aligned}
&\mathfrak{Re}\left\langle \begin{pmatrix} r \\ s \end{pmatrix} \middle| A^* \begin{pmatrix} r \\ s \end{pmatrix} \right\rangle_{H_0(\sqrt{A_0}+\mathrm{i})^2} \\
&= \mathfrak{Re}\left\langle \begin{pmatrix} r \\ s \end{pmatrix} \middle| \begin{pmatrix} 0 & \sqrt{A_0} \\ -\sqrt{A_0} & \frac{1}{2}C_0^\diamond C_0 \end{pmatrix} \begin{pmatrix} r \\ s \end{pmatrix} \right\rangle_{H_0(\sqrt{A_0}+\mathrm{i})^2} \\
&= \mathfrak{Re}\langle r | \sqrt{A_0} s\rangle_{H_0(\sqrt{A_0}+\mathrm{i})} - \mathfrak{Re}\left\langle s \middle| \sqrt{A_0} r - \frac{1}{2}\big(C_0^\diamond C_0\big)s \right\rangle_{H_0(\sqrt{A_0}+\mathrm{i})} \\
&= \mathfrak{Re}\left\langle s \middle| \frac{1}{2}\big(C_0^\diamond C_0\big)s \right\rangle_{H_0(\sqrt{A_0}+\mathrm{i})} \\
&= \frac{1}{2}\mathfrak{Re}\langle C_0 s | C_0 s\rangle_U \\
&= \frac{1}{2}\left\langle \begin{pmatrix} 0 & C_0 \end{pmatrix} \begin{pmatrix} r \\ s \end{pmatrix} \middle| \begin{pmatrix} 0 & C_0 \end{pmatrix} \begin{pmatrix} r \\ s \end{pmatrix} \right\rangle_U .
\end{aligned}$$

□

Remark 3

1. Lemma 3 especially implies that A and A^* are monotone or accretive operators. Hence $-A$ is a generator of a contraction semigroup. Furthermore $(\partial_0 + A)^{-1}$ and $(\partial_0^* + A^*)^{-1}$ are bounded linear operators on $H_{\varrho,0}(\mathbb{R}, H_0(A))$ and can be extended to bounded operators on the associated spaces $H_{\varrho,k}(\mathbb{R}, H_s(A))$ and $H_{\varrho,k}(\mathbb{R}, H_s(A^*))$ respectively, where $k, s \in \mathbb{Z}$.
2. From the equalities we also read off that

$$\begin{pmatrix} 0 & C_0 \end{pmatrix} (\partial_0 + A)^{-1} : H_{\varrho,1}\big(\mathbb{R}, H_1(A)\big) \subseteq H_{\varrho,0}\big(\mathbb{R}, H_0(A)\big) \to H_{\varrho,0}(\mathbb{R}, U)$$

is continuous, since for $u \in H_{\varrho,1}(\mathbb{R}, H_1(A))$ we estimate

$$\begin{aligned}\mathfrak{Re}\big\langle (\mathrm{i}t + \varrho + A)u(t) | u(t) \big\rangle_{H_0(A)} &= \varrho \big|u(t)\big|^2_{H_0(A)} + \big|\begin{pmatrix} 0 & C_0 \end{pmatrix} u(t)\big|^2_U \\ &\geq \big|\begin{pmatrix} 0 & C_0 \end{pmatrix} u(t)\big|^2_U\end{aligned}$$

for every $t \in \mathbb{R}$ and from this we derive the stated continuity. Analogously we get

$$\begin{pmatrix} 0 & C_0 \end{pmatrix} \big(\partial_0^* + A^*\big)^{-1} : H_{\varrho,1}\big(\mathbb{R}, H_1\big(A^*\big)\big) \subseteq H_{\varrho,0}\big(\mathbb{R}, H_0\big(A^*\big)\big) \to H_{\varrho,0}(\mathbb{R}, U)$$

is continuous. Thus we can extend these operators continuously to $H_{\varrho,k}(\mathbb{R}, H_0(A))$ and $H_{\varrho,k}(\mathbb{R}, H_0(A^*))$ respectively taking values in $H_{\varrho,k}(\mathbb{R}, U)$ for all $k \in \mathbb{Z}$. From this it is possible to derive the continuity of the composition operator $(\partial_0 + A)^{-1}\binom{0}{C_0^\diamond}$ as a mapping from $H_{\varrho,k}(\mathbb{R}, U)$ to $H_{\varrho,k}(\mathbb{R}, H_0(A))$, which in the terminology of [15] means that $C_0^\diamond$ is admissible. However, in our setting this property is not needed.

Recall that our equation (10) is valid in $H_{\varrho,-2}(\mathbb{R}, H_0(\sqrt{A_0} + \mathrm{i}) \oplus H_{-1}(\sqrt{A_0} + \mathrm{i}))$. We show now that this implies the validity in $H_{\varrho,-2}(\mathbb{R}, H_{-1}(A))$.

Lemma 4 *The Sobolev-chains of* $\sqrt{A_0}$ *and* A^* *are related by*

$$H_1\big(A^*\big) \hookrightarrow H_0(\sqrt{A_0} + \mathrm{i}) \oplus H_1(\sqrt{A_0} + \mathrm{i}).$$

Proof Since

$$(A^*)^{-1} \subseteq \begin{pmatrix} \frac{1}{2} A_0^{-1/2} C_0^\diamond C_0 A_0^{-1/2} & -A_0^{-1/2} \\ A_0^{-1/2} & 0 \end{pmatrix}$$

we conclude that the inclusion $H_1(A^*) \subseteq H_0(\sqrt{A_0} + \mathrm{i}) \oplus H_1(\sqrt{A_0} \oplus \mathrm{i})$ holds. The Hilbert spaces $H_0(\sqrt{A_0} + \mathrm{i}) \oplus H_1(\sqrt{A_0} + \mathrm{i})$ and $H_1(A^*)$ are both continuously embedded in $H_0(\sqrt{A_0} + \mathrm{i}) \oplus H_0(\sqrt{A_0} + \mathrm{i}) = H_0(A^*)$ and hence the assertion follows by the Closed Graph Theorem. □

Remark 4 As a direct consequence of Lemma 4 we get

$$H_0(\sqrt{A_0}+\mathrm{i}) \oplus H_{-1}(\sqrt{A_0}+\mathrm{i}) \hookrightarrow H_{-1}(A)$$

since $H_{-1}(A)$ is unitary equivalent to the dual space $H_1(A^*)^*$.

With this we conclude that the equation

$$\partial_0 \begin{pmatrix} \zeta \\ v \end{pmatrix} + A \begin{pmatrix} \zeta \\ v \end{pmatrix} = \delta \otimes \begin{pmatrix} \sqrt{A_0} z^{(0)} \\ z^{(1)} \end{pmatrix}$$

holds in $H_{\varrho,-2}(\mathbb{R}, H_{-1}(A))$. From this we get

$$\begin{pmatrix} \zeta \\ v \end{pmatrix} - \chi_{\mathbb{R}_{>0}} \otimes \begin{pmatrix} \sqrt{A_0} z^{(0)} \\ z^{(1)} \end{pmatrix} = (\partial_0 + A)^{-1} \left(\chi_{\mathbb{R}_{>0}} \otimes A \begin{pmatrix} \sqrt{A_0} z^{(0)} \\ z^{(1)} \end{pmatrix} \right).$$

If we assume that $\begin{pmatrix} \sqrt{A_0} z^{(0)} \\ z^{(1)} \end{pmatrix} \in D(A)$, we get, since $-A$ is the generator of a C_0-semigroup, that $(\partial_0 + A)^{-1}\left(\chi_{\mathbb{R}_{>0}} \otimes A\begin{pmatrix} \sqrt{A_0} z^{(0)} \\ z^{(1)} \end{pmatrix}\right) \in H_{\varrho,1}(\mathbb{R}, H_0(A))$, by employing semigroup theory as a regularity result. This shows that the system (9) is globally regularizing with $\mathcal{U} := D(A)$. Thus Theorem 2 is applicable and we can show the conservativity of the system. We summarize our findings of this section in the following theorem.

Theorem 3 *The system* (9) *is well-posed. If* $0 \in \varrho(A_0)$ *it is globally regularizing and conservative in the sense of Theorem* 2*.*

Proof The well-posedness was shown in Sect. 12.5.1 and the regularity was proved above. By comparing the system (9) and the setting in Theorem 2 we see that the conservativity follows with $R = \frac{1}{\sqrt{2}}$ and $\alpha = 1$. □

12.6 Main Observations

In this note, we gave a unified approach to a large class of infinite-dimensional control systems. This perspective enabled us, assuming mild regularizing properties of the solution operator, to construct observation equations such that the respective control systems become conservative in the sense of [15]. Moreover, we studied a particular linear control system, which models wave phenomena and consists of unbounded control and observation operators. It turned out that this system may be rewritten into a form introduced in [8], such that the solution theory becomes easily accessible and unbounded control and observation need not to be treated. Surprisingly enough, the system studied in [15] corresponds to the skew-selfadjoint operator case, which might be a rather special one at first glance.

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Chapter 13
Recent Progress in Smoothing Estimates for Evolution Equations

Michael Ruzhansky and Mitsuru Sugimoto

Abstract This paper is a survey article of results and arguments from authors' papers (Ruzhansky and Sugimoto in Proc. Lond. Math. Soc. 105:393–423, 2012; Ruzhansky and Sugimoto in Smoothing properties of non-dispersive equations; Ruzhansky and Sugimoto in Smoothing properties of inhomogeneous equations via canonical transforms), and describes a new approach to global smoothing problems for dispersive and non-dispersive evolution equations based on ideas of comparison principle and canonical transforms. For operators $a(D_x)$ of order m satisfying the dispersiveness condition $\nabla a(\xi) \neq 0$, the smoothing estimate

$$\left\| \langle x\rangle^{-s} |D_x|^{(m-1)/2} e^{ita(D_x)}\varphi(x)\right\|_{L^2(\mathbb{R}_t\times\mathbb{R}^n_x)} \le C\|\varphi\|_{L^2(\mathbb{R}^n_x)} \quad (s>1/2)$$

is established, while it is known to fail for general non-dispersive operators. Especially, time-global smoothing estimates for the operator $a(D_x)$ with lower order terms are the benefit of our new method. For the case when the dispersiveness breaks, we suggest a form

$$\left\| \langle x\rangle^{-s} \left|\nabla a(D_x)\right|^{1/2} e^{ita(D_x)}\varphi(x)\right\|_{L^2(\mathbb{R}_t\times\mathbb{R}^n_x)} \le C\|\varphi\|_{L^2(\mathbb{R}^n_x)} \quad (s>1/2)$$

which is equivalent to the usual estimate in the dispersive case and is also invariant under canonical transformations for the operator $a(D_x)$. It does continue to hold for a variety of non-dispersive operators $a(D_x)$, where $\nabla a(\xi)$ may become zero on some set. It is remarkable that our method allows us to carry out a global microlocal reduction of equations to the translation invariance property of the Lebesgue measure.

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13.1 Introduction

This survey article is a collection of results and arguments from authors' papers [18, 19], and [20].

Let us consider the following Cauchy problem to the Schrödinger equation:

$$\begin{cases} (i\partial_t + \Delta_x)u(t,x) = 0 & \text{in } \mathbb{R}_t \times \mathbb{R}^n_x, \\ u(0,x) = \varphi(x) & \text{in } \mathbb{R}^n_x. \end{cases}$$

By Plancherel's theorem, the solution $u(t,x) = e^{it\Delta_x}\varphi(x)$ preserves the L^2-norm of the initial data φ, that is, we have $\|u(t,\cdot)\|_{L^2(\mathbb{R}^n_x)} = \|\varphi\|_{L^2(\mathbb{R}^n)}$ for any fixed time $t \in \mathbb{R}$. But if we integrate the solution in t, we get an extra gain of regularity of order $1/2$ in x. For example we have the estimate

$$\left\| \langle x\rangle^{-s} |D_x|^{1/2} e^{it\Delta_x}\varphi \right\|_{L^2(\mathbb{R}_t\times\mathbb{R}^n_x)} \le C\|\varphi\|_{L^2(\mathbb{R}^n_x)} \quad (s > 1/2)$$

for $u = e^{it\Delta_x}\varphi$, where $\langle x\rangle = \sqrt{1+|x|^2}$, and (a sharper version of) this estimate was first given by Kenig, Ponce and Vega [12]. This type of estimate is called a smoothing estimate, and its local version was first proved by Sjölin [23], Constantin and Saut [6], and Vega [26]. We remark that, historically, such a smoothing estimate was first shown to Korteweg-de Vries equation

$$\begin{cases} \partial_t u + \partial_x^3 u + u\partial_x u = 0, \\ u(0,x) = \varphi(x) \in L^2(\mathbb{R}), \end{cases}$$

and Kato [10] proved that the solution $u = u(t,x)$ $(t, x \in \mathbb{R})$ satisfies

$$\int_{-T}^{T}\int_{-R}^{R} \left|\partial_x u(x,t)\right|^2 dx\,dt \le c\big(T, R, \|\varphi\|_{L^2}\big).$$

Similar smoothing estimates have been observed for generalised equations

$$\begin{cases} (i\partial_t + a(D_x))u(t,x) = 0, \\ u(0,x) = \varphi(x) \in L^2(\mathbb{R}^n), \end{cases}$$

which come from equations of fundamental importance in mathematical physics as their principal parts:

- $a(\xi) = |\xi|^2 \cdots$ Schrödinger

$$i\partial_t u - \Delta_x u = 0$$

- $a(\xi) = \sqrt{|\xi|^2 + 1} \cdots$ Relativistic Schrödinger

$$i\partial_t u + \sqrt{-\Delta_x + 1}\,u = 0$$

- $a(\xi) = \xi^3$ $(n = 1) \cdots$ Korteweg-de Vries (shallow water wave)

$$\partial_t u + \partial_x^3 u + u\partial_x u = 0$$

- $a(\xi) = |\xi|\xi$ $(n = 1) \cdots$ Benjamin-Ono (deep water wave)

$$\partial_t u - \partial_x |D_x| u + u \partial_x u = 0$$

- $a(\xi) = \xi_1^2 - \xi_2^2$ $(n = 2) \cdots$ Davey-Stewartson (shallow water wave of 2D)

$$\begin{cases} i\partial_t u - \partial_x^2 u + \partial_y^2 u = c_1 |u|^2 u + c_2 u \partial_x v \\ \partial_x^2 v - \partial_y^2 v = \partial_x |u|^2 \end{cases}$$

- $a(\xi) = \xi_1^3 + \xi_2^3, \xi_1^3 + 3\xi_2^2, \xi_1^2 + \xi_1 \xi_2^2 \cdots$ Shrira (deep water wave of 2D)
- $a(\xi) =$ quadratic form $(n \geq 3) \cdots$ Zakharov-Schulman (interaction of sound wave and low amplitudes high frequency wave)

There has already been a lot of literature on this subject from various points of view. See, Ben-Artzi and Devinatz [2, 3], Ben-Artzi and Klainerman [4], Chihara [5], Hoshiro [7, 8], Kato and Yajima [11], Kenig, Ponce and Vega [12–16], Linares and Ponce [17], Simon [22], Sugimoto [24, 25], Walther [27, 28], and many others. We note that for a given operator A the following are equivalent to each other based on classical works by Agmon [1] and Kato [9]:

- Smoothing estimate

$$\left\| A e^{-it\Delta_x} \varphi(x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^n)} \leq C \|\varphi\|_{L^2(\mathbb{R}_x^n)} \quad \text{where } A = A(X, D_x),$$

- Restriction estimate

$$\left\| \widehat{A^* f}_{|S_\rho^{n-1}} \right\|_{L^2(S_\rho^{n-1})} \leq C\sqrt{\rho} \|f\|_{L^2(\mathbb{R}^n)}, \quad \text{where } S_\rho^{n-1} = \left\{ \xi; |\xi| = \rho \right\} \ (\rho > 0),$$

- Resolvent estimate

$$\sup_{\operatorname{Im} \zeta > 0} \left| \left(R(\zeta) A^* f, A^* f \right) \right| \leq C \|f\|_{L^2(\mathbb{R}^n)}^2, \quad \text{where } R(\zeta) = (-\triangle - \zeta)^{-1}.$$

Most of the literature so far use the above equivalence to show smoothing estimates for dispersive equations by showing restriction or resolvent estimates instead.

But here we develop a completely different strategy. We investigate smoothing estimates by using methods of comparison and canonical transform which are quite efficient for this problem:

1. *Comparison principle* $\cdots$ comparison of symbols implies that of estimates,
2. *Canonical transform* $\cdots$ transform an equation to another simple one.

They work not only for all the dispersive equations (that is, the case $\nabla a \neq 0$) but also for some non-dispersive equations, and induce smoothing estimates of an invariant form. Smoothing estimates for inhomogeneous equations can be also discussed by a similar treatment. We will explain them in due order.

13.2 Comparison Principle

Here we list theorems exemplifying the comparison principle, which have been established in Sect. 2 in [18]:

Theorem 1 (1D case) *Let* $f, g \in C^1(\mathbb{R})$ *be real-valued and strictly monotone. If* $\sigma, \tau \in C^0(\mathbb{R})$ *satisfy*

$$\frac{|\sigma(\xi)|}{|f'(\xi)|^{1/2}} \le A \frac{|\tau(\xi)|}{|g'(\xi)|^{1/2}}$$

then we have

$$\left\|\sigma(D_x)e^{itf(D_x)}\varphi(x)\right\|_{L^2(\mathbb{R}_t)} \le A\left\|\tau(D_x)e^{itg(D_x)}\varphi(x)\right\|_{L^2(\mathbb{R}_t)}$$

for all $x \in \mathbb{R}$.

Theorem 2 (2D case) *Let* $f(\xi,\eta), g(\xi,\eta) \in C^1(\mathbb{R}^2)$ *be real-valued and strictly monotone in* $\xi \in \mathbb{R}$ *for each fixed* $\eta \in \mathbb{R}$. *If* $\sigma, \tau \in C^0(\mathbb{R}^2)$ *satisfy*

$$\frac{|\sigma(\xi,\eta)|}{|f_\xi(\xi,\eta)|^{1/2}} \le A \frac{|\tau(\xi,\eta)|}{|g_\xi(\xi,\eta)|^{1/2}}$$

then we have

$$\begin{aligned}&\left\|\sigma(D_x,D_y)e^{itf(D_x,D_y)}\varphi(x,y)\right\|_{L^2(\mathbb{R}_t\times\mathbb{R}_y)}\\ &\quad\le A\left\|\tau(D_x,D_y)e^{itg(D_x,D_y)}\varphi(x,y)\right\|_{L^2(\mathbb{R}_t\times\mathbb{R}_y)}\end{aligned}$$

for all $x \in \mathbb{R}$.

Theorem 3 (Radially Symmetric case) *Let* $f, g \in C^1(\mathbb{R}_+)$ *be real-valued and strictly monotone. If* $\sigma, \tau \in C^0(\mathbb{R}_+)$ *satisfy*

$$\frac{|\sigma(\rho)|}{|f'(\rho)|^{1/2}} \le A \frac{|\tau(\rho)|}{|g'(\rho)|^{1/2}}$$

then we have

$$\left\|\sigma\big(|D_x|\big)e^{itf(|D_x|)}\varphi(x)\right\|_{L^2(\mathbb{R}_t)} \le A\left\|\tau\big(|D_x|\big)e^{itg(|D_x|)}\varphi(x)\right\|_{L^2(\mathbb{R}_t)}$$

for all $x \in \mathbb{R}^n$.

13.3 Canonical Transforms

Next we will review the idea of canonical transforms discussed in Sect. 4 in [18]. It is based on the so-called Egorov's theorem.

Let $\psi : \Gamma \to \widetilde{\Gamma}$ be a C^∞-diffeomorphism between open sets $\Gamma \subset \mathbb{R}^n$ and $\widetilde{\Gamma} \subset \mathbb{R}^n$. We always assume that

$$C^{-1} \le \left|\det \partial\psi(\xi)\right| \le C \quad (\xi \in \Gamma),$$

for some $C > 0$. We set formally

$$I_\psi u(x) = \mathcal{F}^{-1}\big[\mathcal{F}u\big(\psi(\xi)\big)\big](x) = (2\pi)^{-n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i(x\cdot\xi - y\cdot\psi(\xi))} u(y)dyd\xi.$$

The operators I_ψ can be justified by using cut-off functions $\gamma \in C^\infty(\Gamma)$ and $\widetilde{\gamma} = \gamma \circ \psi^{-1} \in C^\infty(\widetilde{\Gamma})$ which satisfy $\operatorname{supp}\gamma \subset \Gamma$, $\operatorname{supp}\widetilde{\gamma} \subset \widetilde{\Gamma}$. We set

$$\begin{aligned} I_{\psi,\gamma} u(x) &= \mathcal{F}^{-1}\big[\gamma(\xi)\mathcal{F}u\big(\psi(\xi)\big)\big](x) \\ &= (2\pi)^{-n} \int_{\mathbb{R}^n}\int_{\Gamma} e^{i(x\cdot\xi - y\cdot\psi(\xi))} \gamma(\xi) u(y)dyd\xi. \end{aligned} \tag{1}$$

In the case that $\Gamma, \widetilde{\Gamma} \subset \mathbb{R}^n \setminus 0$ are open cones, we may consider the homogeneous functions ψ and γ which satisfy $\operatorname{supp}\gamma \cap \mathbb{S}^{n-1} \subset \Gamma \cap \mathbb{S}^{n-1}$ and $\operatorname{supp}\widetilde{\gamma} \cap \mathbb{S}^{n-1} \subset \widetilde{\Gamma} \cap \mathbb{S}^{n-1}$, where $\mathbb{S}^{n-1} = \{\xi \in \mathbb{R}^n : |\xi| = 1\}$. Then we have the expressions for compositions

$$I_{\psi,\gamma} = \gamma(D_x) \cdot I_\psi = I_\psi \cdot \widetilde{\gamma}(D_x)$$

and also the formula

$$I_{\psi,\gamma} \cdot \sigma(D_x) = (\sigma \circ \psi)(D_x) \cdot I_{\psi,\gamma}. \tag{2}$$

We also introduce the weighted L^2-spaces. For a weight function $w(x)$, let $L^2(\mathbb{R}^n; w)$ be the set of measurable functions $f : \mathbb{R}^n \to \mathbb{C}$ such that the norm

$$\|f\|_{L^2(\mathbb{R}^n;w)} = \left(\int_{\mathbb{R}^n} \left|w(x)f(x)\right|^2 dx\right)^{1/2}$$

is finite. Then, on account of the relations (2), we obtain the following fundamental theorem (Theorem 4.1 in [18]):

Theorem 4 *Assume that the operator $I_{\psi,\gamma}$ defined by (1) is $L^2(\mathbb{R}^n; w)$-bounded. Suppose that we have the estimate*

$$\left\|w(x)\rho(D_x)e^{it\sigma(D_x)}\varphi(x)\right\|_{L^2(\mathbb{R}_t\times\mathbb{R}^n_x)} \le C\|\varphi\|_{L^2(\mathbb{R}^n_x)}$$

for all φ such that $\operatorname{supp}\widehat{\varphi} \subset \operatorname{supp}\widetilde{\gamma}$, where $\widetilde{\gamma} = \gamma \circ \psi^{-1}$. Assume also that the function

$$q(\xi) = \frac{\gamma \cdot \zeta}{\rho \circ \psi}(\xi)$$

is bounded. Then we have

$$\left\| w(x)\zeta(D_x)e^{ita(D_x)}\varphi(x)\right\|_{L^2(\mathbb{R}_t\times\mathbb{R}^n_x)} \le C\|\varphi\|_{L^2(\mathbb{R}^n_x)}$$

for all φ *such that* $\operatorname{supp}\widehat{\varphi}\subset\operatorname{supp}\gamma$, *where* $a(\xi)=(\sigma\circ\psi)(\xi)$.

Note that $e^{ita(D_x)}\varphi(x)$ and $e^{it\sigma(D_x)}\varphi(x)$ are solutions to

$$\begin{cases}(i\partial_t+a(D_x))u(t,x)=0,\\ u(0,x)=\varphi(x),\end{cases}\quad\text{and}\quad\begin{cases}(i\partial_t+\sigma(D_x))v(t,x)=0,\\ v(0,x)=g(x),\end{cases}$$

respectively. Theorem 4 means that smoothing estimates for the equation with $\sigma(D_x)$ implies those with $a(D_x)$ if the canonical transformations which relate them are bounded on weighted L^2-spaces.

As for the $L^2(\mathbb{R}^n;w)$-boundedness of the operator $I_{\psi,\gamma}$, we have criteria for some special weight functions. For $\kappa\in\mathbb{R}$, let $L^2_\kappa(\mathbb{R}^n)$ be the set of measurable functions f such that the norm

$$\|f\|_{L^2_\kappa(\mathbb{R}^n)}=\left(\int_{\mathbb{R}^n}\left|\langle x\rangle^\kappa f(x)\right|^2dx\right)^{1/2}$$

is finite. Then we have the following mapping properties (Theorems 4.2, 4.3 in [18]).

Theorem 5 *Let* $\Gamma,\widetilde{\Gamma}\subset\mathbb{R}^n\setminus 0$ *be open cones. Suppose* $|\kappa|<n/2$. *Assume* $\psi(\lambda\xi)=\lambda\psi(\xi)$, $\gamma(\lambda\xi)=\gamma(\xi)$ *for all* $\lambda>0$ *and* $\xi\in\Gamma$. *Then the operator* $I_{\psi,\gamma}$ *defined by* (1) *is* $L^2_\kappa(\mathbb{R}^n)$*-bounded.*

Theorem 6 *Suppose* $\kappa\in\mathbb{R}$. *Assume that all the derivatives of entries of the* $n\times n$ *matrix* $\partial\psi$ *and those of* γ *are bounded. Then the operator* $I_{\psi,\gamma}$ *defined by* (1) *are* $L^2_\kappa(\mathbb{R}^n)$*-bounded.*

13.4 Smoothing Estimates for Dispersive Equations

We consider smoothing estimates for solutions $u(t,x)=e^{ita(D_x)}\varphi(x)$ to general equations

$$\begin{cases}(i\partial_t+a(D_x))u(t,x)=0,\\ u(0,x)=\varphi(x)\in L^2(\mathbb{R}^n).\end{cases}$$

Let $a_m(\xi)$ be the principal term of $a(\xi)$ satisfying

$$a_m(\xi)\in C^\infty\big(\mathbb{R}^n\setminus 0\big),\qquad\text{real-valued},\qquad a_m(\lambda\xi)=\lambda^m a_m(\xi)\quad(\lambda>0,\xi\ne 0).$$

We assume that $a(\xi)$ is *dispersive* in the following sense:

$$a(\xi)=a_m(\xi),\qquad \nabla a_m(\xi)\ne 0\quad\big(\xi\in\mathbb{R}^n\setminus 0\big),\tag{H}$$

or, otherwise, we assume

$$\begin{aligned} &a(\xi) \in C^\infty\big(\mathbb{R}^n\big), \qquad \nabla a(\xi) \neq 0 \quad \big(\xi \in \mathbb{R}^n\big), \qquad \nabla a_m(\xi) \neq 0 \quad \big(\xi \in \mathbb{R}^n \setminus 0\big), \\ &\big|\partial^\alpha\big(a(\xi) - a_m(\xi)\big)\big| \le C_\alpha |\xi|^{m-1-|\alpha|} \quad \text{for all multi-indices } \alpha \text{ and all } |\xi| \ge 1. \end{aligned} \tag{L}$$

Example 1 $a(\xi) = \xi_1^3 + \cdots + \xi_n^3 + \xi_1$ satisfies (L).

The dispersiveness means that the classical orbit, that is, the solution of the Hamilton-Jacobi equations

$$\begin{cases} \dot{x}(t) = (\nabla a)(\xi(t)), & \dot{\xi}(t) = 0, \\ x(0) = 0, & \xi(0) = k, \end{cases}$$

does not stop, and the singularity of $u(t,x) = e^{ita(D_x)}\varphi(x)$ travels to infinity along this orbit. Hence we can expect the smoothing, and indeed we have the following result (Theorem 5.1, Corollary 5.5 in [18]):

Theorem 7 *Assume* (H) *or* (L). *Suppose* $m \ge 1$ *and* $s > 1/2$. *Then we have*

$$\big\| \langle x \rangle^{-s} |D_x|^{(m-1)/2} e^{ita(D_x)} \varphi(x) \big\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^n)} \le C \|\varphi\|_{L^2(\mathbb{R}^n)}.$$

Remark 1 Theorem 7 with polynomials $a(\xi)$ follows immediately from a sharp version of local smoothing estimate proved by Theorem 4.1 of Kenig, Ponce and Vega [12], and any polynomial $a(\xi)$ which satisfies the estimate in Theorem 7 has to be dispersive, that is $\nabla a_m(\xi) \neq 0$ $(\xi \neq 0)$ (see Hoshiro [8]). Theorem 7 with $a(\xi) = |\xi|^2$ and $n \ge 3$ was also stated by Ben-Artzi and Klainerman [4], and with the case (H) and $m > 1$ by Chihara [5] in different contexts.

13.5 Proof by New Methods

We explain how to prove Theorem 7 under the condition (H) by our new method. The main strategy is that we obtain estimates for low dimensional model cases from some trivial estimate by the comparison principle, and reduce general case to such model cases by the method of canonical transforms.

13.5.1 Low Dimensional Model Estimates

By the comparison principle, we can show the equivalence of low dimensional estimates of various type. In the 1D case, we have (for $l, m > 0$)

$$\sqrt{m} \big\| |D_x|^{(m-1)/2} e^{it|D_x|^m} \varphi(x) \big\|_{L^2(\mathbb{R}_t)} = \sqrt{l} \big\| |D_x|^{(l-1)/2} e^{it|D_x|^l} \varphi(x) \big\|_{L^2(\mathbb{R}_t)} \tag{3}$$

for all $x \in \mathbb{R}$. Here $\operatorname{supp} \widehat{\varphi} \subset [0, +\infty)$ or $(-\infty, 0]$.

In the 2D case, we have (for $l, m > 0$)

$$\begin{aligned} &\left\| |D_y|^{(m-1)/2} e^{itD_x|D_y|^{m-1}} \varphi(x,y) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_y)} \\ &= \left\| |D_y|^{(l-1)/2} e^{itD_x|D_y|^{l-1}\varphi(x,y)} \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_y)} \end{aligned} \tag{4}$$

for all $x \in \mathbb{R}$. On the other hand, in 1D case, we have trivially

$$\left\| e^{itD_x} \varphi(x) \right\|_{L^2(\mathbb{R}_t)} = \left\| \varphi(x+t) \right\|_{L^2(\mathbb{R}_x)} = \|\varphi\|_{L^2(\mathbb{R}_x)} \tag{5}$$

for all $x \in \mathbb{R}$. Using the equality (5), the right hand sides of (3) and (4) with $l = 1$ can be estimated, and we have for all $x \in \mathbb{R}$:

- (1D Case)

$$\left\| |D_x|^{(m-1)/2} e^{it|D_x|^m} \varphi(x) \right\|_{L^2(\mathbb{R}_t)} \le C\|\varphi\|_{L^2(\mathbb{R}_x)},$$

- (2D Case)

$$\left\| |D_y|^{(m-1)/2} e^{itD_x|D_y|^{m-1}} \varphi(x,y) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_y)} \le C\|\varphi\|_{L^2(\mathbb{R}^2_{x,y})}.$$

Remark 2 In the case $m = 2$, these estimates were proved by Kenig, Ponce & Vega [12] (1D case) and Linares & Ponce [17] (2D case).

The following is a straightforward consequence from these estimates:

Proposition 1 *Suppose $m > 0$ and $s > 1/2$. Then for $n \ge 1$ we have*

$$\left\| \langle x \rangle^{-s} |D_n|^{(m-1)/2} e^{it|D_n|^m} \varphi(x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}^n_x)} \le C\|\varphi\|_{L^2(\mathbb{R}^n_x)}$$

and for $n \ge 2$ we have

$$\left\| \langle x \rangle^{-s} |D_n|^{(m-1)/2} e^{itD_1|D_n|^{m-1}} \varphi(x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}^n_x)} \le C\|\varphi\|_{L^2(\mathbb{R}^n_x)},$$

where $D_x = (D_1, \ldots, D_n)$.

13.5.2 Reduction to Model Estimates

On account of the method of canonical transform (Theorem 4), smoothing estimates for dispersive equations (Theorem 7) can be reduced to low dimensional model estimates (Proposition 1) by the canonical transformation if we find a homogeneous change of variable ψ such that

$$a(\xi) = (\sigma \circ \psi)(\xi), \qquad \sigma(D) = |D_n|^m \quad \text{or} \quad \sigma(D) = D_1|D_n|^{m-1}.$$

We show how to select such ψ under the assumption (H). The argument for the case (L) is similar. By microlocalisation and rotation, we may assume that the initial data φ satisfies $\operatorname{supp}\hat{\varphi} \subset \Gamma$, where $\Gamma \subset \mathbb{R}^n \setminus 0$ is a sufficiently small conic neighbourhood of $e_n = (0, \ldots, 0, 1)$. Furthermore, we have Euler's identity

$$a(\xi) = a_m(\xi) = \frac{1}{m}\xi \cdot \nabla a(\xi),$$

and the dispersiveness $\nabla a(e_n) \neq 0$ implies the following two cases:

(I) $\partial_n a(e_n) \neq 0 \cdots$ (elliptic). By Euler's identity, we have $a(e_n) \neq 0$. Hence, in this case, we may assume $a(\xi) > 0$ ($\xi \in \Gamma$), $\partial_n a(e_n) \neq 0$.
(II) $\partial_n a(e_n) = 0 \cdots$ (non-elliptic). By assumption $\nabla a(e_n) \neq 0$, there exits $j \neq n$ such that $\partial_j a(e_n) \neq 0$. Hence, in this case, we may assume $\partial_1 a(e_n) \neq 0$.

In the elliptic case (I), we take

$$\sigma(\eta) = |\eta_n|^m, \qquad \psi(\xi) = \left(\xi_1, \ldots, \xi_{n-1}, a(\xi)^{1/m}\right).$$

Then we have $a(\xi) = (\sigma \circ \psi)(\xi)$, and ψ is surely a change of variables on Γ since

$$\det \partial\psi(e_n) = \begin{vmatrix} E_{n-1} & 0 \\ * & \frac{1}{m}a(e_n)^{1/m-1}\partial_n a(e_n) \end{vmatrix} \neq 0$$

where E_{n-1} is the identity matrix. In the non-elliptic case (II), we take

$$\sigma(\eta) = \eta_1|\eta_n|^{m-1}, \qquad \psi(\xi) = \left(\frac{a(\xi)}{|\xi_n|^{m-1}}, \xi_2, \ldots, \xi_n\right).$$

Then we have again $a(\xi) = (\sigma \circ \psi)(\xi)$ and

$$\det \partial\psi(e_n) = \begin{vmatrix} \partial_1 a(e_n) & * \\ 0 & E_{n-1} \end{vmatrix} \neq 0.$$

Thus, we successfully showed Theorem 7 in both cases.

13.6 Non-dispersive Case

Now we consider what happens if the equation does not satisfy the dispersiveness assumption $\nabla a(\xi) \neq 0$ ($\xi \in \mathbb{R}^n$). All the precise results and arguments in this section are to appear in our forthcoming paper [19].

Although we cannot have smoothing estimates (see Remark 1), such case appears naturally in physics. For example, let us consider a coupled system of Schrödinger equations

$$i\partial_t v = \Delta_x v + b(D_x)w, \qquad i\partial_t w = \Delta_x w + c(D_x)v,$$

which represents a linearised model of wave packets with two modes. Assume that this system is diagonalised and regard it as a single equations for the eigenvalues:

$$a(\xi) = -|\xi|^2 \pm \sqrt{b(\xi)c(\xi)}.$$

Then there could exist points ξ such that $\nabla a(\xi) = 0$ because of the lower order terms $b(\xi)$, $c(\xi)$. Another interesting examples are Shrira equations, in which case:

$$a(\xi) = \xi_1^3 + \xi_2^3, \qquad \xi_1^3 + 3\xi_2^2, \qquad \xi_1^2 + \xi_1\xi_2^2.$$

Although $a(\xi) = \xi_1^3 + \xi_2^3$ satisfies assumption (H), $a(\xi) = \xi_1^3 + 3\xi_2^2$ and $a(\xi) = \xi_1^2 + \xi_1\xi_2^2$ do not satisfy assumption (L) because $\nabla a(0) = 0$.

We suggest an estimate which we expect to hold for non-dispersive equations:

$$\left\| \langle x\rangle^{-s} \left|\nabla a(D_x)\right|^{1/2} e^{ita(D_x)}\varphi(x)\right\|_{L^2(\mathbb{R}_t\times\mathbb{R}^n_x)} \le C\|\varphi\|_{L^2(\mathbb{R}^n_x)} \quad (s > 1/2) \tag{6}$$

and let us call it *invariant estimate*. This estimate has a number of advantages:

- in the dispersive case $\nabla a(\xi) \neq 0$, it is equivalent to Theorem 7;
- it is invariant under canonical transformations for the operator $a(D_x)$;
- it does continue to hold for a variety of non-dispersive operators $a(D_x)$, where $\nabla a(\xi)$ may become zero on some set and when the usual estimate fails;
- it does take into account zeros of the gradient $\nabla a(\xi)$, which is also responsible for the interface between dispersive and non-dispersive zone (e.g. how quickly the gradient vanishes).

13.6.1 Secondary Comparison

By using comparison principle again to the smoothing estimates obtained from the comparison principle, we can have new estimates. This is a powerful tool to induce the invariant estimates (6) for non-dispersive equations. For example, we have just obtained the estimate

$$\left\| \langle x\rangle^{-s} |D_x|^{(m-1)/2} e^{it|D_x|^m}\varphi\right\|_{L^2(\mathbb{R}_t\times\mathbb{R}^n_x)} \le C\|\varphi\|_{L^2(\mathbb{R}^n_x)}$$

(Theorem 7 with $a(\xi) = |\xi|^m$) from comparison principle and canonical transformation. If we set $g(\rho) = \rho^m$, $\tau(\rho) = \rho^{(m-1)/2}$, then we have $|\tau(\rho)|/|g'(\rho)|^{1/2} = 1/\sqrt{m}$. Hence by the comparison result again for the radially symmetric case (Theorem 3), we have

Theorem 8 *Suppose $s > 1/2$. Let $f \in C^1(\mathbb{R}_+)$ be real-valued and strictly monotone. If $\sigma \in C^0(\mathbb{R}_+)$ satisfy*

$$\left|\sigma(\rho)\right| \le A\left|f'(\rho)\right|^{1/2},$$

then we have

$$\left\| \langle x \rangle^{-s} \sigma\big(|D_x|\big) e^{itf(|D_x|)} \varphi(x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}^n_x)} \le C \|\varphi\|_{L^2(\mathbb{R}^n_x)}.$$

From this secondary comparison, we obtain immediately the following invariant estimate since a radial function $a(\xi) = f(|\xi|)$ always satisfies $|\nabla a(\xi)| = |f'(|\xi|)|$.

Theorem 9 *Suppose $s > 1/2$. Let $a(\xi) = f(|\xi|)$ and $f \in C^\infty(\mathbb{R}_+)$ be real-valued. Then we have*

$$\left\| \langle x \rangle^{-s} \left| \nabla a(D_x) \right|^{1/2} e^{ita(D_x)} \varphi(x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}^n_x)} \le C \|\varphi\|_{L^2(\mathbb{R}^n_x)}.$$

Example 2 $a(\xi) = (|\xi|^2 - 1)^2$ is non-dispersive because

$$\nabla a(\xi) = 4\big(|\xi|^2 - 1\big)\xi = 0$$

if $|\xi| = 0, 1$. But we have the invariant estimate by Theorem 9.

For the non-radially symmetric case, we compare again to the low dimensional model estimates (Proposition 1) and obtain

Theorem 10 (1D secondary comparison) *Suppose $s > 1/2$. Let $f \in C^1(\mathbb{R})$ be real-valued and strictly monotone. If $\sigma \in C^0(\mathbb{R})$ satisfies*

$$\big|\sigma(\xi)\big| \le A \big|f'(\xi)\big|^{1/2},$$

then we have

$$\left\| \langle x \rangle^{-s} \sigma(D_x) e^{itf(D_x)} \varphi(x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x)} \le AC \left\| \varphi(x) \right\|_{L^2(\mathbb{R}_x)}.$$

Theorem 11 (2D secondary comparison) *Suppose $s > 1/2$. Let $f \in C^1(\mathbb{R}^2)$ be real-valued and $f(\xi, \eta)$ be strictly monotone in $\xi \in \mathbb{R}$ for every fixed $\eta \in \mathbb{R}$. If $\sigma \in C^0(\mathbb{R}^2)$ satisfies*

$$\big|\sigma(\xi, \eta)\big| \le A \big|\partial f / \partial \xi(\xi, \eta)\big|^{1/2},$$

then we have

$$\left\| \langle x \rangle^{-s} \sigma(D_x, D_y) e^{itf(D_x, D_y)} \varphi(x, y) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}^2_{x,y})} \le AC \left\| \varphi(x, y) \right\|_{L^2(\mathbb{R}^2_{x,y})}.$$

Example 3 By using secondary comparison for non-radially symmetric case, we have invariant estimates for Shrira equations. In fact, for $a(\xi) = \xi_1^3 + 3\xi_2^2$, we have by 1D secondary comparison (Theorem 10)

$$\left\| \langle x_1 \rangle^{-s} |D_1| e^{itD_1^3} \varphi(x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}^2_x)} \le C \|\varphi\|_{L^2(\mathbb{R}^2_x)},$$

$$\left\| \langle x_2 \rangle^{-s} |D_2|^{1/2} e^{it3D_2^2} \varphi(x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^2)} \le C \|\varphi\|_{L^2(\mathbb{R}_x^2)},$$

for $s > 1/2$. Hence by $\langle x \rangle^{-s} \le \langle x_k \rangle^{-s}$ $(k = 1, 2)$ we have

$$\left\| \langle x \rangle^{-s} \big(|D_1| + |D_2|^{1/2} \big) e^{ita(D_x)} \varphi(x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^2)} \le C \|\varphi\|_{L^2(\mathbb{R}_x^2)}$$

and hence we have

$$\left\| \langle x \rangle^{-s} \left| \nabla a(D_x) \right|^{1/2} e^{ita(D_x)} \varphi(x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^2)} \le C \|\varphi\|_{L^2(\mathbb{R}_x^2)}.$$

For $a(\xi) = \xi_1^2 + \xi_1 \xi_2^2$, we have by 2D secondary comparison (Theorem 11)

$$\left\| \langle x_1 \rangle^{-s} \left| 2D_1 + D_2^2 \right|^{1/2} e^{ita(D_1, D_2)} \varphi(x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^2)} \le C \|\varphi\|_{L^2(\mathbb{R}_x^2)},$$

$$\left\| \langle x_2 \rangle^{-s} |D_1 D_2|^{1/2} e^{ita(D_1, D_2)} \varphi(x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^2)} \le C \|\varphi\|_{L^2(\mathbb{R}_x^2)},$$

for $s > 1/2$, hence we have similarly

$$\left\| \langle x \rangle^{-s} \left| \nabla a(D_x) \right|^{1/2} e^{ita(D_x)} \varphi(x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^2)} \le C \|\varphi\|_{L^2(\mathbb{R}_x^2)}.$$

13.6.2 Non-dispersive Case Controlled by Hessian

We will show that in the non-dispersive situation the rank of $\nabla^2 a(\xi)$ still has a responsibility for smoothing properties.

First let us consider the case when dispersiveness (L) is true only for large ξ:

$$\begin{aligned} &\left| \nabla a(\xi) \right| \ge C \langle \xi \rangle^{m-1} \quad \big(|\xi| \gg 1 \big), \\ &\left| \partial^\alpha \big(a(\xi) - a_m(\xi) \big) \right| \le C \langle \xi \rangle^{m-1-|\alpha|} \quad \big(|\xi| \gg 1 \big). \end{aligned} \tag{L$'$}$$

Theorem 12 *Suppose $n \ge 1$, $m \ge 1$, and $s > 1/2$. Let $a \in C^\infty(\mathbb{R}^n)$ be real-valued and assume that it has finitely many critical points. Assume* (L$'$) *and*

$$\nabla a(\xi) = 0 \quad \Rightarrow \quad \det \nabla^2 a(\xi) \ne 0.$$

Then we have

$$\left\| \langle x \rangle^{-s} \left| \nabla a(D_x) \right|^{1/2} e^{ita(D_x)} \varphi(x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^n)} \le C \|\varphi\|_{L^2(\mathbb{R}_x^n)}.$$

Example 4 $a(\xi) = \xi_1^4 + \cdots + \xi_n^4 + |\xi|^2$ satisfies assumptions in Theorem 12.

We outline the proof of Theorem 12. For the region where $\nabla a(\xi) \ne 0$, we can use a smoothing estimate for dispersive equations. Near the points ξ where $\nabla a(\xi) = 0$, there exists a change of variable ψ by Morse's lemma such that $a(\xi) = (\sigma \circ \psi)(\xi)$

where $\sigma(\eta)$ is a non-degenerate quadratic form, and satisfies dispersiveness (H). Hence the estimate can be reduced to the dispersive case by the method of canonical transformation.

Next we consider the case when $a(\xi)$ is homogeneous (of oder m). Then, by Euler's identity, we have

$$\nabla a(\xi) = \frac{1}{m-1}\xi \nabla^2 a(\xi) \quad (\xi \neq 0),$$

hence

$$\nabla a(\xi) = 0 \quad \Rightarrow \quad \det \nabla^2 a(\xi) = 0 \quad (\xi \neq 0).$$

Therefore assumption in Theorem 12 does not make any sense in this case, but we can have the following result if we use the idea of canonical transform again:

Theorem 13 *Suppose $n \geq 2$ and $s > 1/2$. Let $a \in C^\infty(\mathbb{R}^n \setminus 0)$ be real-valued and satisfy $a(\lambda\xi) = \lambda^2 a(\xi)$ $(\lambda > 0, \xi \neq 0)$. Assume that*

$$\nabla a(\xi) = 0 \quad \Rightarrow \quad \operatorname{rank} \nabla^2 a(\xi) = n - 1 \quad (\xi \neq 0).$$

Then we have

$$\left\| \langle x \rangle^{-s} \left| \nabla a(D_x) \right|^{1/2} e^{ita(D_x)} \varphi(x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}^n_x)} \leq C \|\varphi\|_{L^2(\mathbb{R}^n_x)}.$$

Example 5 $a(\xi) = \frac{\xi_1^2 \xi_2^2}{\xi_1^2 + \xi_2^2} + \xi_3^2 + \cdots + \xi_n^2$ satisfies the assumptions in Theorem 13. In the case $n = 2$, this is an illustration of a smoothing estimate for the Cauchy problem for an equation like

$$i\partial_t u + D_1^2 D_2^2 \Delta^{-1} u = 0$$

which is regarded as a mixture of Davey-Stewartson and Benjamin-Ono type equations.

13.7 Concluding Remarks

13.7.1 Summary

Finally we summarise what is explained in this article in a diagram below. It is remarkable that all the results of smoothing estimates so far is derived from just the translation invariance of Lebesgue measure:

- Trivial estimate $\|\varphi(x+t)\|_{L^2(\mathbb{R}_t)} = \|\varphi\|_{L^2(\mathbb{R}_x)}$

$$\Downarrow \quad \text{(comparison principle)}$$

- Low dimensional model estimates (Proposition 1)

$\Downarrow$ (canonical transform)

- Smoothing estimates for dispersive equations (Theorem 7)

$\Downarrow$ (secondary comparison & canonical transform)

- Invariant estimates for non-dispersive equations at least for
  - $*$ radially symmetric $a(\xi) = f(|\xi|)$, $f \in C^1(\mathbb{R}_+)$,
  - $*$ Shrira equation $a(\xi) = \xi_1^3 + 3\xi_2^2, \xi_1^2 + \xi_1\xi_2^2$,
  - $*$ non-dispersive $a(\xi)$ controlled by its Hessian.

13.7.2 Smoothing Estimates for Inhomogeneous Equations

We finish this article by mentioning some results for inhomogeneous equations. Let us consider the solution

$$u(t, x) = -i \int_0^t e^{i(t-\tau)a(D_x)} f(\tau, x) d\tau$$

to the equation

$$\begin{cases} (i\partial_t + a(D_x))u(t, x) = f(t, x) & \text{in } \mathbb{R}_t \times \mathbb{R}^n_x, \\ u(0, x) = 0 & \text{in } \mathbb{R}^n_x. \end{cases}$$

Although smoothing estimates for such equation are necessary for nonlinear applications (see [21] for example), there are considerably less results on this topic available in the literature. But the method of canonical transform also works to this problem, and we will list here some recent achievement given in our forthcoming paper [20]. The following result is a counter part of Theorem 7. Especially, this kind of time-global estimate for the operator $a(D_x)$ with lower order terms are the benefit of our new method:

Theorem 14 *Assume* (H) *or* (L). *Suppose $n \geq 2$, $m \geq 1$, and $s > 1/2$. Then we have*

$$\left\| \langle x \rangle^{-s} |D_x|^{m-1} \int_0^t e^{i(t-\tau)a(D_x)} f(\tau, x) d\tau \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}^n_x)} \leq C \left\| \langle x \rangle^s f(t, x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}^n_x)}.$$

The proof of Theorem 14 is carried out by reducing it to model estimates below via canonical transform:

Proposition 2 *Suppose $n = 1$ and $m > 0$. Let $a(\xi) \in C^\infty(\mathbb{R} \setminus 0)$ be a real-valued function which satisfies $a(\lambda\xi) = \lambda^m a(\xi)$ for all $\lambda > 0$ and $\xi \neq 0$. Then we have*

$$\left\| a'(D_x) \int_0^t e^{i(t-\tau)a(D_x)} f(\tau, x) d\tau \right\|_{L^2(\mathbb{R}_t)} \leq C \int_{\mathbb{R}} \left\| f(t, x) \right\|_{L^2(\mathbb{R}_t)} dx$$

for all $x \in \mathbb{R}$. *Suppose* $n = 2$ *and* $m > 0$. *Then we have*

$$\left\| |D_x|^{m-1} \int_0^t e^{i(t-\tau)|D_x|^{m-1}D_y} f(\tau, x, y) d\tau \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x)} \le C \int_{\mathbb{R}} \| f(t, x, y) \|_{L^2(\mathbb{R}_t \times \mathbb{R}_x)} dy$$

for all $y \in \mathbb{R}$.

Remark 3 Proposition 2 with the case $n = 1$ is a unification of the results by Kenig, Ponce and Vega who treated the cases $a(\xi) = \xi^2$ (p. 258 in [14]), $a(\xi) = |\xi|\xi$ (p. 160 in [15]), and $a(\xi) = \xi^3$ (p. 533 in [13]).

Since we unfortunately do not know the comparison principle for inhomogeneous equations, we gave a direct proof to Proposition 2 in [20].

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Chapter 14
Differentiability of Inverse Operators

Simon Y. Serovajsky

Abstract The Inverse Function Theorem is a mighty tool of the local nonlinear analysis. It guarantees the existence of the inverse function and its differentiability. However the first property is sometimes not used. It is true, for example, for the optimal control theory and the inverse problems of mathematical physics. The inverse operator can be interpreted as a control-state mapping here. Its existence is a corollary of the state equation properties, and the differentiability of the inverse operator is used for the differentiation of the minimizing functional or the discrepancy. We establish a differentiability criterion of the inverse operator. Moreover, we prove a property which can be interpreted as a weak form of the operator differentiability. The Dirichlet problem for a nonlinear elliptic equation is considered as an example.

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14.1 Introduction

Consider an operator $A\colon V \to Y$, where V and Y are Banach spaces. Suppose that it is continuously differentiable at a neighborhood of a point $y_0 \in Y$. Denote by $A'(y_0)$ the derivative of the operator A at the point y_0. It is well known that the following result holds (see, for example, [1]).

The Inverse Function Theorem *Assume that there exists the continuous inverse operator $A'(y_0)^{-1}$. Then there exists an open neighborhood O of the point y_0 such that the set $O' = A(O)$ is an open neighborhood of the point $v_0 = Ay_0$; moreover, there exists the continuously differentiable inverse map $A^{-1} : O' \to O$, and its derivative is defined by the formula*

$$\big(A^{-1}\big)'(v) = \big\{A'\big[A^{-1}(v)\big]\big\}^{-1} \quad \forall v \in O'.$$

This result has very important applications. It has relationships to the Implicit Function Theorem [2], Newton–Kantorovich Method [1, 2], Lusternik Smooth

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Manifold Approximation Theorem [3, 4], Brower Fixed Point Theorem [1, 5], Morse Smooth Function Singularity Lemma [1], Graves Cover Theorem [4], etc. Extensions of the Inverse Function Theorem to high orders differentiability [6], non-smooth operators [7–9], multiple-valued maps [1, 7, 10], etc., are also known.

In reality the Inverse Function Theorem involve two different results. These are the invertibility of the given operator and the differentiability of the corresponding inverse operator. Sometimes only the second property is important. It is true, for example, for the extremum theory and the inverse problems theory. In particular, consider the system described by the equation

$$Ay = v. \tag{1}$$

The term v can be interpreted here as a control or an identifiable parameter, and y is a state function. Suppose that (1) has a unique solution $y = y(v)$ from the space Y for all values $v \in V$. Then the operator A is invertible. This result can be proved by some tools which are applicable to the given equation. Therefore, it is not necessary to use of the Inverse Function Theorem here.

Let U be a convex closed subset of the space V. The state functional is defined by the formula

$$I(v) = J(v) + K\big[y(v)\big],$$

where J is a functional on the set V, and K is a functional on the set Y. We have the following optimization control problem.

Problem 1 Minimize the functional I on the set U.

A necessary condition for the minimum of a smooth functional F on a convex set W at a point v_0 is the variational inequality (see [11])

$$\big\langle F'(v_0), v - v_0\big\rangle \geq 0 \quad \forall v \in W, \tag{2}$$

where $\langle \lambda, \varphi \rangle$ is the value of the linear continuous functional λ at the point φ.

The functional I is the sum of J and the map $v \to K[y(v)]$. The last mapping is the superposition of the functional K and the map $v \to y(v)$, which is, in fact, the inverse operator A^{-1}. Then the proof of the differentiability of the given functional requires the differentiation of the inverse operator. This result can be obtained using the Inverse Function Theorem.

Lemma 1 *Suppose that the operator A has a continuous inverse operator, which is continuously differentiable at an open neighborhood of the point $y_0 = y(v_0)$, and there exists the continuous inverse operator $A'(y_0)^{-1}$. Then the map $y(\cdot)\colon V \to Y$ is Gateaux differentiable at the point v_0, and its derivative satisfies the formula*

$$\big\langle \mu, y'(v_0)h\big\rangle = \big\langle p_\mu(v_0), h\big\rangle \quad \forall \mu \in Y^*, h \in V, \tag{3}$$

where Y^* is the adjoint space of Y, and $p_\mu(v_0)$ is the solution of the equation

$$\big[A'(y_0)\big]^* p_\mu(v_0) = \mu. \tag{4}$$

Proof By the Inverse Function Theorem the map $y(\cdot) : V \to Y$ is differentiable at the point v_0, and, moreover,

$$y'(v_0) = \left[A'(y_0)\right]^{-1}.$$

Then we get

$$\left\langle \mu, y'(v_0)h \right\rangle = \left\langle \mu, \left[A'(y_0)\right]^{-1} h \right\rangle = \left\langle \left\{ \left[A'(y_0)\right]^{-1} \right\}^{*} \mu, h \right\rangle \quad \forall \mu \in Y^{*}, h \in V.$$

It is known that each linear operator and its adjoint operator are invertible at the same time (see p. 460 in [2]). Therefore (4) has a unique solution

$$p_{\mu}(v_0) = \left\{ \left[A'(y_0)\right]^{-1} \right\}^{*} \mu$$

from the space V^{*}. So the previous formula can be transformed to (4), and the equality (3) is true. □

Now we can prove the differentiability of the functional I and obtain necessary conditions of optimality. Let v_0 be the solution of the minimization problem for the functional I on the set U. Define $y_0 = y(v_0)$.

Lemma 2 *Under the conditions of Lemma 1 suppose that the functional J is Gateaux differentiable at the point v_0, and the functional K is Frechet differentiable at the point y_0. Then the control v_0 satisfies the variational inequality*

$$\left\langle J'(v_0) - p_0, v - v_0 \right\rangle \geq 0 \quad \forall v \in U, \tag{5}$$

where p_0 is a solution of the adjoint equation

$$\left[A'(y_0)\right]^{*} p_0 = -K'(y_0). \tag{6}$$

Proof Using the Composite Function Theorem (see p. 637 in [2]), we obtain that the Gateaux derivative of the map $v \to K[y(v)]$ exists such that

$$(Ky)'(v_0) = K'(y_0)y'(v_0).$$

By equality (3) we get

$$\left\langle (Ky)'(v_0), h \right\rangle = \left\langle K'(y_0), y'(v_0)h \right\rangle = -\langle p_0, h \rangle \quad \forall h \in V,$$

where p_0 is the solution of (4) for $\mu = -K'(y_0)$. Thus, we obtain the adjoint equation (6). So the derivative of the map $v \to K[y(v)]$ at the point v_0 equals to $-p_0$. Then the functional I has the derivative

$$I'(v_0) = J'(v_0) - p_0$$

at this point. Using (2), we obtain the variational inequality (5). □

Thus the Inverse Function Theorem is a good tool for proving the differentiability of the control-state mapping. This result is the basis for obtaining necessary optimality conditions. Note that we use now the serious assumption of the invertibility of the operator's derivative. It is equivalent to the existence of the unique solution $y \in Y$ for the linearized equation

$$A'(y_0)y = v \tag{7}$$

for all $v \in V$.

Now we have the following questions:

- How large is the class of operators that satisfy the mentioned assumption?
- What is the criterion of the differentiability of the inverse operator at a concrete point?
- Could we prove the differentiability of the inverse operator without using the Inverse Function Theorem?
- Could we prove a weaker form of the differentiability of the inverse operator for obtaining optimality conditions in the case of non-invertibility of the operator's derivative?

We will try to answer these questions.

14.2 Criterion for the Differentiability of the Inverse Operator

Consider an operator $A : Y \to V$. Let it be continuous and differentiable at a neighborhood of a point $y_0 \in Y$.

Theorem 1 *Suppose the existence of an open neighborhood O of the point y_0 such that the set $O' = A(O)$ is an open neighborhood of the point $v_0 = Ay_0$. Suppose that there exists the inverse operator $A^{-1} : O' \to O$, and that (7) has not more than one solution. Then this inverse operator is Gateaux differentiable at v_0 if and only if the derivative $A'(y_0)$ is a surjection.*

Proof Let the derivative $A'(y_0)$ be a surjection. Then it is invertible by the assumptions of the theorem. By Banach Inverse Operator Theorem there exists the continuous inverse operator $A'(y_0)^{-1}$. Therefore, the differentiability of the operator A^{-1} at the point v_0 follows from the Inverse Function Theorem directly.

Suppose now that the operator A^{-1} has the Gateaux derivative D at y_0, and that the derivative $A'(y_0)$ is not a surjection. We get the equality

$$Ay(v_0 + \sigma v) - Ay(v_0) = \sigma v$$

for all $v \in V$ and small enough number σ. Dividing it by σ and passing to the limit as $\sigma \to 0$, using the Composite Function Theorem and differentiability of A^{-1}, we get

$$A'(y_0)Dv = v.$$

Then there exists a point $y = Dv$ from Y such that $A'(y_0)y = v$. So the derivative $A'(y_0)$ is a surjection. However this conclusion contradicts our assumption. Hence, the operator $A'(y_0)$ is a surjection whenever the inverse operator is differentiable. □

Thus Gateaux differentiability of the inverse operator is equivalent to the following property: *the operator* $A'(y_0)$ *is a surjection*. It is called *Lusternik Condition* [4].

Consider as an example the homogeneous Dirichlet Problem for the equation

$$-\Delta y + |y|^\rho y = v \tag{8}$$

in the n-dimensional bounded set Ω, where $\rho > 0$. Denote the space

$$Y = H_0^1(\Omega) \cap L_q(\Omega),$$

where $q = \rho + 2$. Using Monotone Operators Theory [12], we obtain that this boundary problem has the unique solution $y \in Y$ for all v from the set V, which is the adjoint space

$$Y^* = H^{-1}(\Omega) + L_{q'}(\Omega),$$

where $1/q + 1/q' = 1$. Denote the operator $A : Y \to V$ such that Ay equals to the left side of the equality (8). The existence of the operator A^{-1} follows from the one-valued solvability of the boundary problem. Its differentiability can be obtained by using the properties of the linearized equation. It is the homogeneous Dirichlet Problem for the equation

$$-\Delta y + (\rho + 1)|y_0|^\rho y = v. \tag{9}$$

Corollary 1 *The solution of the Dirichlet problem for* (8) *is Gateaux differentiable with respect to the absolute term at the point* $v = v_0$ *iff* (9) *has a solution* $y \in Y$ *for all* $v \in V$.

Indeed, the continuous differentiability of the given operator A is obvious. The existence of the inverse operator follows from the one-valued solvability of the given boundary problem. It is obvious that the Dirichlet problem for the linear equation (9) cannot have two solutions. Then the criterion for the invertibility of the inverse operator is the Lusternik condition, by Theorem 1.

Now we obtain a criterion for the differentiability of the solution of (8) with respect to the absolute term on the space V.

Corollary 2 *The solution of the Dirichlet problem for* (8) *is Gateaux differentiable with respect to the absolute term at an arbitrary point if and only if the embedding* $H_0^1(\Omega) \subset L_q(\Omega)$ *is true.*

Proof Multiply equality (9) by the function y and integrate the result in $x \in \Omega$ using the Green formula and the boundary condition. We get

$$\int_{\Omega} |\nabla y|^2 dx + (\rho+1) \int_{\Omega} |y_0|^{\rho} y^2 dx = \int_{\Omega} v y dx.$$

We have $Y = H_0^1(\Omega)$ by the given assumption, hence $V = H^{-1}(\Omega)$. So the a-priori estimate of the solution of (9) in the sense of Y for all $v \in V$ follows from the obtained equality. Now we get the one-valued solvability of the linearized equation by means of the standard theory of elliptic equations (see, for example, [11]). Thus the differentiability of the solution of (8) with respect to the absolute term at an arbitrary point follows from Corollary 1.

We prove now that the solution of (8) is not differentiable with respect to its absolute term, if the mentioned embedding does not hold. Let y_0 be a continuous function from the space Y. Then the left side of the equality (9) is a point of the space $H^{-1}(\Omega)$ for all $y \in Y$. Therefore, the image of the derivative $A'(y_0)$ is narrower than the set V, if the mentioned embedding does not hold. So (9) does not have any solutions from the space Y for all function v from the difference $V \setminus H^{-1}(\Omega)$. Therefore, the solution of the homogeneous Dirichlet problem for (8) is not Gateaux differentiable at the point

$$v_0 = -\Delta y_0 + |y_0|^{\rho} y_0$$

by Corollary 1. This completes the proof of Corollary 2. □

By Sobolev Theorem the embedding $H_0^1(\Omega) \subset L_q(\Omega)$ is true if $n = 2$ or $\rho \leq 4/(n-2)$ for $n > 2$. Then the solution of (8) is differentiable with respect to the absolute term for small enough values of the set dimension n and nonlinearity parameter ρ. These characteristics determine a degree of the difficulty for the given equation. It is clear that the differentiability of the inverse operator (but not the absence of this property) follows from the Inverse Function Theorem. We will show soon that there exists another technique for proving this property. It is applicable even in the case of nondifferentiability in the sense of Gateaux. However it is important, that it satisfies some property which can be interpreted as a weak form of the differentiability.

The obtained result can be used for the analysis of optimization control problems for the system described by (8). Consider as an example the functional

$$I(v) = \frac{\alpha}{2} \|v\|_*^2 + \frac{1}{2} \left\| y(v) - y_d \right\|^2,$$

where $\alpha > 0$, $y_d \in H^{-1}(\Omega)$, and $y(v)$ is the solution of the Dirichlet problem (8) for the control v, besides $\|\cdot\|$ and $\|\cdot\|_*$ are the norms of the spaces $H_0^1(\Omega)$ and $H^{-1}(\Omega)$. Consider the following optimization problem.

Problem 2 Minimize the functional I on the convex closed subset U of the space V.

The solvability of this problem can be proved by a standard method (see, for example Chap. 1, Theorem 1.1 in [11]) using the weak continuity of the state function with respect to the absolute term. Note that the indeterminacy of the functional I on the complete set U is not an obstacle for the analysis of the optimization problem [13].

Corollary 3 *If $H_0^1(\Omega) \subset L_q(\Omega)$, then the solution v_0 of Problem 2 satisfies the inequality*

$$\int_\Omega (\alpha \Lambda v_0 - p_0)(v - v_0)dx \geq 0 \quad \forall v \in U, \tag{10}$$

where Λ is the canonical isomorphism of the spaces $H^{-1}(\Omega)$ and $H_0^1(\Omega)$, and p_0 is the solution of the homogeneous Dirichlet problem for the equation

$$-\Delta p_0 + (\rho + 1)|y_0|^\rho p_0 = \Delta y_0 - \Delta y_d. \tag{11}$$

Proof The derivative of the functional J (first term of the minimizing functional) is defined by the equality

$$\langle J'(v_0), h\rangle = \alpha(v_0, h)_* \quad \forall h \in H^{-1},$$

where $(\cdot, \cdot)_*$ is the scalar product of the space $H^{-1}(\Omega)$. By Riesz theorem there exists the canonical isomorphism $\Lambda : H^{-1}(\Omega) \to H_0^1(\Omega)$. Then we get

$$J'(v_0) = \alpha \Lambda v_0.$$

The derivative of the functional K (second term of the minimizing functional) is defined by the equality

$$\langle K'(y_0), h\rangle = (y_0 - y_d, h) \quad \forall h \in H_0^1(\Omega),$$

where $(\cdot, \cdot)$ is the scalar product of the space $H_0^1(\Omega)$. Using Green formula, we obtain

$$K'(y_0) = \Delta y_d - \Delta y_0.$$

The operator $A'(y_0)$ is self-adjoint. Then the adjoint equation (6) transforms to (11), and the variational inequality (5) transforms to (10). This completes the proof of the corollary. □

14.3 Differentiation of the Inverse Operator

We will try to prove the differentiability of the inverse operator directly without using of the Inverse function Theorem. Consider again an operator $A : V \to Y$ and a point $v_0 \in V$. Suppose the following assumption.

Property 1 The operator A is invertible in a neighborhood O of the point v_0.

Choose a small enough positive number σ such that the point $v_\sigma = v_0 + \sigma h$ is in O for all $h \in V$. Denote by $y(v)$ the value $A^{-1}v$. Using the equalities $Ay(v_\sigma) = v_\sigma$, $Ay(v_0) = v_0$, we get

$$Ay(v_\sigma) - Ay(v_0) = \sigma h.$$

Assume the following property.

Property 2 The operator A is Gateaux differentiable.

By the Mean Value Theorem we obtain

$$Ay - Ay_0 = \left\{ \int_0^1 A'\big[y_0 + \theta(y - y_0)\big] d\theta \right\} (y - y_0),$$

where $y_0 = y(v_0)$. Then we have

$$G(v_\sigma)\big[y(v_\sigma) - y(v_0)\big] = \sigma h,$$

where the linear continuous operator $G(v) : Y \to V$ is defined by the formula

$$G(v) = \int_0^1 A'\big\{y_0 + \theta\big[y(v) - y_0\big]\big\} d\theta$$

for all $v \in V$. We get

$$\big\langle G(v_\sigma)^* \lambda, \big[y(v_\sigma) - y(v_0)\big]/\sigma \big\rangle = \langle \lambda, h \rangle \quad \forall \lambda \in V^*. \tag{12}$$

Consider the linear operator equation

$$G(v)^* p_\mu(v) = \mu. \tag{13}$$

It transforms to

$$A'(y_0)^* p_\mu(v_0) = \mu \tag{14}$$

for $v = v_0$. We will use the following assumption.

Property 3 Equation (13) has a unique solution $p_\mu(v) \in V^*$ for all $\mu \in Y^*$, $v \in O$.

Defining $\lambda = p_\mu(v_\sigma)$ for small enough σ in (12) we get

$$\big\langle \mu, \big[y(v_0 + \sigma h) - y(v_0)\big]/\sigma \big\rangle = \big\langle p_\mu(v_\sigma), h \big\rangle \quad \forall \mu \in Y^*, h \in V. \tag{15}$$

Define

$$M = \big\{ \mu \in Y^* \,\big|\, \|\mu\| = 1 \big\}.$$

Property 4 The convergence $p_\mu(v_\sigma) \to p_\mu(v_0)$ *-weakly in V^* uniformly with respect to $\mu \in M$ as $\sigma \to 0$ is true for all $v \in V$.

Theorem 2 *Let us suppose the Properties 1–4. Then the operator A^{-1} has the Gateaux derivative D at the point v_0 such that*

$$\langle \mu, Dh\rangle = \big\langle p_\mu(v_0), h\big\rangle \quad \forall \mu \in Y^*, h \in V. \tag{16}$$

Proof Let the operator D be defined by (16). It is a map from V to Y. Besides it is linear continuous. Using (15) and (16) we get

$$\begin{aligned}\big\|\big[y(v_0+\sigma h) - y(v_0)\big]/\sigma - Dh\big\|_V &= \sup_{\mu\in M}\big|\big\langle \mu, \big[y(v_0+\sigma h) - y(v_0)\big]/\sigma - Dh\big\rangle\big| \\ &= \sup_{\mu\in M}\big|\big\langle p_\mu(v_\sigma) - p_\mu(v_0), h\big\rangle\big|\end{aligned}$$

by the definition of the norm. Then we obtain $p_\mu(v_\sigma) \to p_\mu(v_0)$ *-weakly in V^* uniformly with respect to $\mu \in M$ for all $h \in V$ because of Property 4. Passing to the limit in the last equality as $\sigma \to 0$, we get the convergence

$$\big[y(v_0+\sigma h) - y(v_0)\big]/\sigma \to Dh \quad \text{in } V \text{ for all } h \in V.$$

So the operator D is the Gateaux derivative of the operator A^{-1} at the point v_0. □

Let us explain applications of this result.

Lemma 3 *The operator A for (8) satisfies the Properties 1–4 if $H_0^1(\Omega) \subset L_q(\Omega)$.*

Proof Property 1 is the one-valued solvability of (8). The differentiability of the operator A (Property 2) is obvious, moreover, its derivative is defined by the equality

$$A'(y)h = -\Delta h + (\rho+1)|y|^\rho h \quad \forall h \in Y.$$

Thus it is necessary to use Properties 3 and 4 and properties of the adjoint equation (13).

We have

$$\begin{aligned} G(v)y &= \left\{\int_0^1 A'\big\{y_0 + \theta\big[y(v) - y_0\big]\big\}d\theta\right\}y \\ &= -\Delta y + \left\{\int_0^1 \big|y_0 + \theta\big[y(v) - y_0\big]\big|^\rho d\theta\right\}y \\ &= -\Delta y + \big|y_0 + \varepsilon\big[y(v) - y_0\big]\big|^\rho y \quad \forall y \in Y,\end{aligned}$$

where $\varepsilon \in [0, 1]$. Define

$$g(v) = \big|y_0 + \varepsilon\big[y(v) - y_0\big]\big|^{\rho/2},$$

so that we get

$$G(v)y = -\Delta y + g(v)^2 y.$$

Then we obtain the equality

$$\langle G(v)^* p, y\rangle = \langle p, G(v)y\rangle = \int_\Omega \left[-\Delta y + g(v)^2 y\right] p dx = \int_\Omega \left[-\Delta p + g(v)^2 p\right] y dx$$

for all $y \in Y$, $p \in V^*$, $v \in V$. So we get

$$G(v)^* p = -\Delta p + g(v)^2 p,$$

and (13) is transformed to

$$-\Delta p_\mu(v_\sigma) + g(v_\sigma)^2 p_\mu(v_\sigma) = \mu. \tag{17}$$

Multiplying (17) by $p_\mu(v_\sigma)$ and integrating in $x \in \Omega$ we have

$$\int_\Omega \left|\nabla p_\mu(v_\sigma)\right|^2 dx + \int_\Omega \left|g(v_\sigma) p_\mu(v_\sigma)\right|^2 dx = \int_\Omega \mu p_\mu(v_\sigma) dx.$$

Then we obtain the inequality

$$\left\| p_\mu(v_\sigma)\right\|^2 + \left\| g(v_\sigma) p_\mu(v_\sigma)\right\|_2^2 \le \|\mu\|_* \left\| p_\mu(v_\sigma)\right\|,$$

where $\|\cdot\|_p$ is the norm in $L_p(\Omega)$. So we get

$$\left\| p_\mu(v_\sigma)\right\| \le \|\mu\|_*, \qquad \left\| g(v_\sigma) p_\mu(v_\sigma)\right\|_2 \le \|\mu\|_*. \tag{18}$$

Then (17) has the unique solution $p_\mu(v_\sigma) \in V^*$ for all $\mu \in Y^*$, $h \in V$, and σ, and hence Property 3 holds.

The space V is reflexive, so it is sufficient to prove that $p_\mu(v_\sigma) \to p_\mu(v_0)$ weakly in V^* uniformly with respect to μ as $\sigma \to 0$ for all $h \in V$. The set $\{p_\mu(v_\sigma)\}$ is bounded in the space $H_0^1(\Omega)$, and the set $\{g(v_\sigma) p_\mu(v_\sigma)\}$ is bounded in the space $L_2(\Omega)$ uniformly with respect to $\mu \in M$ for all $h \in V$ because of the inequalities (18). Using the Banach–Alaogly Theorem we get $p_\mu(v_\sigma) \to p$ weakly in $H_0^1(\Omega)$ uniformly with respect to $\mu \in M$ for all $h \in V$. Applying the Rellich–Kondrashov Theorem we get $p_\mu(v_\sigma) \to p$ strongly in $L_2(\Omega)$ and a.e. on Ω. Using the continuity of the solution of (8) with respect to the absolute term, we obtain $y(v_\sigma) \to y(v_0)$ in $H_0^1(\Omega)$ and a.e. on Ω. Then

$$\left|g(v_\sigma)\right|^2 p_\mu(v_\sigma) \to (\rho+1)|y_0|^\rho p \quad \text{a.e. on } \Omega.$$

The sets $\{p_\mu(v_\sigma)\}$, $\{y(v_\sigma)\}$, and $\{g(v_\sigma)^{2/\rho}\}$ are uniformly bounded in $L_q(\Omega)$. We have

$$\begin{aligned}\left\| g(v_\sigma)^2 p_\mu(v_\sigma)\right\|_{q'} &\le \left\| g(v_\sigma) p_\mu(v_\sigma)\right\|_2 \left\| g(v_\sigma)\right\|_{2q/\rho} \\ &= \left\| g(v_\sigma) p_\mu(v_\sigma)\right\|_2 \left\| y_0 + \varepsilon\left[y(v_\sigma) - y_0\right]\right\|_q^{\rho/2}.\end{aligned}$$

So the set $\{g(v_\sigma)^2 p_\mu(v_\sigma)\}$ is uniformly bounded in $L_{q'}(\Omega)$. Using Lemma 1.3 (see Chap. 1 in [12]), we get

$$g(v_\sigma)^2 p_\mu(v_\sigma) \to (\rho+1)|y_0|^\rho p \quad \text{weakly in } L_{q'}(\Omega)$$

uniformly with respect to $\mu \in M$ for all $h \in V$.

Let us multiply (16) by a function $\lambda \in H_0^1(\Omega)$. After integration we get

$$\int_\Omega \big[-\Delta p_\mu(v_\sigma) + g(v_\sigma)^2 p_\mu(v_\sigma)\big]\lambda dx = \int_\Omega \lambda\mu dx.$$

Passing to the limit as $\sigma \to 0$, we obtain, that the function $p = p_\mu(v_0)$ satisfies the equation

$$-\Delta p_\mu(v_0) + (\rho+1)|y_0|^\rho p_\mu(v_0) = \mu. \tag{19}$$

Thus $p_\mu(v_\sigma) \to p_\mu(v_0)$ weakly in $H_0^1(\Omega)$ uniformly with respect to $\mu \in M$ for all $h \in V$, notably the Property 4 is true. □

By Lemma 3 the differentiability of the solution of (8) with respect to the absolute term follows from Theorem 2 if the embedding $H_0^1(\Omega) \subset L_q(\Omega)$ holds.

Lemma 4 *Properties 1–4 follow from the assumptions of the Inverse Function Theorem.*

Proof The existence of the inverse operator is a corollary of the Inverse Function Theorem. The differentiability of the operator A is the assumption of this theorem. So our general difficulty is the analysis of (13), namely the justification of Assumptions 3 and 4. Equation (13) can be transformed to

$$G(v_0)^* p_\mu(v_\sigma) = A'(y_0)^* p_\mu(v_\sigma) = \big[G(v_0)^* - G(v_\sigma)^*\big] p_\mu(v_\sigma) + \mu.$$

The derivative $A'(y_0)$ is invertible by the Inverse Function Theorem. So its adjoint operator is invertible too. Then (13) can be transformed to the equality

$$p_\mu(v_\sigma) = L_\mu(\sigma h) p_\mu(v_\sigma), \tag{20}$$

where the map $L_\mu(\sigma h) : V^* \to V^*$ is defined by the formula

$$L_\mu(\sigma h) p = \big[A'(y_0)^*\big]^{-1} \big\{\big[G(v_0)^* - G(v_\sigma)^*\big] p + \mu\big\}.$$

Using properties of the operator norm we get the inequality

$$\begin{aligned}\big\|L_\mu(\sigma h)p_1 - L_\mu(\sigma h)p_2\big\|_{V^*} &= \big\|\big[A'(y_0)^*\big]^{-1}\big[G(v_0)^* - G(v_\sigma)^*\big](p_1 - p_2)\big\|_{V^*} \\ &\le \big\|\big[A'(y_0)^*\big]^{-1}\big\| \big\|G(v_0)^* - G(v_\sigma)^*\big\| \|p_1 - p_2\|_{V^*}\end{aligned}$$

for all $p_1, p_2 \in V^*$. Then we obtain

$$\begin{aligned}
&\left\| L_\mu(\sigma h) p_1 - L_\mu(\sigma h) p_2 \right\|_{V^*} \\
&\quad \le \left\| A'(y_0)^{-1} \right\| \left\| G(v_0) - G(v_\sigma) \right\| \| p_1 - p_2 \|_{V^*} \quad \forall p_1, p_2 \in V^*
\end{aligned}$$

because of the equality of the norms for adjoint operators. The operator A^{-1} is continuous at the point v_0 by the Inverse Function Theorem. Therefore we get the convergence $y(v_0 + \sigma h) \to y_0$ in Y as $\sigma \to 0$ for all $h \in V$. Using the continuous differentiability of the operator A at the point y_0, we get $G(v_\sigma) \to G(v_0)$ in the sense of the corresponding operator norm. The value σ can be chosen small enough such that

$$\left\| G(v_\sigma) - G(v_0) \right\| \le \chi \left\| A'(y_0)^{-1} \right\|^{-1},$$

where $0 < \chi < 1$. So we obtain the estimate

$$\left\| L_\mu(\sigma h) p_1 - L_\mu(\sigma h) p_2 \right\|_{V^*} \le \chi \| p_1 - p_2 \|_{V^*} \quad \forall p_1, p_2 \in V^*.$$

Thus the operator $L_\mu(\sigma h)$ is contracting. Then (20) has a unique solution $p_\mu(v_\sigma) \in V^*$ because of the Contracting Mapping Theorem.

We get $G(v_\sigma) \to G(v_0)$ as $\sigma \to 0$. So $G(v_\sigma)\lambda \to G(v_0)\lambda$ in V for all $\lambda \in Y$. Using the obtained inequalities, we get

$$\begin{aligned}
\left\| p_\mu(v_\sigma) \right\|_{V^*} &= \left\| L_\mu(\sigma h) p_\mu(v_\sigma) \right\|_{V^*} \\
&\le \left\| \left[A'(y_0)^* \right]^{-1} \right\| \left\| \left[G(v_0)^* - G(v_\sigma)^* \right] p_\mu(v_\sigma) + \mu \right\|_{Y^*} \\
&\le \left\| A'(y_0)^{-1} \right\| \left[\left\| G(v_0) - G(v_\sigma) \right\| \left\| p_\mu(v_\sigma) \right\|_{V^*} + \| \mu \|_{Y^*} \right] \\
&\le \chi \left\| p_\mu(v_\sigma) \right\|_{V^*} + \left\| A'(y_0)^{-1} \right\| \| \mu \|_{Y^*}.
\end{aligned}$$

So we have

$$(1 - \chi) \left\| p_\mu(v_\sigma) \right\|_{V^*} \le \left\| A'(y_0)^{-1} \right\| \| \mu \|_{Y^*}.$$

Then $p_\mu(v_\sigma) \to p$ *-weakly in V^* for all $h \in V$ as $\sigma \to 0$.

Using inequality (13) we get

$$\left\langle p_\mu(v_\sigma), G(v_\sigma)\lambda \right\rangle = \langle \mu, \lambda \rangle \quad \forall \lambda \in Y.$$

As a consequence $\{p_\mu(v_\sigma)\}$ converges *-weakly, and $\{G(v_\sigma)\}$ converges strongly. After passing to the limit we have $A'(y_0)^* p = \mu$, and $p = p_\mu(v_0)$. □

Thus the assumptions of Theorem 2 follow from the assumptions of the Inverse Operator Theorem. However assertions of Theorem 2 may be true if assumptions of the Inverse Operator Theorem are not satisfied.

14.4 Extended Differentiation of the Inverse Operator

The solution of (8) is differentiable with respect to the absolute term for small enough values of the set dimension n and nonlinearity parameter ρ. But it is not differentiable for large enough values of these parameters. Suppose $n \geq 3$. By Sobolev Theorem the embedding $H_0^1(\Omega) \subset L_q(\Omega)$ is true if $\rho \leq 4/(n-2)$. It guarantees the differentiability of the considered inverse operator. However this embedding fails if the parameter ρ increases. Then the solution of the equation becomes nondifferentiable with respect to the absolute term. It seems to be a strange situation. Properties of the inverse operator change with a jump at the neighborhood of some value ρ. The differentiability of the operator disappears after the passage of this value. This situation seems not likely. We could suppose the existence of a weaker operator differentiability than the Gateaux derivative. We would like also to determine the extension of the operator derivative because the solvability of our optimization problem was proved for all values of the set dimension and the nonlinearity parameter.

There exist extensions of classical operator differentiation, for example, subdifferential calculus [14], Clarke derivatives [15], quasidifferential calculus [7]. They are used also for the resolution of nonsmooth optimization problems. These results are effective enough for the analysis of operators with nonsmooth terms, for example, the absolute value or the maximum of functions. However similar terms are absent in our case. So we will try to define another form of operator derivatives extension.

It is known that "*the general idea of the differential calculus is a local approximation of a function by a linear function*" (see p. 170 in [16]). The differentiation is a tool of the local approximation of the analyzed object. The desired form of an operator derivative can be obtained by weakening of topological approximation properties of the differentiation. Then we get the extended operator derivative (see [17–19]).

Definition An operator $L : V \to Y$ is called $(V_0, Y_0; V_1, Y_1)$-*extended differentiable in the sense of Gateaux* at the point $v_0 \in V$ if there exist linear topological spaces V_0, Y_0, V_1, Y_1 with continuous embeddings

$$V_1 \subset V_0 \subset V, \qquad Y \subset Y_0 \subset Y_1,$$

and a linear continuous operator $D : V_0 \to Y_0$ such that

$$\left[L(v_0 + \sigma h) - L(v_0)\right]/\sigma \to Dh \quad \text{in } Y_1 \text{ for all } h \in V_1$$

as $\sigma \to 0$.

It is obvious that the $(V, Y; V, Y)$-derivative is the standard Gateaux derivative. The following result is known (see Theorem 4 in [18]; Theorem 5.4 in [19]).

Lemma 5 *The operator* A^{-1} *for* (8) *is* $(V_0, Y_0; V_1, Y_1)$*-extended differentiable in the sense of Gateaux at an arbitrary point* $v_0 \in V$, *where*

$$Y_1 = H_0^1(\Omega), \qquad Y_0 = Y_1 \cap \{y \,|\, |y_0|^{\rho/2} y \in L_2(\Omega)\},$$

$$V_1 = H^{-1}(\Omega), \qquad V_0 = V_1 + \{v \,|\, v = |y_0|^{\rho/2}\varphi, \varphi \in L_2(\Omega)\}, \quad y_0 = y(v_0),$$

moreover, its derivative D *satisfies the equality*

$$\int_\Omega \mu D h dx = \int_\Omega p_\mu(v_0) h dx \quad \forall \mu \in Y_0^*, h \in V_0, \tag{21}$$

and $p_\mu(v_0)$ *is the solution of the homogeneous Dirichlet problem for* (19).

Thus the inverse operator for the given example is extended differentiable for all values of the set dimensions and nonlinearity parameters. Its extended derivative is transformed to the Gateaux one for small enough values of these characteristics. However the Gateaux derivative does not exist for its large enough values, notably in the case of the high enough degree of the difficulty for the problem. Besides the difference between standard derivative and extended one is determined by this degree of the difficulty. Thus the inverse operator is extended differentiable without any constraints. However the extended derivative differs from the classical one after the augmentation of the parameters that determine the degree of the difficulty for the problem. Then we obtain the gradual change of the inverse operator properties after the gradual change of its parameters, although the standard derivatives theory permits the change with a jump.

We will prove that the obtained result is sufficient for the analysis of the given optimization problem without any constraints.

Corollary 4 *The solution of the minimization problem of the functional* I *on the set* U *for* (8) *satisfies the variational inequality*

$$\int_\Omega (\alpha \Lambda v_0 - p_0)(v - v_0) dx \geq 0 \quad \forall v \in U_1, \tag{22}$$

where $U_1 = U \cap (v_0 + V_1)$, *and* p_0 *is a solution of* (11).

Indeed, if v_0 is a solution of the optimization problem, then

$$I\big[v_0 + \sigma(v - v_0)\big] - I(v_0) \geq 0 \quad \forall v \in U.$$

Let us choose $v \in v_0 + V_1$. Passing to the limit and using Lemma 5 after division by σ we get

$$\int_\Omega \alpha \Lambda v_0 (v - v_0) dx + \int_\Omega \nabla(y_0 - y_d) \nabla D(v - v_0) dx \geq 0 \quad \forall v \in U_1.$$

Then the inequality (22) is true.

If $H_0^1(\Omega) \subset L_q(\Omega)$, then $U_1 = U$, and the variational inequalities (10) and (22) are equal. Thus necessary conditions of optimality can be obtained without any assumptions by means of the extended derivatives theory. Optimization problems for elliptic equations with power nonlinearity without Gateaux differentiability of the control-state mapping were considered in [18, 19]. But the control space was narrower, and the state functional was more regular there. This technique was used for the analysis of optimization problems for others equations in [20].

Note that Lemma 5 uses the technique of the proof of Theorem 2. We can suppose that it is possible to obtain the extended differentiability of the inverse operator in the general case. Consider Banach spaces Y, V, a map $A : Y \to V$, and points $y_0 \in Y$, $v_0 = Ay_0$. Let V_1 be a Banach subspace of V with a neighborhood O_1 of zero. Then $O = v_0 + O_1$ is a neighborhood of v_0. We suppose the following assertion.

Property 5 The operator A is invertible on the set O.

Define $y(v) = A^{-1}v$. We get the equality

$$Ay(v_\sigma) - Ay(v_0) = \sigma h$$

for all $v \in V_1$ and small enough σ, where $v_\sigma = v_0 + \sigma h$. Let $G(v)$ be the operator from the proof of Theorem 2. We have

$$G(v_\sigma)\big[y(v_\sigma) - y(v_0)\big] = \sigma h,$$

so

$$\big\langle \lambda, G(v_\sigma)\big[y(v_\sigma) - y(v_0)\big]\big\rangle = \sigma \langle \lambda, h\rangle \quad \forall \lambda \in V^*.$$

Consider Banach spaces $V(v)$ and $Y(v)$ such that the embeddings of the spaces Y, Y_1 and $Y(v)$ to $V(v)$, $Y(v)$ and V, respectively, are continuous for all $v \in O$. Let the following assumption be true.

Property 6 The operator A is Gateaux differentiable, moreover, there exists the continuous extension $\overline{G}(v)$ of the operator $G(v)$ to $Y(v)$ such that its image is a subset of $V(v)$ for all $v \in O$.

Using the properties $y(v) \in y_0 + Y(v)$ and $V(v)^* \subset V^*$ we get

$$\big\langle \overline{G}(v_\sigma)^*\lambda, \big[y(v_\sigma) - y(v_0)\big]\big\rangle = \sigma \langle \lambda, h\rangle \quad \forall \lambda \in V^*. \tag{23}$$

It is an analogue of (12). Consider the linear operator equation

$$\overline{G}(v_\sigma)^* p_\mu(v_\sigma) = \mu, \tag{24}$$

which is an analogue of (13). It can be transformed to

$$\overline{A}'(y_0) p_\mu(v_0) = \mu$$

for $v = v_0$, where $\overline{A}'(y_0) = \overline{G}(v_0)$ is the extension of the operator $A'(y_0) = G(v_0)$ to the set $Y(v_0)$.

Consider Banach space V_1 such that the embedding $Y(v) \subset V_1$ is continuous and dense for all $v \in O$. We suppose the following condition.

Property 7 Equation (24) has the unique solution $p_\mu(v) \in V(v)^*$ for all $v \in O$, $\mu \in Y(v)^*$.

Defining in (23) $\lambda = p_\mu(v_\sigma)$ for a small enough σ we get

$$\langle \mu, [y(v_0 + \sigma h) - y(v_0)]/\sigma \rangle = \langle p_\mu(v_\sigma), h \rangle \quad \forall \mu \in Y(v_\sigma)^*, h \in V_1. \tag{25}$$

We will use the additional assumption.

Property 8 The convergence $p_\mu(v_\sigma) \to p_\mu(v_0)$ holds *-weakly in V_1^* uniformly with respect to $\mu \in M$ as $\sigma \to 0$ for all $h \in V_1$.

The extended differentiability of the inverse operator is guaranteed by the following result.

Theorem 3 *Let us suppose the Properties 5–8. Then the operator A^{-1} has the $(V(v_0), Y(v_0); V_1, Y_1)$-extended Gateaux derivative D at the point v_0 such that*

$$\langle \mu, Dh \rangle = \langle p_\mu(v_0), h \rangle \quad \forall \mu \in Y(v_0)^*, h \in V(v_0). \tag{26}$$

Proof By (25), (26) we get

$$\begin{aligned} &\langle \mu, [y(v_0 + \sigma h) - y(v_0)]/\sigma - Dh \rangle \\ &\quad = \langle p_\mu(v_\sigma) - p_\mu(v_0), h \rangle \quad \forall \mu \in M, h \in V(v_1). \end{aligned} \tag{27}$$

Then

$$\big\| [y(v_0 + \sigma h) - y(v_0)]/\sigma - Dh \big\|_{V_1} = \sup_{\mu \in M} \big| \langle p_\mu(v_\sigma) - p_\mu(v_0), h \rangle \big|.$$

We have $p_\mu(v_\sigma) \to p_\mu(v_0)$*-weakly in V_1 uniformly with respect to $\mu \in M$ for all $h \in V_1$ as $\sigma \to 0$ by Property 8. Passing to the limit in the last equality we obtain

$$[y(v_0 + \sigma h) - y(v_0)]/\sigma \to Dh \quad \text{in } Y_1$$

for all $h \in V_1$. Thus D is an extended derivative of the inverse operator. □

A result of the extended differentiability of the inverse operator for nonnormalized spaces was obtained in [18].

Let us prove that the assumptions of the Theorem 3 are true for the considered example.

Lemma 6 *The operator A, which is defined by* (8), *satisfies the Properties* 5–8.

Proof The Property 5 is the solvability of (8) at a neighborhood of the given point. It is obviously that this assumption is true. The differentiability of the operator A is clear. The operator $G(v)$ for our case is determined by the equality

$$G(v)y = -\Delta y + g(v)^2 y \quad \forall y \in Y,$$

where

$$g(v)^2 = (\rho+1)\big|y_0 + \varepsilon\big[y(v) - y(v_0)\big]\big|^{\rho}, \quad \varepsilon \in [0, 1].$$

Let the spaces Y_1, V_1, $Y(v)$, $V(v)$ be those defined in the proof of Lemma 5. Define the map $\overline{G}(v)$ by the equality

$$\overline{G}(v)y = -\Delta y + g(v)^2 y \quad \forall y \in Y(v).$$

Then Property 6 is true. Reliability of Properties 7 and 8 was obtained in the proof of Lemma 5. □

Thus extended differentiability of the inverse operator for (8) follows from Theorem 3.

Lemma 7 *the Properties* 5–8 *follow from the assumptions of the Inverse Function Theorem.*

Indeed, Property 5 is a direct corollary of this theorem. Let us define the spaces $V = V_1$, $Y = Y_1$. Then we get $Y(v) = Y$, $V(v) = V$. So the operator $\overline{G}(v)$ is equal to $G(v)$, and Property 6 is trivial. Therefore Properties 7 and 8 are transformed to Properties 3 and 4. Its validity was proved before.

Thus Theorem 3 is a generalization of the Theorem 2. The obtained results can be used for other applications if it is necessary to differentiate an inverse operator. For example the extended differentiable submanifolds of Banach spaces are defined in [21, 22]. Optimization control problems on differentiable submanifolds are considered there. Analogical results could be obtained for the implicit operator, including the case of nonnormalized spaces (see [23]). Banach spaces with extended differentiable operators form a category, and necessary conditions of optimality have a category interpretation (see [24]).

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Chapter 15
Solution of the Cauchy Problem for Generalized Euler-Poisson-Darboux Equation by the Method of Fractional Integrals

A.K. Urinov and S.T. Karimov

Abstract In this work the singular Cauchy problem for the multi-dimensional Euler-Poisson-Darboux equation with spectral parameter has been investigated with the help of the generalized Erdelyi-Kober fractional operator. Solution of the considered problem is found in explicit form for various values of the parameter p of the equation.

Mathematics Subject Classification 35L10 · 35Q05 · 26A33

15.1 Introduction

For the first time the equation

$$u_{xy} - \frac{\alpha}{x-y}u_x + \frac{\beta}{x-y}u_y + \frac{\gamma}{(x-y)^2}u = 0, \tag{1}$$

where $\alpha, \beta, \gamma = \text{const}$, was obtained by Euler [1] in connection with the study of the air flow in pipes of different cross sections and the vibrations of strings of variable thickness. He gave a solution of this equation for $\alpha = \beta = m$, $\gamma = n$ (m, n are natural numbers).

The same equation, but in another form

$$E^-_{q,p}(u) \equiv u_{xx} - u_{yy} - \frac{2q}{y}u_x - \frac{2p}{y}u_y = 0, \tag{2}$$

where $q, p = \text{const}$, was solved by Poisson [2] for $q = 0$. He found a hyperbolic analogue of the representation of solution for this equation. In the same work he

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considered the equation

$$L_p(u) \equiv \sum_{k=1}^{n} \frac{\partial^2 u}{\partial x_k^2} - \frac{\partial^2 u}{\partial t^2} - \frac{2p}{t}\frac{\partial u}{\partial t} = 0, \tag{3}$$

with $n = 3,\ p = 1$.

The general solution of (1) with $\alpha = \beta$ was found by Riemann [3]. He constructed the solution of the Cauchy problem with the help of auxiliary function using the method which is now called after him.

Much later, (2) with $q = 0,\ 0 < p < 1$ appeared in the monograph by Darboux [4] in connection with studying curvature of surfaces, where it was called the Euler-Poisson equation. Subsequently, many authors began to cite equations of the forms (1), (2), (3) and their elliptic analogs, as the equations of Euler-Poisson-Darboux.

After the publication of the first issue of the book by Tricomi [5], where the problem for mixed elliptic-hyperbolic equation $yu_{xx} + u_{yy} = 0$, later called as Tricomi equation, was formulated and investigated, the interest in such equations greatly increased. When studying this problem the key role is played by the equation of the form (2) and

$$E_{q,p}^{+}(u) \equiv u_{xx} + u_{yy} + \frac{2q}{y}u_x + \frac{2p}{y}u_y = 0, \tag{4}$$

where $q = 0,\ p = (1/6)$.

More bibliography in this direction can be found in the monographs by Bitsadze [6] and Smirnov [7].

The theory of equations with singular coefficients is directly connected to the theory of equation degenerating on the boundary. Using a change of variables, a wide class of degenerate equations can be reduced to equations with singular coefficients. For instance, the equation with degeneration of type and order,

$$y^m \sum_{k=1}^{n} \frac{\partial^2 u}{\partial x_k^2} - y^k \frac{\partial^2 u}{\partial y^2} - \alpha y^{k-1}\frac{\partial u}{\partial y} - \lambda^2 y^k u = 0$$

by the change of variables $t = \frac{2}{m-k-2} y^{(m-k+2)/2}$ can be reduced to the equation

$$L_p^{\lambda}(u) \equiv \sum_{k=1}^{n} \frac{\partial^2 u}{\partial x_k^2} - \frac{\partial^2 u}{\partial t^2} - \frac{2p}{t}\frac{\partial u}{\partial t} - \lambda^2 u = 0. \tag{5}$$

The main role in creating the theory of Euler-Poisson-Darboux equations was played by works of Weinstein [8–11]. In these works Weinstein investigated the Cauchy problem for (3) with various values of the parameter p, with half-homogeneous initial conditions

$$u(x,0) = \tau(x), \qquad u_t(x,0) = 0, \quad x \in R^n, \tag{6}$$

and found its solution in an explicit form.

There he showed also the matching formulae of the form

$$E^{+}_{q,p}\big(y^{1-2p}u\big) = y^{1-2p}E^{+}_{q,1-p}(u), \tag{7}$$

considering (4) with $q = 0$, $0 < p < (1/2)$. Note that the formula of the form (7) can be found in the work of Darboux [4].

In the work by Young [12] one can find the survey of the investigations of the singular Cauchy problem {(3), (6)}. In the works of Diaz, Weinberger [13], Blum [14], the problem {(3), (6)} was studied for various values of the parameter p.

Kapilevich [15] investigated the Cauchy problem with initial conditions

$$u(x,0) = \tau(x), \qquad \lim_{t\to+0} t^{2p}u_t(x,t) = \nu(x), \quad x \in R^n \tag{8}$$

for (5), when $\lambda \neq 0$, $0 < p < (1/2)$ and $n = 1, 2$.

The uniqueness of the solution of the Cauchy problem {(5), (8)} was proved in the works by Fox [16], Blum [17], Bresters [18]. However, as it was shown by Bresters [18], the solution is not unique when $p < 0$.

In the present work, using fractional integrals, we investigate the Cauchy problem {(5), (8)} for various values of the parameters $p \geq 0$ and $\lambda \neq 0$.

15.2 Generalized Erdelyi-Kober Operator

In the paper [10] Weinstein found a formulae in which the connection of the solution of (2) for $q = 0$ with fractional integrals was made for various values of the parameter p. This idea was substantially developed in the work of Erdelyi [19–22], who continued investigations by Weinstein [11], and studied properties of the differential operator

$$B^{(x)}_{\eta} = x^{-2\eta-1}\frac{d}{dx}x^{2\eta+1}\frac{d}{dx} = \frac{d^2}{dx^2} + \frac{2\eta+1}{x}\frac{d}{dx}. \tag{9}$$

In the work of Erdelyi [22] the apparatus of fractional integration was used for developing the result by Friedlander and Heins [23], where (2) was considered for $q = 0$.

The results of Erdelyi were generalized by Lowndes [24–26], where a generalized Erdelyi-Kober operator

$$J_{\lambda}(\eta,\alpha)f(x) = 2^{\alpha}\lambda^{1-\alpha}x^{-2\alpha-2\eta} \times \int_0^x t^{2\eta+1}\big(x^2-t^2\big)^{(\alpha-1)/2}J_{\alpha-1}\big(\lambda\sqrt{x^2-t^2}\big)f(t)dt \tag{10}$$

was introduced and studied. Here $\eta, \alpha, \lambda \in R$, such that $\alpha > 0$, $\eta \geq -(1/2)$, and $J_{\nu}(z)$ is the Bessel function of the first kind of order ν [27–29].

Further we need the following properties of the operator (10), which were proved in [25]:

1. It is obvious that for $\lambda \to 0$ the operator (10) coincides with the regular Erdelyi-Kober operator

$$I_{\eta,\alpha} f(x) = \frac{2x^{-2(\eta+\alpha)}}{\Gamma(\alpha)} \int_0^x \left(x^2 - t^2\right)^{\alpha-1} t^{2\eta+1} f(t)dt,$$

   where $\Gamma(\alpha)$ is Euler's Gamma function.
2. The following equalities hold true:

$$J_{i\lambda}(\eta+\alpha,\beta) J_\lambda(\eta,\alpha) = J_\lambda(\eta+\alpha,\beta) J_{i\lambda}(\eta,\alpha) = I_{\eta,\alpha+\beta},$$

   where i is the imaginary unit, $\alpha, \beta, \lambda \in R$.
3. From the latter equality, using the property $J_0(\eta, 0) = E$, where E is unique operator, one can pre-define the operator $J_\lambda(\eta,\alpha)$ for $\alpha < 0$ in the following way:

$$J_\lambda(\eta,\alpha) f(x) = x^{-2(\eta+\alpha)} \left(\frac{d}{2xdx}\right)^m x^{2(\eta+\alpha+m)} J_\lambda(\eta, \alpha+m) f(x), \qquad (11)$$

   where $-m < \alpha < 0$, $m = 1, 2, \ldots$.
4. From the property 3, the relations for inverse operator

$$J_{i\lambda}^{-1}(\eta,\alpha) = J_\lambda(\eta+\alpha, -\alpha), \qquad J_\lambda^{-1}(\eta,\alpha) = J_{i\lambda}(\eta+\alpha, -\alpha)$$

   follow.

In the work [26] Lowndes proved the following lemma:

Lemma 1 *Let $\alpha > 0$, $f(x) \in C^2(0, b)$, $b > 0$, let the function $x^{2\eta+1} f(x)$ be integrable in a neighborhood and let $x^{2\eta+1} f'(x) \to 0$ as $x \to 0$. Then*

$$J_\lambda^{(x)}(\eta,\alpha) B_\eta^{(x)} f(x) = \left(B_{\eta+\alpha}^{(x)} + \lambda^2\right) J_\lambda^{(x)}(\eta,\alpha) f(x), \qquad (12)$$

where $B_\eta^{(x)}$ is the operator of Bessel which is defined by (9).

Using this lemma Lowndes solved the Cauchy problem {(5), (8)} for $p = 0$.
Further we need the following form of the formula (10):

$$J_\lambda(\eta,\alpha) f(x) = \frac{2x^{-2(\alpha+\eta)}}{\Gamma(\alpha)} \int_0^x t^{2\eta+1} \left(x^2 - t^2\right)^{\alpha-1} \bar{J}_{\alpha-1}\left(\lambda\sqrt{x^2 - t^2}\right) f(t)dt, \quad (13)$$

where $\bar{J}_\nu(z)$ is the Bessel-Clifford function, which can be written by the Bessel function as: $\bar{J}_\nu(z) = \Gamma(\nu+1)(z/2)^{-\nu} J_\nu(z)$.

15.3 Application of the Erdelyi-Kober Operator for Solving the Cauchy Problem

For the construction of the solution of the problem {(5), (8)}, corresponding to various values of the parameter p, first we give some properties of the solution of (5), [9].

We denote by $u(x,t;p)$, $w(x,t;p)$ the solutions of (5) for a given value of p.

1. If $u(x,t;1-p)$ is a solution of the equation $L^{\lambda}_{1-p}(u)=0$, then the function $w(x,t;p)=t^{1-2p}u(x,t;1-p)$ will be a solution of the equation $L^{\lambda}_{p}(w)=0$ and vice versa, if $w(x,t;p)$ is a solution of the equation $L^{\lambda}_{p}(w)=0$, then $u(x,t;1-p)=t^{2p-1}w(x,t;p)$ will be a solution of the equation $L^{\lambda}_{1-p}(u)=0$.
2. If $u(x,t;p)$ is a solution of the equation $L^{\lambda}_{p}(u)=0$, then the function

$$u(x,t;1+p)=\left(\frac{1}{t}\frac{\partial}{\partial t}\right)u(x,t;p)$$

will be a solution of the equation $L^{\lambda}_{1+p}(u)=0$ and vice versa, if $u(x,t;1+p)$ is a solution of the equation $L^{\lambda}_{1+p}(u)=0$, then there exists always a solution $u(x,t;p)$ of the equation $L^{\lambda}_{p}(u)=0$.

Now we begin the investigation of the problem {(5), (8)}. Assume that the solution of the problem {(5), (6)} exists. We look for this solution as a generalized Erdelyi-Kober operator:

$$\begin{aligned} u(x,t) &= J^{(t)}_{\lambda}(\eta,\alpha)V(x,t) \\ &= \frac{2t^{-2(\eta+\alpha)}}{\Gamma(\alpha)}\int_0^t s^{2\eta+1}\left(t^2-s^2\right)^{\alpha-1}\bar{J}_{\alpha-1}\left(\lambda\sqrt{t^2-s^2}\right)V(x,s)ds, \end{aligned} \quad (14)$$

where $\alpha,\eta\in R$ are numbers to be specified later and, moreover, $\alpha>0$, $\eta\geq-(1/2)$, $V(x,t)$ is a twice continuously differentiable unknown function.

Substituting (14) into (5) and initial condition (6), and applying Lemma 1 we find the unknown function $V(x,s)$, so that it satisfies the equation

$$\sum_{k=1}^{n}\frac{\partial^2 V}{\partial x_k^2}-\frac{\partial^2 V}{\partial s^2}-\frac{2\eta+1}{s}\frac{\partial V}{\partial s}=0 \quad (15)$$

and the initial conditions

$$\begin{aligned} &V(x,0)=k_0\tau(x), \qquad V_s(x,0)=0, \\ &x\in R^n, k_0=\Gamma(\alpha+\eta+1)/\Gamma(\eta+1). \end{aligned} \quad (16)$$

Further, we choose parameters α, η such that the function $u(x,t)$ defined by (14) satisfies (5) and the initial conditions (8).

Let $\eta = (n/2) - 1$, $\alpha = p - (n-1)/2$ and $p > (n-1)/2$. Then (15) is transformed to the Darboux equation. It is known from [30] that the solution of the problem {(15), (16)} in this case is unique and represented by $M_n(x,t;\tau)$, which is the spherical mean of the function $\tau(x)$ in the space R^n, by the formula

$$V(x,s) = k_0 M_n(x,s;\tau) = \frac{k_0}{\omega_n s^{n-1}} \int_{|\xi - x| = s} \tau(\xi) d\sigma_\xi = \frac{k_0}{\omega_n} \int_{|y|=1} \tau(x+sy) d\omega, \tag{17}$$

where $|\xi - x|^2 = \sum_{k=1}^n (\xi_k - x_k)^2$, $d\omega$ is the area-element of the unit sphere, and $\omega_n = 2\pi^{n/2}/\Gamma(n/2)$ is the area of its surface.

It is easy to verify that the function $M_n(x,s;\tau)$ satisfies the initial conditions

$$\lim_{s\to 0} M_n(x,s;\tau) = \tau(x), \qquad \lim_{s\to 0} \frac{\partial M_n(x,s;\tau)}{\partial s} = 0, \quad x \in R^n. \tag{18}$$

Substituting (17) into the equality (14) we obtain

$$u(x,t) = \frac{\Gamma(p+1/2)t^{1-2p}}{\pi^{n/2}\Gamma(p-(n-1)/2)} \times \int_{|\xi - x| \le t} \tau(\xi) \frac{\bar{J}_{p-(n+1)/2}(\lambda\sqrt{t^2 - |\xi - x|^2})}{[t^2 - |\xi - x|^2]^{(n+1)/2-p}} d\xi. \tag{19}$$

If $\tau(x) \in C^2(R^n)$, then by virtue of Lemma 1, the function (19) will be a regular solution of (5) satisfying the initial conditions (6).

Note that in the case when $p < (n-1)/2$, the function (19) will be the solution of the problem {(5), (6)}, if one uses the pre-definition of the operator (14) for $\alpha < 0$ based on (11):

$$\begin{aligned} J_\lambda^{(t)}(\eta,\alpha)V(x,t) &= t^{-2(\eta+\alpha)} \left(\frac{\partial}{2t\partial t}\right)^m t^{2(\eta+\alpha+m)} J_\lambda^{(t)}(\eta, \alpha+m) \\ &= \frac{2t^{-2(\eta+\alpha)}}{\Gamma(\alpha+m)} \left(\frac{\partial}{2t\partial t}\right)^m \\ &\quad \times \int_0^t s^{2\eta+1} \left(t^2 - s^2\right)^{\alpha+m-1} J_{\alpha+m-1}\left(\lambda\sqrt{t^2-s^2}\right) V(x,s) ds, \end{aligned}$$

where $-m < \alpha < 0$, $m = 1,2,3\ldots$. In this case we choose m to be the smallest positive integer satisfying the inequality $p + m > (n-1)/2$.

Here one can see that the range of the parameter p depends on the dimension of the space R^n. There is a question: how to find the solution of the considered problem for any n, if the range of the parameter p is fixed in advance, for instance, $0 < p < 1/2$?

In this case we choose the parameter η so that the function $u(x,t)$ which is defined by (14) satisfies (5) and the initial conditions (6).

Let $\eta = -1/2$, then the parameter $\alpha = p$ and (15) can be transformed to the n-dimensional wave equation.

In this case the solution of the problem $\{(15), (16)\}$ for odd n has the form ([30]):

$$V(x,s) = \gamma_1 k_0 \frac{\partial}{\partial s}\left(\frac{1}{s}\frac{\partial}{\partial s}\right)^{(n-3)/2}\left(s^{n-2}M_n(x,s;\tau)\right), \tag{20}$$

where $\gamma_1 = 1/[1 \cdot 3 \cdot \dots \cdot (n-2)]$.

Let $n = 2m+1$, then the solution (20) has the form

$$V(x,s) = \gamma_1 k_0 \frac{\partial}{\partial s}\left(\frac{1}{s}\frac{\partial}{\partial s}\right)^{m-1}\left(s^{2m-1}M_{2m+1}(x,s;\tau)\right). \tag{21}$$

The solution of the problem $\{(15), (16)\}$ for even n can be written as ([30]):

$$V(x,s) = \gamma_2 k_0 \frac{\partial}{\partial s}\left(\frac{1}{s}\frac{\partial}{\partial s}\right)^{(n-2)/2}\left(\int_0^s M_n(x,\rho;\tau)\frac{\rho^{n-1}d\rho}{\sqrt{s^2-\rho^2}}\right), \tag{22}$$

where $\gamma_2 = 1/[2 \cdot 4 \cdot \dots \cdot (n-2)]$.

Let $n = 2m$, then the solution (22) will have the form

$$V(x,s) = \gamma_2 k_0 \frac{\partial}{\partial s}\left(\frac{1}{s}\frac{\partial}{\partial s}\right)^{m-1}\left(\int_0^s M_{2m}(x,\rho;\tau)\frac{\rho^{2m-1}d\rho}{\sqrt{s^2-\rho^2}}\right). \tag{23}$$

Combining solutions (21) and (23), we obtain

$$V(x,s) = \gamma \frac{\partial}{\partial s}\left(\frac{1}{s}\frac{\partial}{\partial s}\right)^{m-1} T(x,s), \tag{24}$$

where

$$\gamma = \begin{cases} \gamma_1 k_0, & n = 2m+1, \\ \gamma_2 k_0, & n = 2m, \end{cases}$$

$$T(x,s) = \begin{cases} s^{2m-1}M_{2m+1}(x,s;\tau), & n = 2m+1, \\ \int_0^s M_{2m}(x,\rho;\tau)\frac{\rho^{2m-1}d\rho}{\sqrt{s^2-\rho^2}}, & n = 2m. \end{cases}$$

Substituting (24) into the formula (14) we have

$$u(x,t) = \frac{2\gamma t^{1-2p}}{\Gamma(p)}\int_0^t \frac{\bar{J}_{p-1}(\lambda\sqrt{t^2-s^2})}{(t^2-s^2)^{1-p}}\frac{\partial}{\partial s}\left(\frac{1}{s}\frac{\partial}{\partial s}\right)^{m-1} T(x,s)ds. \tag{25}$$

The following lemmas hold true:

Lemma 2 *If $\tau(x)$ is m times continuously differentiable, then*

$$\lim_{s\to 0}\left(\frac{1}{s}\frac{\partial}{\partial s}\right)^{m-1} T(x,s) = 0, \quad m = 1, 2, \dots.$$

Proof Considering (17) and (18) we rewrite the function $T(x,s)$ as $T(x,s) = s^{2m-1}T_0(x,s)$, where

$$T_0(x,s) = \begin{cases} M_{2m+1}(x,s;\tau), & n = 2m+1, \\ \int_0^1 M_{2m}(x,s\zeta;\tau)\frac{\zeta^{2m-1}d\zeta}{\sqrt{1-\zeta^2}}, & n = 2m. \end{cases}$$

Then

$$\left(\frac{1}{s}\frac{\partial}{\partial s}\right)^{m-1} T(x,s) = T_0(x,s)\left(\frac{1}{s}\frac{\partial}{\partial s}\right)^{m-1} s^{2m-1} + s^{2m-1}\left(\frac{1}{s}\frac{\partial}{\partial s}\right)^{m-1} T_0(x,s).$$

Further, considering the equality

$$\left(\frac{1}{s}\frac{\partial}{\partial s}\right)^{m-1} s^{2m-1} = s\prod_{k=1}^{m-1}\left[2m-(2k-1)\right],$$

$$\left(\frac{1}{s}\frac{\partial}{\partial s}\right)^{m-1} T_0(x,s) = O\left(s^{-2m+3}\right),$$

we obtain $(\frac{1}{s}\frac{\partial}{\partial s})^{m-1}T(x,s) = O(s)$, from which the statement of the Lemma 2 follows. □

Lemma 3 *Under the conditions of Lemma 2 the equality*

$$\int_0^t \left(t^2-s^2\right)^{p-1}\bar{J}_{p-1}\left(\lambda\sqrt{t^2-s^2}\right)\left[\left(\frac{1}{s}\frac{\partial}{\partial s}\right)^m T(x,s)\right]sds$$
$$= \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^m \int_0^t \left(t^2-s^2\right)^{p-1}\bar{J}_{p-1}\left(\lambda\sqrt{t^2-s^2}\right)T(x,s)sds \qquad (26)$$

holds true.

Proof We prove this lemma using the method of mathematical induction. First, we prove that (25) is true for $m = 1$.

Consider the function

$$u_\varepsilon(x,t) = \int_0^{t-\varepsilon} \left(t^2-s^2\right)^{p-1}\bar{J}_{p-1}\left(\lambda\sqrt{t^2-s^2}\right)\frac{\partial}{\partial s}T(x,s)ds,$$

where ε is a small enough positive real number.

Applying integration by parts to the latter integral and considering statements of the Lemma 2, we obtain

$$u_\varepsilon(x,t) = \left[t^2-(t-\varepsilon)^2\right]^{p-1}\bar{J}_{p-1}\left(\lambda\sqrt{t^2-(t-\varepsilon)^2}\right)T(x,t-\varepsilon)$$
$$-\int_0^{t-\varepsilon}\frac{\partial}{\partial s}\left[\left(t^2-s^2\right)^{p-1}\bar{J}_{p-1}\left(\lambda\sqrt{t^2-s^2}\right)\right]T(x,s)ds.$$

Further, taking into account the following easily checkable equalities

$$\frac{\partial}{\partial s}\big[\big(t^2-s^2\big)^{p-1}\bar{J}_{p-1}\big(\lambda\sqrt{t^2-s^2}\big)\big]=-\frac{s}{t}\frac{\partial}{\partial t}\big[\big(t^2-s^2\big)^{p-1}\bar{J}_{p-1}\big(\lambda\sqrt{t^2-s^2}\big)\big],$$

$$\int_0^{t-\varepsilon}\frac{1}{t}\frac{\partial}{\partial t}\big[\big(t^2-s^2\big)^{p-1}\bar{J}_{p-1}\big(\lambda\sqrt{t^2-s^2}\big)\big]T(x,s)sds$$
$$=\left(\frac{1}{t}\frac{\partial}{\partial t}\right)\int_0^{t-\varepsilon}\big(t^2-s^2\big)^{p-1}\bar{J}_{p-1}\big(\lambda\sqrt{t^2-s^2}\big)T(x,s)sds$$
$$-\frac{t-\varepsilon}{t}\big[t^2-(t-\varepsilon)^2\big]^{p-1}\bar{J}_{p-1}\big(\lambda\sqrt{t^2-(t-\varepsilon)^2}\big)T(x,t-\varepsilon),$$

we have

$$u_\varepsilon=\frac{1}{t}\big[t-(t-\varepsilon)\big]^p\big[t+(t-\varepsilon)\big]^{p-1}\bar{J}_{p-1}\big(\lambda\sqrt{t^2-(t-\varepsilon)^2}\big)T(x,t-\varepsilon)$$
$$+\left(\frac{1}{t}\frac{\partial}{\partial t}\right)\int_0^{t-\varepsilon}\big(t^2-s^2\big)^{p-1}\bar{J}_{p-1}\big(\lambda\sqrt{t^2-s^2}\big)T(x,s)sds.$$

From here, by virtue of $p>0$, after $\varepsilon\to 0$ we get

$$\int_0^t\big(t^2-s^2\big)^{p-1}\bar{J}_{p-1}\big(\lambda\sqrt{t^2-s^2}\big)\left(\frac{1}{s}\frac{\partial}{\partial s}\right)T(x,s)sds$$
$$=\left(\frac{1}{t}\frac{\partial}{\partial t}\right)\int_0^t\big(t^2-s^2\big)^{p-1}\bar{J}_{p-1}\big(\lambda\sqrt{t^2-s^2}\big)T(x,s)sds. \qquad (27)$$

Assume that formula (26) holds for $m=k-1$. We prove that it is valid also for $m=k$:

$$\int_0^t\big(t^2-s^2\big)^{p-1}\bar{J}_{p-1}\big(\lambda\sqrt{t^2-s^2}\big)\left(\frac{1}{s}\frac{\partial}{\partial s}\right)^k T(x,s)sds$$
$$=\int_0^t\big(t^2-s^2\big)^{p-1}\bar{J}_{p-1}\big(\lambda\sqrt{t^2-s^2}\big)\left(\frac{1}{s}\frac{\partial}{\partial s}\right)^{k-1}\left[\frac{1}{s}\frac{\partial}{\partial s}T(x,s)\right]sds$$
$$=\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{k-1}\int_0^t\big(t^2-s^2\big)^{p-1}\bar{J}_{p-1}\big(\lambda\sqrt{t^2-s^2}\big)\left[\frac{1}{s}\frac{\partial}{\partial s}T(x,s)\right]sds.$$

Further, considering (27) we get the statement of Lemma 3. □

Now, applying Lemma 3 to (25) we obtain

$$u(x,t)=\frac{2\gamma t^{1-2p}}{\Gamma(p)}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^m\int_0^t\big(t^2-s^2\big)^{p-1}\bar{J}_{p-1}\big(\lambda\sqrt{t^2-s^2}\big)T(x,s)sds. \quad (28)$$

Let $n=2m+1$. Then, substituting the value of the function $T(x,s)$ into (28) we deduce

$$u(x,t)=\frac{\gamma_1\Gamma(p+1/2)\Gamma(n/2)}{\pi^{(n+1)/2}\Gamma(p)}t^{1-2p}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{(n-1)/2}$$
$$\times\int_{|\xi-x|\le t}\tau(\xi)\left(t^2-|x-\xi|^2\right)^{p-1}\bar{J}_{p-1}\left(\lambda\sqrt{t^2-|x-\xi|^2}\right)d\xi. \quad (29)$$

We now construct a solution of (5) for odd n satisfying the conditions

$$w(x,0)=0, \qquad \lim_{t\to+0}t^{2p}w_t(x,t)=\nu(x), \quad x\in R^n, \quad (30)$$

where $\nu(x)\in C^{[n/2]+1}(R^n)$ is a given function, and $[n/2]$ means the integer part of the number $n/2$.

Let function $u(x,t;1-p)$ be a solution of the equation $L^{\lambda}_{1-p}(u)=0$ satisfying conditions (6). Then by virtue of the property 1 of (5), the function $w(x,t;p)=t^{1-2p}u(x,t;1-p)$ will be a solution of the equation $L^{\lambda}_{p}(w)=0$, satisfying conditions (30). Further, substituting $(1-2p)\tau(x)$ to $\nu(x)$, we get

$$w(x,t;p)=t^{1-2p}u(x,t;1-p)$$
$$=\frac{\gamma_1\Gamma[(1/2)-p]\Gamma(n/2)}{\pi^{(n+1)/2}\Gamma(1-p)}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{(n-1)/2}$$
$$\times\int_{|\xi-x|\le t}\nu(\xi)\left(t^2-|x-\xi|^2\right)^{-p}\bar{J}_{-p}\left(\lambda\sqrt{t^2-|x-\xi|^2}\right)d\xi. \quad (31)$$

Thus, if $\tau(x)\in C^{[n/2]+2}(R^n)$, $\nu(x)\in C^{[n/2]+1}(R^n)$, then the sum of the functions (29) and (31) for odd n is a solution of (5), satisfying conditions (8).

Let $n=2m$. Then, substituting the value of the function $T(x,s)$ into formula (28) we obtain

$$u(x,t)=\frac{2\gamma_2k_0t^{1-2p}}{\Gamma(p)}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{m}\int_0^t\left(t^2-s^2\right)^{p-1}\bar{J}_{p-1}\left(\lambda\sqrt{t^2-s^2}\right)$$
$$\times\left\{\int_0^s M_{2m}(x,\rho;\tau)\frac{\rho^{2m-1}d\rho}{\sqrt{s^2-\rho^2}}\right\}sds.$$

Changing the order of integration by the Dirichlet formula we have

$$u(x,t)=\frac{2\gamma_2k_0t^{1-2p}}{\Gamma(p)}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{m}\int_0^t M_{2m}(x,\rho;\tau)\rho^{2m-1}d\rho$$
$$\times\int_\rho^t\left(s^2-\rho^2\right)^{-(1/2)}\left(t^2-s^2\right)^{p-1}\bar{J}_{p-1}\left(\lambda\sqrt{t^2-s^2}\right)sds. \quad (32)$$

We now evaluate the inner integral. Using the expansion of the Bessel-Clifford function into a series, and calculating the obtained integral we get

$$\int_\rho^t \left(s^2-\rho^2\right)^{-(1/2)}\left(t^2-s^2\right)^{p-1}\bar{J}_{p-1}\left(\lambda\sqrt{t^2-s^2}\right)sds$$
$$=\frac{1}{2}\frac{\Gamma(p)\Gamma(1/2)}{\Gamma[p+(1/2)]}\left(t^2-\rho^2\right)^{p-1/2}\bar{J}_{p-(1/2)}\left(\lambda\sqrt{t^2-\rho^2}\right). \tag{33}$$

Substituting (33) into (32) we obtain

$$u(x,t)=\frac{\gamma_2}{\omega_n}t^{1-2p}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{(n/2)}\int_{|\xi-x|\le t}\tau(\xi)\left[t^2-|\xi-x|^2\right]^{p-1/2}$$
$$\times\bar{J}_{p-(1/2)}\left(\lambda\sqrt{t^2-|\xi-x|^2}\right)d\xi. \tag{34}$$

Similarly, as in the case when n is odd, for even n we get a solution of the problem (5), (30) as

$$w(x,t)=\frac{\gamma_2}{\omega_n(1-2p)}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{(n/2)}\int_{|\xi-x|\le t}\nu(\xi)\left[t^2-|\xi-x|^2\right]^{(1/2)-p}$$
$$\times\bar{J}_{(1/2)-p}\left(\lambda\sqrt{t^2-|\xi-x|^2}\right)d\xi. \tag{35}$$

Thus, if $\tau(x)\in C^{[n/2]+2}(R^n)$, $\nu(x)\in C^{[n/2]+1}(R^n)$, then the sum of the functions (34) and (35) for even n will be the solution of (5) satisfying conditions (8).

The formulae (29) and (31) for odd n, and formulae (34) and (35) for even n were obtained for $0<p<(1/2)$. For other values of the parameter $p\neq(1/2),(3/2),\ldots,$ the solution will be defined by the analytic continuation of the operator $J_\lambda(\eta,\alpha)$ in the parameter $\alpha=p$.

When $\tau(x)$ and $\nu(x)$ are arbitrary functions, then the sum of the functions (29) and (31) for odd n, and formulae (34), (35) for even n, respectively, give the general solution of (5). Assume that $p=(1/2)$. Then these sums contain only one arbitrary function. Therefore, it is not a general solution of (5) for $p=(1/2)$.

Naturally, it is interesting to find a general solution of (5) for $p=(1/2)$, because with the help of the general solution for any equation one can find information on correct initial and boundary problems for this equation.

Let n be odd. Then by virtue of $\bar{J}_{(-1/2)}(z)=\cos(z)$ from the formula (29) for arbitrary $\varphi(x)\in C^{[n/2]+2}(R^n)$, it follows that the function $u(x,t)$ defined by the formula

$$u(x,t)=\frac{\gamma_1\Gamma(n/2)}{\pi^{(n+3)/2}}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{(n-1)/2}$$
$$\times\int_{|\xi-x|\le t}\varphi(\xi)\frac{\cos(\lambda\sqrt{t^2-|\xi-x|^2})}{\sqrt{t^2-|\xi-x|^2}}d\xi \tag{36}$$

will be a solution of (5).

In order to construct a second linear-independent solution of (5), we replace in formulae (29) and (31) the functions $\tau(x)$ and $\nu(x)$ by an arbitrary function $g(x) \in C^{[n/2]+2}(R^n)$ and rewrite it as

$$u(x,t) = \frac{2\Gamma[(1/2)+p]}{\sqrt{\pi}\Gamma(p)} \times \int_0^1 \left(1-z^2\right)^{p-1} \bar{J}_{p-1}\left(\lambda t\sqrt{1-z^2}\right) P_n(x,tz;g)dz, \tag{37}$$

$$w(x,t) = \frac{2\Gamma[(3/2)-p]t^{1-2p}}{\sqrt{\pi}\Gamma(1-p)(1-2p)} \times \int_0^1 \left(1-z^2\right)^{-p} \bar{J}_{-p}\left(\lambda t\sqrt{1-z^2}\right) P_n(x,tz;g)dz, \tag{38}$$

where

$$P_n(x,s;g) = \gamma_1 \frac{\partial}{\partial s}\left(\frac{1}{s}\frac{\partial}{\partial s}\right)^{(n-3)/2} \left(s^{n-2} M_n\left(x,s;g(x)\right)\right). \tag{39}$$

Calculating the derivatives appearing in equality (39) we deduce

$$P_n(x,s;g) = M_n(x,s;g) + A_1 s\frac{\partial M_n}{\partial s} + A_2 s^2 \frac{\partial^2 M_n}{\partial s^2} + \cdots + A_n s^{n-3} \frac{\partial^{(n-3)/2} M_n}{\partial s^{(n-3)/2}},$$

where A_k $(k = \overline{1,n})$ are some constants.

By virtue of (18), from the latter equality it follows that the function $P_n(x,s;g)$ satisfies the conditions

$$\lim_{s\to 0} P_n(x,s;g) = g(x), \qquad \lim_{s\to 0} \frac{\partial P_n(x,s;g)}{\partial s} = 0, \quad x \in R^n.$$

It is obvious that the linear combination of the expressions (37) and (38) of the form

$$W(x,t) = \frac{u(x,t)}{1-2p} - \frac{\Gamma(1-p)\Gamma[(1/2)+p]}{\Gamma[(3/2)-p]\Gamma(p)} w(x,t)$$

will be a solution of (5). We rewrite this combination as

$$W(x,t) = \frac{2\Gamma[(1/2)+p]}{\sqrt{\pi}\Gamma(p)} \int_0^1 \left(1-z^2\right)^{p-1} \frac{1}{1-2p} \times \left\{\bar{J}_{p-1}\left(\lambda t\sqrt{1-z^2}\right) - \left[t\left(1-z^2\right)\right]^{1-2p} \bar{J}_{-p}\left(\lambda t\sqrt{1-z^2}\right)\right\} \times P_n(s,tz;g)dz. \tag{40}$$

Considering

$$\begin{aligned}&\frac{\bar{J}_{p-1}(\lambda t\sqrt{1-z^2})-[t(1-z^2)]^{1-2p}\bar{J}_{-p}(\lambda t\sqrt{1-z^2})}{1-2p}\\&\quad=\frac{\bar{J}_{p-1}(\lambda t\sqrt{1-z^2})-\bar{J}_{-p}(\lambda t\sqrt{1-z^2})}{1-2p}\\&\qquad+\frac{1-[t(1-z^2)]^{1-2p}}{1-2p}\bar{J}_{-p}\big(\lambda t\sqrt{1-z^2}\big),\end{aligned}$$

passing to the limit as $p\to(1/2)$, and taking into consideration the equalities

$$\lim_{p\to(1/2)}\frac{1-[t(1-z^2)]^{1-2p}}{1-2p}\bar{J}_{-p}\big(\lambda t\sqrt{1-z^2}\big)=-\cos\big(\lambda t\sqrt{1-z^2}\big)\ln\big[t\big(1-z^2\big)\big],$$

$$\lim_{p\to(1/2)}\frac{\bar{J}_{p-1}(\lambda t\sqrt{1-z^2})-\bar{J}_{-p}(\lambda t\sqrt{1-z^2})}{1-2p}=-B_{-(1/2)}\big(\lambda t\sqrt{1-z^2}\big),$$

we deduce from (40) that

$$\begin{aligned}W_1(x,t;g)&=\lim_{p\to(1/2)}W(x,t)\\&=-\frac{2}{\pi}\int_0^1\big\{\cos\big(\lambda t\sqrt{1-z^2}\big)\ln\big[t\big(1-z^2\big)\big]+B_{-(1/2)}\big(\lambda t\sqrt{1-z^2}\big)\big\}\\&\quad\times\big(1-z^2\big)^{-(1/2)}P_n(x,tz;g)dz.\end{aligned}\tag{41}$$

Here

$$B_\nu(\sigma)=\Gamma(\nu+1)\sum_{k=1}^{\infty}\frac{(-1)^k(\sigma/2)^{2k}}{k!\Gamma(\nu+k+1)}\big[\psi(\nu+1)-\psi(k+\nu+1)\big],\tag{42}$$

and $\psi(z)=[\Gamma'(z)/\Gamma(z)]$ is the logarithmic derivative of the Gamma-function ([27]).

Consequently, in the case $p=(1/2)$ and odd n, the general solution of (5), in accordance with (36) and (41), has the form

$$\begin{aligned}u(x,t)&=\frac{2}{\pi}\int_0^1\big(1-z^2\big)^{-(1/2)}\cos\big(\lambda t\sqrt{1-z^2}\big)P_n\big(x,tz;\varphi(x)\big)dz\\&\quad-\frac{2}{\pi}\int_0^1\big\{\cos\big(\lambda t\sqrt{1-z^2}\big)\ln\big[t\big(1-z^2\big)\big]+B_{-(1/2)}\big(\lambda t\sqrt{1-z^2}\big)\big\}\\&\quad\times\big(1-z^2\big)^{-(1/2)}P_n\big(x,tz;g(x)\big)dz,\end{aligned}\tag{43}$$

where $P_n(x,s;f)$ is the function, defined by (39), and $\varphi(x)$, $g(x)$ are arbitrary functions from the class of functions $C^{[n/2]+2}(R^n)$.

Now consider the case when n is even, $p=(1/2)$. In this case one of the solutions of (5) will be the function:

$$u(x,t)=\frac{\gamma_2}{\omega_n}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{(n/2)}\int_{|\xi-x|\le t}\varphi(\xi)J_0\big(\lambda\sqrt{t^2-|\xi-x|^2}\big)d\xi, \tag{44}$$

which follows from (34) at $p=(1/2)$, here $\varphi(x)\in C^{[n/2]+2}(R^n)$ is an arbitrary function.

With the aim to find a second linearly-independent solution of (5), we replace in formulae (34), (35), the functions $\tau(x)$ and $\nu(x)$ by an arbitrary function $g(x)\in C^{[n/2]+2}(R^n)$, and rewrite them as

$$\begin{aligned}u(x,t)=&\int_0^1\big[2\bar{J}_{p-(1/2)}(\sigma)-(1-2p)\bar{J}_{p-(3/2)}(\sigma)\big]\\&\times Q_n(x,tz;g)\big(1-z^2\big)^{p-(1/2)}zdz\\&+\int_0^1\bar{J}_{p-(1/2)}(\sigma)t\frac{\partial Q_n(x,tz;g)}{\partial t}\big(1-z^2\big)^{p-(1/2)}zdz,\end{aligned} \tag{45}$$

$$\begin{aligned}w(x,t)=&\frac{t^{1-2p}}{1-2p}\int_0^1\big[2\bar{J}_{(1/2)-p}(\sigma)+(1-2p)\bar{J}_{-p-(1/2)}(\sigma)\big]\\&\times Q_n(x,tz;g)\big(1-z^2\big)^{(1/2)-p}zdz\\&+\frac{t^{1-2p}}{1-2p}\int_0^1\bar{J}_{(1/2)-p}(\sigma)t\frac{\partial Q_n(x,tz;g)}{\partial t}\big(1-z^2\big)^{(1/2)-p}zdz,\end{aligned} \tag{46}$$

where $\sigma=\lambda t\sqrt{1-z^2}$,

$$Q_n(x,s;g)=\gamma_2\frac{\partial}{\partial s}\left(\frac{1}{s}\frac{\partial}{\partial s}\right)^{(n-2)/2}\big(s^{n-2}M_n\big(x,s;g(x)\big)\big). \tag{47}$$

Calculating all the necessary derivatives in (47) we obtain

$$\begin{aligned}Q_n(x,s;g)=&M_n(x,s;g)+C_1s\frac{\partial M_n}{\partial s}\\&+C_2s^2\frac{\partial^2M_n}{\partial s^2}+\cdots+C_ns^{n-2}\frac{\partial^{(n-2)/2}M_n}{\partial s^{(n-2)/2}},\end{aligned}$$

where C_k $(k=\overline{1,n})$ are some well-defined constants.

By virtue of (18), from the latter equality it follows that the function $Q_n(x,s;g)$ satisfies the conditions

$$\lim_{s\to 0}Q_n(x,s;g)=g(x),\qquad \lim_{s\to 0}\frac{\partial Q_n(x,s;g)}{\partial s}=0,\quad x\in R^n.$$

The following linear combination of the functions (45), (46),

$$\begin{aligned} W^*(x,t) &= \frac{u(x,t)}{1-2p} - w(x,t) \\ &= \int_0^1 \frac{(1-z^2)^{p-(1/2)}}{1-2p} \left\{ \bar{J}_{p-(1/2)}(\sigma) - \left[t\left(1-z^2\right)\right]^{1-2p} \bar{J}_{(1/2)-p}(\sigma) \right\} \\ &\quad \times \left[2Q_n(x,tz;g) + t\frac{\partial Q_n(x,tz;g)}{\partial t} \right] z dz \\ &\quad - \int_0^1 \left\{ \bar{J}_{p-(3/2)}(\sigma) + \left[t\left(1-z^2\right)\right]^{1-2p} \bar{J}_{-p-(1/2)}(\sigma) \right\} \\ &\quad \times Q_n(x,tz;g)\left(1-z^2\right)^{p-(1/2)} z dz \end{aligned} \tag{48}$$

will be a solution of (5).

In the equality (48) we pass to the limit as $p \to (1/2)$, and we have

$$\begin{aligned} W_2(x,t;g) &= \lim_{p\to(1/2)} W^*(x,t) \\ &= -\int_0^1 \left\{ J_0(\sigma) \ln\left[t\left(1-z^2\right)\right] + B_0(\sigma) \right\} \\ &\quad \times \left[2Q_n(x,tz;g) + t\frac{\partial Q_n(x,tz;g)}{\partial t} \right] z dz \\ &\quad - 2\int_0^1 \left\{ 1 + B^*(\sigma) \right\} Q_n(x,tz;g) z dz, \end{aligned} \tag{49}$$

where $B_0(\sigma)$ is the function defined by (42), $\sigma = \lambda t\sqrt{1-z^2}$,

$$B^*(\sigma) = \sum_{k=1}^{\infty} \frac{(-1)^k (\sigma/2)^{2k}}{k!\Gamma(k)} \left\{ \psi(1) - \psi(k) + \ln\left[t\left(1-z^2\right)\right] \right\},$$

such that $B^*(\sigma) = O(\sigma^2[C + \ln\sigma])$, $C = \text{const}$.

Consequently, in the case $p = (1/2)$ and even n, the general solution of (5), in accordance with (44), (49), has the form

$$\begin{aligned} u(x,t) &= \int_0^1 J_0(\sigma) \left[2Q_n(x,tz;\varphi) + t\frac{\partial Q_n(x,tz;\varphi)}{\partial t} \right] z dz \\ &\quad - \int_0^1 \left\{ J_0(\sigma) \ln\left[t\left(1-z^2\right)\right] + B_0(\sigma) \right\} \\ &\quad \times \left[2Q_n(x,tz;g) + t\frac{\partial Q_n(x,tz;g)}{\partial t} \right] z dz \\ &\quad - 2\int_0^1 \left\{ 1 + B^*(\sigma) \right\} Q_n(x,tz;g) z dz, \end{aligned} \tag{50}$$

where $Q_n(x, s; \varphi)$ is the function defined by (47), and $\varphi(x)$, $g(x)$ are arbitrary functions from $C^{[n/2]+2}(R^n)$.

From formulae (43) and (50), which give the general solution of (5) for $p = (1/2)$, it follows that the Cauchy problem for this equation with initial conditions (8) is not correctly formulated. In this case the initial conditions should be given in a modified form. Precisely, in the case when n is odd, they should be given in the form of

$$\lim_{t\to+0} \frac{u(x,t)}{(-\ln t)} = \tau(x), \qquad \lim_{t\to+0} t(\ln t)^2 \frac{\partial}{\partial t}\left[\frac{u(x,t) - W_1(x,t;\tau)}{(-\ln t)}\right] = \nu(x), \quad (51)$$

and in the case when n is even, in the form of

$$\lim_{t\to+0} \frac{u(x,t)}{(-\ln t)} = \tau(x), \qquad \lim_{t\to+0} t(\ln t)^2 \frac{\partial}{\partial t}\left[\frac{u(x,t) - W_2(x,t;\tau)}{(-\ln t)}\right] = \nu(x). \quad (52)$$

Here W_1 and W_2 are functions which are defined by (41) and (49), respectively.

Remark Using property 2 of (5) in the case when $p = l + (1/2)$, $l = 1, 2, \ldots,$ one can find a formula for a general solution of this equation.

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Chapter 16
Quasi-symmetrizer and Hyperbolic Equations

Giovanni Taglialatela

Abstract Given a matrix A, a *symmetrizer* for A is a symmetric matrix $\mathcal{Q}$ such that $\mathcal{Q}A$ is symmetric. The symmetrizer is a useful tool to obtain a-priori estimates for the solutions to hyperbolic equations. If $\mathcal{Q}$ is not positive definite, it is more convenient to consider a *quasi-symmetrizer*: a sequence of symmetric and positive defined matrices $\{\mathcal{Q}_\varepsilon\}_{\varepsilon\in]0,1]}$ such that $\mathcal{Q}_\varepsilon A$ approaches a symmetric matrix.

In these notes we make a short survey of the basic notions of symmetrizer and quasi-symmetrizer and we give some applications to the well-posedness for the hyperbolic Cauchy problem.

Mathematics Subject Classification 35L30

16.1 The Cauchy Problem

These notes are a short survey on some results concerning the well-posedness of the Cauchy problem

$$\begin{cases} L(t;\partial_t,\partial_x)u(t,x)=f(t,x), & (t,x)\in]-T,T[\times\mathbb{R}, \\ \partial_t^j u(0,x)=u_j(x), & x\in\mathbb{R},\ j=0,\dots,\mathrm{N}-1, \end{cases} \tag{CP}$$

where $L(t;\partial_t,\partial_x)$ is a hyperbolic operator of order N with coefficients depending only on the time variable:

$$L(t;\partial_t,\partial_x)u\equiv\partial_t^{\mathrm{N}}u-\sum_{j+|\alpha|=\mathrm{N}}a_\alpha(t)\partial_t^j\partial_x^\alpha u-\sum_{j+|\alpha|<N}b_{j,\nu}(t)\partial_t^j\partial_x^\alpha u.$$

We say that the Cauchy problem (CP) is well-posed in $\mathcal{C}^\infty$ (resp. γ^s), if for any $u_j\in\mathcal{C}^\infty(\mathbb{R}^n)$ (resp. $u_j\in\gamma^s(\mathbb{R}^n)$), $j=0,\dots,m-1$, and any $f\in\mathcal{C}([-T,T];\mathcal{C}^\infty(\mathbb{R}^n))$ (resp. $f\in\mathcal{C}([-T,T];\gamma^s(\mathbb{R}^n))$), the Cauchy problem (CP)

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admits a unique solution $u \in \mathcal{C}^{\mathrm{N}}([-T, T]; \mathcal{C}^{\infty}(\mathbb{R}^n))$ (resp. $u \in \mathcal{C}^{\mathrm{N}}([-T, T]; \gamma^s(\mathbb{R}^n)))$.

Here $\gamma^s = \gamma^s(\mathbb{R}^n)$ is the space of Gevrey functions, that is, the space of functions $f \in \mathcal{C}^{\infty}(\mathbb{R}^n)$ such that for any compact st K of $\mathbb{R}^n$ there exists C_K such that

$$\sup_{x \in K} \left|\partial_x^{\alpha} f(x)\right| \leq C_K |\alpha|!^s, \quad \text{for any } \alpha \in \mathbb{N}^n.$$

We begin with a rapid survey on some well-known results on hyperbolic equations. In this paper we restrict our study to operators whose smooth coefficients depend only on the time variable, but the following general results are valid also for space dependent coefficients.

The classical Lax-Mizohata theorem states that, in order the Cauchy problem (CP) to be well-posed in $\mathcal{C}^{\infty}$ or in Gevrey spaces, the principal symbol of L:

$$P(t; \tau, \xi) \equiv \tau^{\mathrm{N}} - \sum_{j+|\alpha|=\mathrm{N}} a_{\alpha}(t) \tau^j \xi^{\alpha}$$

should be hyperbolic; this means that the characteristic roots $\tau_1(t, \xi), \ldots, \tau_{\mathrm{N}}(t, \xi)$, i.e. the solutions in τ of the characteristic equation

$$P(t; \tau, \xi) = 0$$

are all real.

On the other side if L is *strictly* (*regularly*) *hyperbolic*, i.e. the characteristic roots are distinct and verify

$$\left|\tau_j(t; \xi) - \tau_k(t; \xi)\right| \geq \delta|\xi|,$$

for some $\delta > 0$ if $j \neq k$, then (CP) is well-posed in $\mathcal{C}^{\infty}$ and in all Gevrey classes.

If L is *weakly hyperbolic*, i.e. $\delta = 0$, so that the characteristic roots may coincide, the situation becomes more difficult. A complete characterization of the $\mathcal{C}^{\infty}$ and γ^s well-posedness has been obtained for operators with constant coefficients principal part [14, 26, 28], or when the multiplicity of the characteristic roots is constant [2, 8, 13]. Bronšteĭn [1] proved that the Cauchy problem (CP) is well-posed in γ^d for $1 < d < d_B = \frac{r}{r-1}$, where r is the largest multiplicity of the characteristic roots. In general the bound d_B is sharp, unless one assumes further conditions on the principal symbol and on the lower order terms.

To illustrate the difficulties in the weakly hyperbolic case, we recall the following examples.

Example 1 Colombini and Spagnolo constructed in [4] a function $a \in \mathcal{C}^{\infty}$ verifying $a(t) > 0$ for $t \neq 0$ and exhibiting infinite oscillations as $t \to 0^+$ such that the Cauchy problem at $t = 0$ for the operator

$$\partial_t^2 - a(t)\partial_x^2$$

is well-posed in all the Gevrey classes, but not well-posed in $\mathcal{C}^{\infty}$.

To prevent the oscillations in the principal symbols one can assume the coefficients to be analytic. However, the next example shows that even if we assume analytic coefficients the well-posedness may fail. Thus some extra conditions are needed.

Example 2 The Cauchy problem for the operator

$$\partial_t^2 - 2t\partial_t\partial_x + t^2\partial_x^2$$

is well-posed in γ^s for $s < 2$ and not well-posed in γ^s for $s > 2$ and in $\mathcal{C}^\infty$.

The last example shows that the influence of the lower-order terms may destroy the well-posedness.

Example 3 The Cauchy problem for the operator

$$\partial_t^2 - \partial_x$$

is well-posed in γ^s for $s < 2$ and not well-posed in γ^s for $s > 2$ and in $\mathcal{C}^\infty$.

16.2 Reduction to a System Depending on a Parameter $\xi \in \mathbb{R}^n$

We will limit our presentation to third order equations in only one space variable, but the method can be applied to the general case in a similar way.

Consider the Cauchy problem

$$\begin{cases} L(t;\partial_t,\partial_x)u \\ \quad \equiv \partial_t^3 u - \sum_{j+|\alpha|=3} a_{j,\alpha}(t)\partial_t^{3-j}\partial_x^\alpha u - \sum_{j+|\alpha|\le 2} b_{j,\alpha}(t)\partial_t^{3-j}\partial_x^\alpha u \\ \quad = 0, \\ u(0,x) = u_0(x),\, u_t(0,x) = u_1(x),\, u_{tt}(0,x) = u_2(x). \end{cases} \tag{16.1}$$

The standard strategy to get well-posedness results for (16.1) goes back to [6] and [7], and it is to show an a-priori estimate for the Fourier transform of the solution.

By the classical Ovciannikov theorem the Cauchy problem is well-posed in the space of the real analytic functionals. Moreover, by the finite speed of propagation property, we can assume that the data have compact support. Thus we only need to prove that, for fixed t, the solution belongs to $\mathcal{C}^\infty$ or to a Gevrey space.

Let u be a solution to (16.1). We set

$$V(t;\xi) := \begin{pmatrix} (i|\xi|)^2 v(t;\xi) \\ i|\xi|\partial_t v(t;\xi) \\ \partial_t^2 v(t;\xi) \end{pmatrix}, \qquad V_0(t;\xi) := \begin{pmatrix} (i|\xi|)^2 v_0(\xi) \\ i|\xi| v_1(\xi) \\ v_2(\xi) \end{pmatrix},$$

where $v(t;\xi) := \mathcal{F}_{x\to\xi}(u(t;x))$ is the Fourier transform with respect to the space variables x of u and $v_j(\xi) := \mathcal{F}_{x\to\xi}(u_j(x))$, $j = 0, 1, 2$, are the Fourier transforms of the data. The Cauchy problem (16.1) is transformed into a Cauchy problem for a first order linear system depending on a parameter $\xi \in \mathbb{R}^n$

$$\begin{cases} V'(t;\xi) = i|\xi|A(t;\xi)V(t;\xi) + B(t;\xi)V(t;\xi), \\ V(t_0;\xi) = V_0(\xi), \end{cases}$$

where

$$A(t;\xi) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a_3 & a_2 & a_1 \end{pmatrix}, \tag{16.2}$$

is the Sylvester matrix associated to P, and

$$B(t;\xi) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ b_{0,2} & b_{1,1} & b_{2,0} \end{pmatrix} + |\xi|^{-1} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ b_{0,1} & b_{1,0} & 0 \end{pmatrix} + |\xi|^{-2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ b_{0,0} & 0 & 0 \end{pmatrix}. \tag{16.3}$$

Note that the characteristic polynomial of A is the polynomial P, and the eigenvalues of A are the characteristic roots τ_j.

The Paley-Wiener theorem for $\mathcal{C}^\infty$ functions states that U_0 belongs to $\mathcal{C}_0^\infty$ if, and only if, for any $N \in \mathbb{N}$ there exists a constant C_{N} such that

$$\left|V_0(\xi)\right| \leq C_{\mathrm{N}}|\xi|^{-N} \quad \text{for } |\xi| \geq 1.$$

Thus to verify that $U \in \mathcal{C}^2([0,T];\mathcal{C}_0^\infty)$ it will be sufficient to show that there exist constants C and ν such that

$$\left|V(t;\xi)\right| \leq C|\xi|^\nu \left|V(0;\xi)\right| \quad \text{for } |\xi| \geq 1, \tag{16.4}$$

for any $t \in [0,T]$.

Similarly, the Paley-Wiener theorem for Gevrey functions states that a function f belong to $\gamma^s \cap \mathcal{C}_0^\infty$ if, and only if, there exists constants C_0, δ_0 such that

$$\left|\widehat{f}(\xi)\right| \leq C_0 e^{-\delta_0|\xi|^{1/s}} \quad \text{for } |\xi| \geq 1.$$

To verify that $U \in \mathcal{C}^2([0,T];\gamma^s)$ it will be sufficient to show that there exist constants C_0, δ_0, ν_0 such that

$$\left|V(t;\xi)\right| \leq C_0|\xi|^{\nu_0} e^{-\delta_0|\xi|^{1/s}} \left|V(0;\xi)\right|, \quad \text{for } |\xi| \geq 1 \tag{16.5}$$

for any $t \in [0,T]$.

To derive (16.4) or (16.5) we will use an energy estimate. In order to construct a suitable energy functional, we preliminarily need to consider a symmetrizer.

16.3 Jannelli's Symmetrizer of a Sylvester-Type Matrix

We recall the construction of the Jannelli's symmetrizer for a 3×3 Sylvester matrix. The symmetrizer in the general case can be constructed in a similar way (see [18]). The construction being an algebraic procedure, we omit the dependence on the variables. Let

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a_3 & a_2 & a_1 \end{pmatrix}$$

be a Sylvester 3×3 matrix, and let τ_1, τ_2, τ_3 be the eigenvalues of A:

$$P(\tau) = \det(\tau I - A) = (\tau - \tau_1)(\tau - \tau_2)(\tau - \tau_3).$$

Let

$$\mathcal{W} = \begin{pmatrix} \tau_2\tau_3 & -(\tau_2+\tau_3) & 1 \\ \tau_1\tau_3 & -(\tau_1+\tau_3) & 1 \\ \tau_1\tau_2 & -(\tau_1+\tau_2) & 1 \end{pmatrix} \tag{16.6}$$

and denote by w_k the k-th row of $\mathcal{W}$. Note that the entries of w_k are the coefficients (in reversed order) of the polynomial

$$P_{\widehat{k}}(\tau) := \frac{P(\tau)}{\tau - \tau_k}.$$

Moreover w_k is a left eigenvector of A, that is:

$$w_k A = \tau_k w_k.$$

Let

$$\begin{aligned} \mathcal{Q} := \mathcal{W}^*\mathcal{W} &= \sum_{1 \le i < j \le 3} \begin{pmatrix} \tau_i^2\tau_j^2 & -\tau_i^2\tau_j - \tau_i\tau_j^2 & \tau_i\tau_j \\ -\tau_i^2\tau_j - \tau_i\tau_j^2 & (\tau_i+\tau_j)^2 & -2\tau_i \\ \tau_i\tau_j & -2\tau_i & 1 \end{pmatrix} \\ &= \begin{pmatrix} a_2^2 - 2a_1a_3 & a_1a_2 + 3a_3 & -a_2 \\ a_1a_2 + 3a_3 & 2a_1^2 + 2a_2 & -2a_1 \\ -a_2 & -2a_1 & 3 \end{pmatrix}, \end{aligned} \tag{16.7}$$

where we used Vieta's formulas:

$$\tau_1 + \tau_2 + \tau_3 = a_1, \qquad \tau_1\tau_2 + \tau_2\tau_3 + \tau_3\tau_1 = -a_2, \qquad \tau_1\tau_2\tau_3 = a_3.$$

Then

$$\mathcal{Q}A = \mathcal{W}^*\mathcal{W}A = \mathcal{W}^*D\mathcal{W} = (\mathcal{Q}A)^*,$$

where $D := \mathrm{diag}(\tau_1, \tau_2, \tau_3)$. We say that $\mathcal{Q}$ is the *standard symmetrizer* of A.

We gather the fundamental properties of the standard symmetrizer in the following proposition.

Proposition 1 ([17, 18]) *Let*

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ & 0 & 1 & \ddots & \vdots \\ & & \ddots & \ddots & 0 \\ & & & 0 & 1 \\ a_{\mathrm{N}} & & \dots & & a_1 \end{pmatrix}$$

be a N × N *Sylvester type matrix, then there exists a symmetric matrix* $\mathcal{Q}$ *such that*

- *the matrix* $\mathcal{Q}$ *symmetrize* A, *that is*

$$(\mathcal{Q}A)^* = \mathcal{Q}A;$$

- $\mathcal{Q}$ *is the Bezout matrix of* $(P, \partial_\tau P)$;
- *there is an explicit formula to compute the entries of* $\mathcal{Q}$:

$$\mathcal{Q}_{jk} = j a_{\mathrm{N}-j} a_{\mathrm{N}-k} - \sum_{p=1}^{\mathrm{N}-k} (k - j + 2p) a_{\mathrm{N}-j+p} a_{\mathrm{N}-k-p} \quad (j \le k);$$

- *the determinant of* $\mathcal{Q}$ *is the discriminant of the characteristic polynomial of* A:

$$\det \mathcal{Q} = \Delta := \prod_{j \ne k} (\tau_j - \tau_k)^2;$$

- *a lower bound of* $\mathcal{Q}$ *is given by*

$$(\mathcal{Q}V, V) \ge C\Delta |V|^2, \tag{16.8}$$

where the constant C *depends only on the maximum of the coefficients of* $P(\tau)$.

16.3.1 Leray's Symmetrizer

In [22] Leray constructed a symmetrizer for any hyperbolic Sylvester type matrix, and used it to prove the well-posedness in $\mathcal{C}^\infty$ for strictly hyperbolic equations (with coefficients depending also on the space variables). Nevertheless, as we remarked in [24], in the weakly hyperbolic case Jannelli's symmetrizer seems to be more powerful. In this section we compare these two symmetrizers, limiting our discussion, for the sake of simplicity, to 3 × 3 matrices.

Let

$$\mathcal{V} := \begin{pmatrix} 1 & 1 & 1 \\ \tau_1 & \tau_2 & \tau_3 \\ \tau_1^2 & \tau_2^2 & \tau_3^2 \end{pmatrix}$$

be the Vandermonde matrix associated to $\tau_1,\ \tau_2,\ \tau_3$. It is easy to check that the column vectors of $\mathcal{V}$ are right eigenvectors of A, thus

$$A\mathcal{V} = \mathcal{V}D,$$

where $D := \operatorname{diag}(\tau_1, \tau_2, \tau_3)$. Moreover, the matrix $A\mathcal{V}\mathcal{V}^*$ is symmetric. Let $\mathcal{Q}^{\mathrm{L}}$ be the cofactor matrix of $\mathcal{V}\mathcal{V}^*$:

$$\mathcal{Q}^{\mathrm{L}} := \big(\mathcal{V}\mathcal{V}^*\big)^{co} = \Delta\big(\mathcal{V}\mathcal{V}^*\big)^{-1},$$

where $\Delta = (\det \mathcal{V})^2$ is the discriminant of $P(\tau)$. The matrix $\mathcal{Q}^{\mathrm{L}}$ is a symmetrizer for A, indeed we have

$$\begin{aligned}\mathcal{Q}^{\mathrm{L}}A &= \mathcal{Q}^{\mathrm{L}}\big(A\mathcal{Q}^{\mathrm{L}-1}\big)\mathcal{Q}^{\mathrm{L}} = \big(\mathcal{Q}^{\mathrm{L}*}\big(A\mathcal{Q}^{\mathrm{L}-1}\big)^*\mathcal{Q}^{\mathrm{L}*}\big)^* \\ &= \big(\mathcal{Q}^{\mathrm{L}}A\mathcal{Q}^{\mathrm{L}-1}\mathcal{Q}^{\mathrm{L}}\big)^* = \big(\mathcal{Q}^{\mathrm{L}}A\big)^*,\end{aligned}$$

that is, $\mathcal{Q}^{\mathrm{L}}A$ is symmetric.

The entries of $\mathcal{Q}^{\mathrm{L}}$ are polynomial functions in the coefficients of the operator. Indeed,

$$\mathcal{V}\mathcal{V}^* = \begin{pmatrix} s_0 & s_1 & s_2 \\ s_1 & s_2 & s_3 \\ s_2 & s_3 & s_4 \end{pmatrix},$$

where s_j are the Newton polynomials

$$s_j = \tau_1^j + \tau_2^j + \tau_3^j,$$

and they can be computed by the inductive formula

$$s_j = \begin{cases} a_1, & \text{if } j = 1, \\ a_1 s_{j-1} + a_2 s_{j-2} + \cdots + a_{j-1}s_1 + j a_j, & \text{if } j > 1, \end{cases}$$

where we set $a_j \equiv 0$ if $j > 3$. We have

$$\mathcal{V}\mathcal{V}^* = \begin{pmatrix} 3 & a_1 & a_1^2 + 2a_2 \\ a_1 & a_1^2 + 2a_2 & a_1^3 + 3a_1a_2 + 3a_3 \\ a_1^2 + 2a_2 & a_1^3 + 3a_1a_2 + 3a_3 & a_1^4 + 4a_1^2a_2 + 4a_1a_3 + 2a_2^2 \end{pmatrix},$$

thus

$$\mathcal{Q}^{\mathrm{L}} = \begin{pmatrix} s_2s_4 - s_3^2 & s_3s_2 - s_1s_4 & s_1s_3 - s_2^2 \\ s_3s_2 - s_1s_4 & 3s_4 - s_2^2 & s_1s_2 - 3s_3 \\ s_1s_3 - s_2^2 & s_1s_2 - 3s_3 & 3s_2 - s_1^2 \end{pmatrix},$$

and

$$\begin{aligned}\mathcal{Q}^{\mathrm{L}}_{1,1} &= -2a_1^3a_3 + a_1^2{a_2}^2 - 10a_2a_1a_3 + 4{a_2}^3 - 9{a_3}^2, \\ \mathcal{Q}^{\mathrm{L}}_{1,2} &= \mathcal{Q}^{\mathrm{L}}_{2,1} = a_1^3a_2 - a_1^2a_3 + 4a_1{a_2}^2 + 6a_3a_2,\end{aligned}$$

$$
\begin{aligned}
\mathcal{Q}^{\mathrm{L}}_{1,3} &= \mathcal{Q}^{\mathrm{L}}_{3,1} = -a_1^2 a_2 + 3a_1a_3 - 4{a_2}^2, \\
\mathcal{Q}^{\mathrm{L}}_{2,2} &= 2a_1^4 + 8a_1^2 a_2 + 12a_1a_3 + 2{a_2}^2, \\
\mathcal{Q}^{\mathrm{L}}_{2,3} &= \mathcal{Q}^{\mathrm{L}}_{3,2} = -2a_1^3 - 7a_1a_2 - 9a_3, \\
\mathcal{Q}^{\mathrm{L}}_{3,3} &= 2a_1^2 + 6a_2.
\end{aligned}
$$

In order to compare Leray's and Jannelli's symmetrizer we note that

$$
\mathcal{W}\mathcal{V} = \mathcal{Z} := \{z_1, z_2, z_3\}, \quad \text{where } z_k := P_{\widehat{k}}(\tau_k) = \prod_{1\le j\le 3,\, j\ne k} (\tau_k - \tau_j),
$$

thus

$$
\begin{aligned}
\left\langle \mathcal{Q}^{\mathrm{L}} V, V\right\rangle &= \Delta\left\langle \left(\mathcal{V}\mathcal{V}^*\right)^{-1} V, V\right\rangle = \Delta\left\langle \left(\mathcal{V}^*\right)\mathcal{V}^{-1} V, V\right\rangle \\
&= \Delta\left\|\mathcal{V}^{-1}V\right\|^2 = \Delta\left\|\mathcal{Z}^{-1}\mathcal{W}V\right\|^2 \\
&= \left\|\mathcal{Z}^{co}\mathcal{W}V\right\|^2 \le \left\|\mathcal{Z}^{co}\right\|^2 \|\mathcal{W}V\|^2 = \left\|\mathcal{Z}^{co}\right\|^2 \langle \mathcal{Q}V, V\rangle.
\end{aligned}
$$

We see that for strictly hyperbolic equations the two symmetrizers are equivalent. But, in the weakly hyperbolic case, Jannelli's symmetrizer seems to be more suitable, since it vanishes on the multiple characteristics at lower order.

Example 4 Consider the polynomial

$$
P(t;\tau,\xi) = \tau^3 - 3t^2\tau\xi^2 + 2t^3\xi^3,
$$

which has the roots

$$
\tau_1 = \tau_2 = t\xi, \qquad \tau_3 = -2t\xi,
$$

we get:

$$
\mathcal{Q} = \begin{pmatrix} 9t^4 & -6t^3 & -3t^2 \\ -6t^3 & 6t^2 & 0 \\ -3t^2 & 0 & 3 \end{pmatrix} \qquad \mathcal{Q}^{\mathrm{L}} = 18t^2 \begin{pmatrix} 4t^4 & -2t^3 & -2t^2 \\ -2t^3 & t^2 & t \\ -2t^2 & t & 1 \end{pmatrix}.
$$

We can see directly that

$$
\left\langle \mathcal{Q}^{\mathrm{L}} V, V\right\rangle \le 18t^2 \langle \mathcal{Q}V, V\rangle,
$$

since the matrix

$$
\mathcal{Q} - \frac{1}{18t^2}\mathcal{Q}^{\mathrm{L}} = \begin{pmatrix} 5t^4 & -4t^3 & -t^2 \\ -4t^3 & 5t^2 & -t \\ -t^2 & -t & 2 \end{pmatrix}
$$

is positive semi-definite.

16.4 Energy Estimate for the Homogeneous Equation

To begin with, let us consider the homogeneous case, i.e. $B(t) \equiv 0$, thus V is a solution of

$$V' = i|\xi|AV. \tag{16.9}$$

Using the symmetrizer we can obtain an a-priori estimate of V. Let

$$E_{\text{hyp}}(t;\xi) := \big\langle \mathcal{Q}(t;\xi)V(t;\xi), V(t;\xi)\big\rangle$$

be the (*hyperbolic*) *energy* of V. Since V is a solution of (16.9) and $\mathcal{Q}A$ is symmetric, we have

$$\begin{aligned} E'_{\text{hyp}} &= \big\langle \mathcal{Q}'V, V\big\rangle + 2\,\mathrm{Re}\big\langle \mathcal{Q}V', V\big\rangle \\ &= \big\langle \mathcal{Q}'V, V\big\rangle + 2\,\mathrm{Re}\big\langle i|\xi|\mathcal{Q}AV, V\big\rangle = \frac{\langle \mathcal{Q}'V, V\rangle}{\langle \mathcal{Q}V, V\rangle} E_{\text{hyp}}, \end{aligned}$$

hence, by Gronwall's Lemma we get

$$E_{\text{hyp}}(t_2) \le \exp\left(\int_{t_1}^{t_2} \frac{|\langle \mathcal{Q}'V, V\rangle|}{\langle \mathcal{Q}V, V\rangle} ds\right) E_{\text{hyp}}(t_1) \quad \text{for any } t_1, t_2 \in \,]-T, T[. \tag{16.10}$$

If

$$\frac{\langle \mathcal{Q}'V, V\rangle}{\langle \mathcal{Q}V, V\rangle} \in L^\infty \tag{16.11}$$

we get the *energy estimate*

$$E_{\text{hyp}}(t_2) \le C E_{\text{hyp}}(t_1). \tag{16.12}$$

Now, some natural questions arise:

1. Does the estimate (16.12) imply an estimate for V as (16.4) or (16.5)?
2. Can we assume a condition weaker than (16.11)?
3. Can we give a more explicit form of (16.11)?

Concerning the first question, the answer is negative, in general, since $\mathcal{Q}$ is only positive semi-definite, if the characteristic roots may coincide (cf. (16.8)). One may ask if another positive definite symmetrizer is possible. The answer is negative.

Example 5 Consider the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

and let $\mathcal{Q} = (q_{jk})_{jk=1,2,3}$ be any symmetric matrix. Then, since

$$\mathcal{Q}A = \begin{pmatrix} 0 & q_{11} & q_{12} \\ 0 & q_{12} & q_{22} \\ 0 & q_{13} & q_{32} \end{pmatrix}$$

we see that in order that $\mathcal{Q}A$ to be symmetric we need to assume $q_{11} = 0$, which implies that $\mathcal{Q}$ is not positive definite.

To deal with this problem the strategy is to modify the energy near the zeroes of Δ. To illustrate the method we assume, for simplicity, that the discriminant Δ may vanishes only at $t = 0$, with finite order, i.e. the following inequality is satisfied:

$$\Delta(t) \geq Ct^{\nu} \tag{16.13}$$

with a positive constant C and an even number ν. Moreover, let us assume the following condition:

$$|t| \frac{\langle \mathcal{Q}' V, V \rangle}{\langle \mathcal{Q} V, V \rangle} \in L^{\infty} \tag{16.14}$$

which is weaker than (16.11). Define

$$E_{\text{kov}}(t;\xi) := \left| V(t;\xi) \right|^2, \quad \text{if } |t||\xi| < 1,$$
$$E_{\text{hyp}}(t;\xi) := \left| \big\langle \mathcal{Q}(t;\xi) V(t;\xi), V(t;\xi) \big\rangle \right|^2, \quad \text{if } |t||\xi| \geq 1.$$

E_{kov} is called the *pseudodifferential* (or *kovalevskian*) *energy*, E_{hyp} is called the *hyperbolic energy*.

Since V is a solution of (16.9) we have

$$E'_{\text{kov}}(t;\xi) := 2\,\text{Re}\big\langle V(t;\xi), V'(t;\xi) \big\rangle$$
$$= 2\,\text{Re}\big\langle V(t;\xi), i|\xi|A(t)V(t;\xi) \big\rangle \leq C|\xi| E_{\text{kov}}(t;\xi),$$

where here and in the following C denotes a constant which can vary from line to line, but which is independent of t and ξ. Then by Gronwall's Lemma we get

$$E_{\text{kov}}(t_2;\xi) \leq e^{2C|\xi||t_2 - t_1|} E_{\text{kov}}(t_1;\xi) \leq e^{4C} E_{\text{kov}}(t_1;\xi) \tag{16.15}$$

if $|t_1||\xi|, |t_2||\xi| < 1$.

From (16.10), by using (16.14), we get

$$E_{\text{hyp}}(t_2;\xi) \leq \exp\left(\int_{|s| \geq 1/|\xi|} \frac{C}{|s|} ds \right) E_{\text{hyp}}(t_1;\xi) \leq C|\xi|^C E_{\text{hyp}}(t_1;\xi). \tag{16.16}$$

Moreover,

$$E_{\text{hyp}}(t;\xi) \geq C\Delta(t)|V|^2 \geq Ct^{\nu}|V|^2 \geq C|\xi|^{-\nu}|V|^2, \tag{16.17}$$

thus combining (16.15), (16.16) and (16.17) we get (16.4).

Summing up we have proved that after assuming (16.13) and (16.14) the Cauchy Problem (CP) with $b_{j,k} \equiv 0$ is well-posed in $\mathcal{C}^\infty$.

Thus we have given simultaneously a (partial) answer to questions 1 and 2.

We are now interested in question 3. More precisely: can we put the hypothesis (16.13) and (16.14) into an intrinsic form?

It is clear that (16.13) can be replaced by the hypothesis of analyticity of the coefficients, since, at least in the one dimensional case, if Δ is analytic, then we can decompose $[0, T]$ in a finite number of subintervals $[t_j - \delta, t_j + \delta]$, in which we have

$$\Delta \geq C_j |t - t_j|^{\nu_j}, \tag{16.18}$$

and we can repeat a similar reasoning.

We remark however that the extension of this method to several variables presents some difficulties, since we have to prove that the constants C_j, ν_j in (16.18) are uniformly bounded for $\xi \in \mathbb{R}^n$. We refer to Sect. 4 in [19] for further details.

Concerning condition (16.14) we can explicit it in terms of the characteristic roots, or in terms of the coefficients of the operator.

16.4.1 Expressing (16.14) in Terms of the Characteristic Roots

Recalling that $\mathcal{Q} := \mathcal{W}^*\mathcal{W}$ we have

$$\langle \mathcal{Q}' V, V\rangle = 2\,\mathrm{Re}\langle \mathcal{W}' V, \mathcal{W} V\rangle,$$

hence

$$\big|\langle \mathcal{Q}' V, V\rangle\big| \leq 2\big|\mathcal{W}' V\big|\,|\mathcal{W} V| = 2\big|\mathcal{W}' V\big|\sqrt{E_{\mathrm{hyp}}},$$

and

$$\big|\mathcal{W}' V\big|^2 = \sum_{j=1}^{3} \big|w_j' V\big|^2,$$

the w_j being the j-th row of $\mathcal{W}$. We have

$$w_1' = \big(\tau_2'\tau_3 + \tau_2\tau_3', -\tau_2' - \tau_3', 0\big) = \tau_2'(\tau_3, -1, 0) + \tau_3'(\tau_2, -1, 0)$$

and since

$$(\tau_3, -1, 0) = \frac{w_1 - w_2}{\tau_2 - \tau_1}, \qquad (\tau_2, -1, 0) = \frac{w_1 - w_3}{\tau_3 - \tau_1},$$

we get

$$w_1' = \frac{\tau_2'}{\tau_2 - \tau_1}(w_1 - w_2) + \frac{\tau_3'}{\tau_3 - \tau_1}(w_1 - w_3),$$

thus

$$\begin{aligned}|w_1' V| &\le \left|\frac{\tau_2'}{\tau_2-\tau_1}\right|\big(|w_1 V|+|w_2 V|\big)+\left|\frac{\tau_3'}{\tau_3-\tau_1}\right|\big(|w_1 V|+|w_2 V|\big)\\ &\le \max\left\{\left|\frac{\tau_2'}{\tau_2-\tau_1}\right|,\left|\frac{\tau_3'}{\tau_3-\tau_1}\right|\right\}\langle \mathcal{Q}V, V\rangle^{1/2}.\end{aligned}$$

Analogous formulas hold true for w_2' and w_3'. Thus we see that (16.14) holds true if we assume

$$|t|\,|\tau_j'| \le C|\tau_j-\tau_k| \quad \text{if } j\neq k. \tag{16.19}$$

Condition (16.19) has been introduced in [3], where they proved that such condition is also necessary, in space dimension $n=1$, then extended in various form in [5, 9, 10] and [16]. Nevertheless it is not easy to check (16.19) for concrete examples, since it involves the derivatives of the characteristic roots (see Sect. 16.5).

Thus we are interested in a more explicit form of (16.14).

16.4.2 Expressing (16.14) in Terms of the Coefficients of the Operator

In view to explicit (16.14) we remark that if B and C are two real symmetric $\mathrm{N}\times\mathrm{N}$ matrices, with C positive definite, then

$$\sup_{V\in\mathbb{R}^n\setminus\{0\}}\frac{\langle BV, V\rangle}{\langle CV, V\rangle}\le \sup\big\{\lambda\in\mathbb{R}\,|\,\det(\lambda C-B)=0\big\}, \tag{16.20}$$

and the polynomial $\det(\lambda C-B)=0$ has only real roots. Moreover, it follows from Newton's formula that if $\sum_{h=0}^{\mathrm{N}} d_h(t)\lambda^{\mathrm{N}-h}$ is a polynomial with real roots for any t, then the roots $\lambda_1(t),\ldots,\lambda_{\mathrm{N}}(t)$ are bounded if, and only if, the ratios $d_1(t)/d_0(t)$, $d_2(t)/d_0(t)$ are bounded, since

$$\sum_{j=1}^{\mathrm{N}}\lambda_j^2(t)=\frac{d_1^2(t)}{d_0^2(t)}-2\frac{d_2(t)}{d_0(t)}.$$

From (16.20) we see that (16.14) is equivalent to the boundness of the solutions in λ of the equation

$$\det\big(\lambda\mathcal{Q}(t)-|t|\mathcal{Q}'(t)\big)=\sum_{k=0}^{\mathrm{N}} d_k(t)|t|^k\lambda^{\mathrm{N}-h}=0. \tag{16.21}$$

The first and the last terms in (16.21) are easily computed

$$d_0=\det(\mathcal{Q}),\qquad d_{\mathrm{N}}=(-1)^{\mathrm{N}}\det\big(\mathcal{Q}'\big).$$

The term d_1 is the sum of the determinants of the matrices obtained from $\mathrm{N}-1$ lines of $\mathcal{Q}$ and one line of $\mathcal{Q}'$. Thus we have (see [15])

$$d_1 = -\big(\det(\mathcal{Q})\big)' = -\Delta'.$$

The term d_2 needs more complicated calculations (see Proposition 2.4 in [19]), however we obtain:

$$d_2 = \frac{1}{2}\,\mathrm{trace}\big(\mathcal{Q}'\big(\mathcal{Q}^{co}\big)'\big),$$

where $\mathcal{Q}^{co}$ is the cofactor matrix of $\mathcal{Q}$. We call d_2 the *check function* of $\mathcal{Q}$, and we denote it by ψ.

Example 6 If $\mathrm{N}=3$, according to (16.7), we have

$$\begin{aligned}\psi = {}& -3a_2^2a_1'^2 + 6a_1a_2a_1'a_2' - a_1^2a_2'^2 + 6a_2a_2'^2 - 8a_1^2a_1'a_3' \\ & - 6a_2a_1'a_3' - 18a_1a_2'a_3' - 27a_3'^2. \end{aligned} \tag{16.22}$$

Thus we get that (16.14) is equivalent to

$$|t|\frac{\det(\mathcal{Q})'}{\det(\mathcal{Q})} \in L^\infty \quad \text{and} \quad t^2\frac{\psi}{\det(\mathcal{Q})} \in L^\infty.$$

The first condition, being $\det(\mathcal{Q}) = \Delta$, is automatically verified, if $\Delta(t)$ is an analytic function. The second condition is not always verified, and should be assumed as hypothesis.

Theorem 1 ([19]) *Assume that the coefficients of $P(t;\partial_t,\partial_x)$ are analytic in $]-T,T[$, and, moreover, $\Delta(t)\not\equiv 0$ in $]-T,T[$.*

Then the condition

$$\big|\psi(t)\big| \le C\widetilde{\Delta}(t) \quad \text{for any } t\in\,]-T,T[, \tag{16.23}$$

where

$$\widetilde{\Delta}(t) := \Delta(t) + \frac{\Delta'^2(t)}{\Delta(t)}$$

is necessary and sufficient for the Cauchy Problem (CP) *to be well-posed in $\mathcal{C}^\infty$.*

Remark 1 Condition (16.23) means that if $\Delta(t)$ vanishes at $\bar{t}$ of order $2k$, then $\psi(t)$ vanishes at $\bar{t}$ of order $2k-2$.

Remark 2 In space dimension greater than 1, condition (16.23) (with a constant C independent of $\xi\in\mathbb{R}^{\mathrm{N}}$) remains sufficient, while it is an open problem to show its necessity.

16.4.3 The Case $\Delta \equiv 0$

If $\Delta \equiv 0$, condition (16.23) is no longer sufficient to assure the $\mathcal{C}^\infty$ well-posedness.

Indeed, if the coefficients of $P(t; \partial_t, \partial_x)$ are analytic, then, two roots either coincide in $]-T, T[$, either coincide only at a finite number of points. Thus it is possible to find a closed interval $I \subset]-T, T[$ such that the restriction of $P(t; \partial_t, \partial_x)$ to I is an operator with characteristics of *constant multiplicity*. For such operators the necessary and sufficient condition for the $\mathcal{C}^\infty$ well-posedness is well known, also in the case of x-depending coefficients (see [2, 8, 13]). In particular, for homogeneous operators, this condition means that the multiple roots should be constant functions: if we write

$$P(t; \tau, \xi) = \prod \big(\tau - \tau_j(t)\xi\big)^{m_j},$$

then in order the Cauchy problem for $P(t; \partial_t, \partial_x)$ to be well-posed in $\mathcal{C}^\infty(I)$ the following condition is necessary:

$$m_j > 1 \quad \Longrightarrow \quad \tau_j'(t) \equiv 0. \tag{16.24}$$

Consider a third order operator with a double characteristic root $\tau_1(t)$, so that

$$P(t; \tau, \xi) = \big(\tau - \tau_1(t)\big)^2 \big(\tau - \tau_2(t)\big).$$

A direct calculation shows that

$$\psi = -4\tau_1'(t)\big(\tau_1(t) - \tau_2(t)\big)^4.$$

Now, if τ_1 is constant, then both (16.23) and (16.24) are satisfied. Remark that in this case it is obvious that the Cauchy problem for $P(t; \partial_t, \partial_x)$ is well-posed since we can write

$$P(t; \partial_t, \partial_x) = \big(\partial_t - \tau_2(t)\partial_x\big)(\partial_t - \tau_1\partial_x)^2.$$

On the other side, if $\tau_1(t) \equiv \tau_2(t)$ is not constant, that is, $P(t; \tau, \xi)$ has a triple root, (16.23) is satisfied, but (16.24) is not satisfied, and the Cauchy problem for $P(t; \partial_t, \partial_x)$ is not well-posed.

Thus, in the case $\Delta \equiv 0$ more refined conditions are needed.

Before to proceed, recall the following criterion to establish the hyperbolicity of a polynomial.

Lemma 1 ([18]) *Let*

$$\mathcal{Q}_j := \begin{pmatrix} \mathcal{Q}_{\mathrm{N}+1-j,\mathrm{N}+1-j} & \cdots & \mathcal{Q}_{\mathrm{N}+1-j,\mathrm{N}} \\ \vdots & \ddots & \vdots \\ \mathcal{Q}_{\mathrm{N},\mathrm{N}+1-j} & \cdots & \mathcal{Q}_{\mathrm{N},\mathrm{N}} \end{pmatrix},$$

so that $\mathcal{Q}_\mathrm{N} := \mathcal{Q}$, and $\mathcal{Q}_j$, $j = 1, \dots, \mathrm{N} - 1$, is the principal $j \times j$ minor of $\mathcal{Q}$ obtained by removing the first $\mathrm{N} - j$ rows and the first $\mathrm{N} - j$ columns of $\mathcal{Q}$. Let Δ_j be the determinant of $\mathcal{Q}_j$. Then

1. *P is strictly hyperbolic if and only if*

$$\Delta_j > 0 \quad \textit{for any } j = 1, \ldots, \mathrm{N}.$$

2. *P is weakly hyperbolic if and only if there exists $r < \mathrm{N}$ such that*

$$\Delta_{\mathrm{N}} = \cdots = \Delta_{r+1} = 0, \quad \Delta_r > 0, \ldots, \Delta_1 > 0.$$

In this case P has exactly r distinct roots.

If $P(t; \partial_t, \partial_x)$ is a third order operator with a triple root $\tau_1(t)$, then

$$Q = Q_3 = \begin{pmatrix} 3\tau_1^4 & -6\tau_1^3 & 3\tau_1^2 \\ -6\tau_1^3 & 12\tau_1^2 & -6\tau_1 \\ 3\tau_1^2 & -6\tau_1 & 3 \end{pmatrix},$$

and

$$Q_2 = \begin{pmatrix} 12\tau_1^2 & -6\tau_1 \\ -6\tau_1 & 3 \end{pmatrix}.$$

Note that $\Delta_2 := \det Q_2 = 0$, and the check function ψ_2 of Q_2 is $-36(\tau_1')^2$. It seems then natural to require that ψ_2 vanishes when Δ_2 vanishes.

Indeed, in the case $\Delta \equiv 0$ we have the following result.

Theorem 2 ([19]) *Assume that the coefficients of $P(t; \partial_t, \partial_x)$ are analytic in $]-T, T[$, and $\Delta(t) \equiv 0$ in $]-T, T[$. According to Lemma 1 let $r < N$ be the greatest integer such that $\Delta_r(t) \not\equiv 0$ in $]-T, T[$. Then the conditions*

$$\psi_{r+1}(t) \equiv 0, \tag{16.25}$$

$$\left|\psi_r(t)\right| \le C\widetilde{\Delta}_r(t), \tag{16.26}$$

where

$$\widetilde{\Delta}_r(t) := \Delta_r(t) + \frac{{\Delta_r'}^2(t)}{\Delta_r(t)},$$

are necessary and sufficient for the Cauchy Problem (CP) to be well-posed in C^∞.

Remark 3 Let $r < N$ be as in Theorem 2. We can write the decomposition

$$P(t; \tau, \xi) = \prod_{j=1}^{s} \big(\tau - \tau_j(t)\xi\big)^{m_j} \prod_{j=s+1}^{r} \big(\tau - \tau_j(t)\xi\big),$$

where $\tau_j(t) \neq \tau_k(t)$ for $j \neq k$ apart at most a finite set Σ. Condition (16.25) means that the multiple roots are constant, i.e. $\tau_j'(t) \equiv 0$ for $j = 1, \ldots, s$, whereas condition

(16.26) is equivalent to condition (16.23) for the operator with symbol

$$\widetilde{P}(t;\tau,\xi)=\prod_{j=1}^{r}\bigl(\tau-\tau_j(t)\xi\bigr).$$

Indeed, up to a multiplicative constant, Δ_r is the discriminant of $\widetilde{P}(t;\tau,\xi)$ (see Proposition 3.1 in [19]).

Remark 4 In space dimension greater than 1, conditions (16.25) and (16.26) (allowing r dependent on $\xi\in\mathbb{R}^N$, but C independent of $\xi\in\mathbb{R}^N$) remains sufficient, while it is an open problem to show its necessity.

16.4.4 Non Analytic Coefficients

If the coefficients are not analytic, then the Cauchy problem can fail to be well-posed in $\mathcal{C}^\infty$ (see Example 1). We expect, however, that if the coefficients belong to some $\mathcal{C}^\kappa$, then the Cauchy problem is well-posed in some Gevrey space, as it has been shown in [6] and [7] for wave type equations. A result of this kind has been obtained in [21], where they assumed a condition which is, in the case of analytic coefficients, more restrictive then (16.23) (or (16.25) and (16.26)).

For third order operators we have the following result. The proof will appear elsewhere [20].

Theorem 3 *Let*

$$P(t;\partial_t,\partial_x)u=\partial_t^3u-\sum_{j=1}^{3}a_j(t)\partial_t^{3-j}\partial_x^j u.$$

We assume that the coefficients are $\mathcal{C}^\infty(]-T,T[)$, *and*, *moreover,*

$$\bigl|\psi(t)\bigr|\le C\Delta(t)+\frac{\Delta'^2(t)}{\Delta(t)},\qquad a_1'^2(t)\le C\Delta_1(t)+\frac{\Delta_1'^2(t)}{\Delta_1(t)},$$

where ψ *is the check function*, *defined in* (16.22), Δ *is the discriminant of* $P(t;\tau,\xi)$ *and*

$$\Delta_1:=(\tau_1-\tau_2)^2+(\tau_2-\tau_3)^2+(\tau_3-\tau_1)^2=2a_1^2+6a_2.$$

Then the Cauchy problem (16.1) *is well-posed in every Gevrey space.*

16.5 Examples of Operators of Higher Order

As previously remarked, condition (16.23) is expressed only in terms of the coefficients of the operator, without needing to compute the characteristic roots. This is of great importance for higher order operators.

Consider for example the following two operators:

$$\begin{aligned}L_1(t;\partial_t,\partial_x) &= \partial_t^{10} - 10\partial_t^8\partial_x^2 - \left(-33+4t^2\right)\partial_t^6\partial_x^4 - \left(2+2t^2\right)\partial_t^5\partial_x^5 \\ &\quad - \left(40-20t^2\right)\partial_t^4\partial_x^6 - \left(-10-8t^2\right)\partial_t^3\partial_x^7 \\ &\quad - \left(-16+20t^2\right)\partial_t^2\partial_x^8 - \left(8+t^2\right)\partial_t\partial_x^9 + \partial_t\partial_x^{10}, \\ L_2(t;\partial_t,\partial_x) &= \partial_t^{10} + 10t\partial_t^9\partial_x - \left(10-45t^2\right)\partial_t^8\partial_x^2 - \left(80t-120t^3\right)\partial_t^7\partial_x^3 \\ &\quad - \left(-33+280t^2\right)\partial_t^6\partial_x^4 - \left(2-198t+560t^3\right)\partial_t^5\partial_x^5 \\ &\quad - \left(40+10t-495t^2+50t^4\right)\partial_t^4\partial_x^6 \\ &\quad - \left(-10+160t+20t^2-660t^3\right)\partial_t^3\partial_x^7 \\ &\quad - \left(-16-30t+240t^2+20t^3\right)\partial_t^2\partial_x^8 \\ &\quad - \left(8-32t-30t^2+160t^3\right)\partial_t\partial_x^9 \\ &\quad - \left(-1+8t-16t^2-10t^3+50t^4\right)\partial_x^{10}.\end{aligned}$$

Are these operators hyperbolic? Is the corresponding Cauchy problem well-posed? Note that, since

$$L_1(0;\tau,1) = L_2(0;\tau,1) = \left(\tau^5 - 5\tau^3 + 4\tau - 1\right)^2$$

it is not possible to compute the characteristic roots, even at $t=0$.

On the other hand, using Theorem 1, we can answer the above questions. All calculations needed may be carried out by means of a symbolic manipulation software. (see the appendix).

The results for L_1 are

$$\begin{aligned}\Delta(t) &= 13436759265055744t^{10} + o\left(t^{10}\right), \\ \Delta_9(t) &= 734001349184512t^8 + o\left(t^8\right), \\ \Delta_8(t) &= 4892992931840t^6 + o\left(t^6\right), \\ \Delta_7(t) &= 33846920192t^4 + o\left(t^4\right), \\ \Delta_6(t) &= 123420800t^2 + o\left(t^2\right), \\ \Delta_5(t) &= 1234208 + o(1), \\ \Delta_4(t) &= 141280 + o(1), \\ \Delta_3(t) &= 5600 + o(1), \\ \Delta_2(t) &= 200, \\ \Delta_1(t) &= 10, \\ \psi(t) &= 537470370602229760t^8 + o\left(t^8\right);\end{aligned}$$

which shows that L_1 is hyperbolic in a neighborhood of the origin and (16.23) holds true.

Similar calculations can be carried out for L_2:

$$\begin{aligned}
\Delta(t) &= 4121196658887895100800000t^{20} + o(t^{20}),\\
\Delta_9(t) &= 926251234245231264000\,0t^{16} + o(t^{16}),\\
\Delta_8(t) &= 79745993273876480000t^{12} + o(t^{12}),\\
\Delta_7(t) &= 1953869747968000t^{8} + o(t^{8}),\\
\Delta_6(t) &= 25177843200t^{4} + o(t^{4}),\\
\Delta_5(t) &= 1234208 + o(1),\\
\Delta_4(t) &= 141280 + o(1),\\
\Delta_3(t) &= 5600 + o(1),\\
\Delta_2(t) &= 200,\\
\Delta_1(t) &= 10,\\
\psi(t) &= -1059243787382464605568 0000t^{16} + o(t^{16});
\end{aligned}$$

which show that L_2 is hyperbolic in a neighborhood of the origin but (16.23) fails to hold.

16.6 The Quasi-symmetrizer

When studying the Cauchy problem for homogeneous operators in the $\mathcal{C}$-infinity framework the symmetrizer $\mathcal{Q}$ is enough to get energy estimates. But, when we consider non homogeneous operators, a more substantial perturbation of the symmetrizer is needed. Thus we are lead to consider the so called *quasi-symmetrizer*, introduced in [17] and [11], and extensively studied in [18].

Roughly speaking, a *quasi-symmetrizer* of a N × N matrix A is a family $\{\mathcal{Q}_\varepsilon\}_{0<\varepsilon\le 1}$ of positive definite matrices such that

$$(\mathcal{Q}_\varepsilon V, V) \ge C\varepsilon^{2(m-1)}|V|^2, \tag{16.27}$$

$$\big|\big(\big(\mathcal{Q}_\varepsilon A - (\mathcal{Q}_\varepsilon A)^*\big)V, V\big)\big| \le C\varepsilon(\mathcal{Q}_\varepsilon V, V), \tag{16.28}$$

where m is maximum multiplicity of the eigenvalues of A.

Example 7 Returning to Example 5, if

$$A = \begin{pmatrix} 0 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0 \end{pmatrix}$$

we choose

$$\mathcal{Q}_\varepsilon = \begin{pmatrix} \varepsilon^4 & 0 & 0 \\ 0 & \varepsilon^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

so that

$$R_\varepsilon := \mathcal{Q}_\varepsilon A - (\mathcal{Q}_\varepsilon A)^* = \begin{pmatrix} 0 & \varepsilon^4 & 0 \\ -\varepsilon^4 & 0 & \varepsilon^2 \\ 0 & -\varepsilon^2 & 0 \end{pmatrix}.$$

Thus, using the elementary inequality $2ab \le a^2 + b^2$, we get

$$\begin{aligned} \left|(R_\varepsilon v, v)\right| &= \left|\varepsilon^4(v_1\overline{v_2} - \overline{v_1}v_2) + \varepsilon^2(v_2\overline{v_3} - \overline{v_2}v_3)\right| \\ &\le 2\varepsilon^4|v_1||v_2| + 2\varepsilon^2|v_2||v_3| \le 2\varepsilon\left(\varepsilon^4 v_1^2 + \varepsilon^2 v_2^2 + v_3^2\right) = 2\varepsilon(\mathcal{Q}_\varepsilon v, v). \end{aligned}$$

We briefly sketch the construction of the quasi-symmetrizer as introduced by D'Ancona & Spagnolo [11]. To simplify the presentation we consider only the case of a 3×3 matrix, and we refer to [11] and [18] for further details. As for the symmetrizer, the construction of the quasi-symmetrizer is just an algebraic procedure, thus we omit the dependence on the variables.

Let

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a_3 & a_2 & a_1 \end{pmatrix}$$

be a Sylvester 3×3 matrix, and let τ_1, τ_2, τ_3 be the eigenvalues of A, then define $\mathcal{W}_0 = \mathcal{W}$ as in (16.6), and

$$\mathcal{W}_1 := \begin{pmatrix} -\tau_1 & 1 & 0 \\ -\tau_2 & 1 & 0 \\ -\tau_3 & 1 & 0 \end{pmatrix} \qquad \mathcal{W}_2 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then

$$\begin{aligned} \mathcal{Q}_\varepsilon &:= \mathcal{W}_0^*\mathcal{W}_0 + \varepsilon^2\mathcal{W}_1^*\mathcal{W}_1 + \varepsilon^4\mathcal{W}_2^*\mathcal{W}_2 \\ &= \begin{pmatrix} a_2^2 - 2a_1a_3 & a_1a_2 + 3a_3 & -a_2 \\ a_1a_2 + 3a_3 & 2a_1^2 + 2a_2 & -2a_1 \\ -a_2 & -2a_1 & 3 \end{pmatrix} \\ &\quad + \varepsilon^2 \begin{pmatrix} a_1^2 + 2a_2 & -a_1 & 0 \\ -a_1 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \varepsilon^4 \begin{pmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

verifies (16.27) and (16.28).

To prove (16.27) with $m = 3$, we remark that for $\varepsilon = 1$ the matrix $\mathcal{Q}_1$ is coercive, since

$$\mathcal{Q}_1 := \mathcal{P}_1^*\mathcal{P}_1 + \mathcal{P}_2^*\mathcal{P}_2 + \mathcal{P}_3^*\mathcal{P}_3,$$

with

$$\mathcal{P}_1 := \begin{pmatrix} 1 & 0 & 0 \\ -\tau_1 & 1 & 0 \\ \tau_1\tau_2 & -\tau_1-\tau_2 & 1 \end{pmatrix}, \qquad \mathcal{P}_2 := \begin{pmatrix} 1 & 0 & 0 \\ -\tau_2 & 1 & 0 \\ \tau_2\tau_3 & -\tau_2-\tau_3 & 1 \end{pmatrix},$$
$$\mathcal{P}_3 := \begin{pmatrix} 1 & 0 & 0 \\ -\tau_3 & 1 & 0 \\ \tau_3\tau_1 & -\tau_3-\tau_1 & 1 \end{pmatrix}.$$

Thus $(\mathcal{Q}_1 V, V) \geq C|V|^2$ for some $C > 0$. We get

$$C\varepsilon^4|V|^2 \leq \varepsilon^4(\mathcal{Q}_1 V, V) \leq (\mathcal{Q}_\varepsilon V, V),$$

which shows (16.27) for $m = 3$. The case $m = 2$ is proved in a similar way.

To prove (16.28), since $\mathcal{W}_0^*\mathcal{W}_0 A$ is symmetric, it will be enough to show that

$$\varepsilon^2\left|\left(\left(\mathcal{W}_1^*\mathcal{W}_1 A - \left(\mathcal{W}_1^*\mathcal{W}_1 A\right)^*\right)V, V\right)\right| \leq C\varepsilon(\mathcal{Q}_\varepsilon V, V), \tag{16.29}$$
$$\varepsilon^4\left|\left(\left(\mathcal{W}_2^*\mathcal{W}_2 A - \left(\mathcal{W}_2^*\mathcal{W}_2 A\right)^*\right)V, V\right)\right| \leq C\varepsilon(\mathcal{Q}_\varepsilon V, V). \tag{16.30}$$

Let $w_{1,1}$ be the first row of $\mathcal{W}_1$, since

$$w_{1,1} = \frac{1}{\tau_2 - \tau_3}(w_2 - w_3),$$

where w_2 and w_3 are the second and third row of $\mathcal{W}_0$, we have

$$w_{1,1}A = \frac{1}{\tau_2-\tau_3}(w_2 A - w_3 A) = \frac{1}{\tau_2-\tau_3}(\tau_2 w_2 - \tau_3 w_3)$$
$$= \frac{1}{\tau_2-\tau_3}(\tau_2 w_2 - \tau_2 w_3 + \tau_2 w_3 - \tau_3 w_3) = \tau_2 w_{1,1} + w_3,$$

and similarly we get

$$w_{1,2}A = \tau_3 w_{1,2} + w_1, \qquad w_{1,3}A = \tau_1 w_{1,3} + w_2.$$

Thus

$$\mathcal{W}_1 A = D\mathcal{W}_1 + \widetilde{\mathcal{W}_0},$$

where $D = \mathrm{diag}(\tau_2, \tau_3, \tau_1)$ and $\widetilde{\mathcal{W}_0}$ is obtained from $\mathcal{W}_0$ after a permutation of the rows. It follows that

$$\begin{aligned}\varepsilon^2\big(\big(\mathcal{W}_1^*\mathcal{W}_1 A - \big(\mathcal{W}_1^*\mathcal{W}_1 A\big)^*\big)V, V\big) &= \varepsilon^2\big(\big(\mathcal{W}_1^*\widetilde{\mathcal{W}_0} - \widetilde{\mathcal{W}_0}^*\mathcal{W}_1\big)V, V\big)\\ &\le 2\varepsilon^2\|\widetilde{\mathcal{W}_0}V\|\|\mathcal{W}_1 V\| = 2\varepsilon^2\|\mathcal{W}_0 V\|\|\mathcal{W}_1 V\|\\ &\le \varepsilon\big(\|\mathcal{W}_0 V\|^2 + \varepsilon^2\|\mathcal{W}_1 V\|^2\big)\end{aligned}$$

which implies (16.29). Equation (16.30) is proved in a similar way.

Remark 5 There are other methods to construct a quasi-symmetrizer.

- According to [17] let $\{P^\varepsilon(\tau)\}_{\varepsilon\in]0,1]}$ be a sequence of polynomials approximating $P(\tau)$, whose roots τ_j^ε satisfy

$$\left|\tau_j^\varepsilon - \tau_k^\varepsilon\right| \ge \varepsilon, \quad \text{if } j \ne k, \quad \text{and} \quad \left|\tau_j - \tau_j^\varepsilon\right| \le C\varepsilon,$$

  then define

$$\mathcal{Q}_\varepsilon^{\mathrm{J}} := \mathcal{Q}_{P^\varepsilon},$$

  $\mathcal{Q}_{P^\varepsilon}$ being the symmetrizer of P^ε.
- Following [23] let

$$P^{(j)}(\tau) = \frac{(\mathrm{N}-j)!}{\mathrm{N}!}\partial_\tau^j P(\tau)$$

  be the normalized derivatives of P, we set

$$\mathcal{Q}_\varepsilon^{\mathrm{P}} := \mathcal{Q}_P + \varepsilon^2\mathcal{Q}_{P^{(1)}} + \varepsilon^4\mathcal{Q}_{P^{(2)}} + \cdots + \varepsilon^{2(\mathrm{N}-1)}\mathcal{Q}_{P^{(\mathrm{N}-1)}},$$

  $\mathcal{Q}_{P^{(j)}}$ being the symmetrizer of $P^{(j)}$ (here we identify a $k \times k$ matrix with the $\mathrm{N}\times\mathrm{N}$ matrix obtained form the given matrix adding $\mathrm{N}-k$ rows and $\mathrm{N}-k$ columns of zeros).

However, we can show ([27]) that the three methods are equivalent in the sense that there exist positive constants C_1, C_2, C_3 depending only on the L^∞-norm of the coefficients such that

$$(\mathcal{Q}_\varepsilon v, v) \le C_1\big(\mathcal{Q}_\varepsilon^{\mathrm{J}} v, v\big) \le C_2\big(\mathcal{Q}_\varepsilon^{\mathrm{P}} v, v\big) \le C_3(\mathcal{Q}_\varepsilon v, v).$$

We gather the properties of the quasi-symmetrizer in the following proposition.

Proposition 2 ([11, 17, 18, 25]) *Let*

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0\\ & 0 & 1 & \ddots & \vdots\\ & & \ddots & \ddots & 0\\ & & & 0 & 1\\ a_{\mathrm{N}} & \dots & \dots & \dots & a_1\end{pmatrix}$$

be a N × N *Sylvester type matrix, then for any* $\varepsilon \in]0, 1]$ *there exists a symmetric matrix* $\mathcal{Q}_\varepsilon$ *such that*

$$\mathcal{Q}_\varepsilon = \sum_{j=0}^{\mathrm{N}-1} \varepsilon^{2j} q_j(a_1, \ldots, a_{\mathrm{N}}),$$

where q_j *are matrices whose entries are polynomials in* $a_1, \ldots, a_{\mathrm{N}}$. *The following properties are satisfied*:

1. $\mathcal{Q}_\varepsilon^* = \mathcal{Q}_\varepsilon$;
2. $\varepsilon^{2(m-1)}|V|^2 \le \langle \mathcal{Q}_\varepsilon V, V\rangle \le |V|^2$, *where* m *is the maximum multiplicity of the eigenvalues*;
3. *if the coefficients* a_j *belong to* $\mathcal{C}^2$, *then*

$$\left|\left\langle \mathcal{Q}_\varepsilon' V, V\right\rangle\right| \le C\varepsilon^{1-m}\langle \mathcal{Q}_\varepsilon V, V\rangle;$$

4. *we have*

$$\mathcal{Q}_\varepsilon A = T_\varepsilon + R_\varepsilon,$$

   where
   a. T_ε *is symmetric*: $T_\varepsilon^* = T_\varepsilon$;
   b. $|\langle T_\varepsilon V, V\rangle| \le C\langle \mathcal{Q}_\varepsilon V, V\rangle$;
   c. *if the coefficients* a_j *belong to* $\mathcal{C}^m$, *then* $|\langle T_\varepsilon' V, V\rangle| \le C\varepsilon^{1-m}\langle \mathcal{Q}_\varepsilon V, V\rangle$;
   d. $|(R_\varepsilon U, V)| \le \varepsilon(\mathcal{Q}_\varepsilon U, U)^{1/2}(\mathcal{Q}_\varepsilon V, V)^{1/2}$;
5. *if the coefficients* a_j *belong to* $\mathcal{C}^m$, *then*

$$\left|\left\langle A^{(k)} V, V\right\rangle\right| \le C\varepsilon^{1-k}\langle \mathcal{Q}_\varepsilon V, V\rangle \quad \textit{for } k = 1, \ldots, m-1.$$

Remark 6 If the coefficients of the operator depend only on the time variable, as in this notes, in order to derive energy estimates we use only properties 1., 2., 3., 4.a. and 4.d. But when the coefficients depend also on the space variables, then properties 4.b., 4.c. and 5. play an essential role (see [24, 25]).

16.7 Non Homogeneous Operators

In this section we use the quasisymmetrizer technique to obtain a sufficient condition for the well-posedness in $\mathcal{C}^\infty$ and in Gevrey spaces for non-homogeneous operators. This result is just a different presentation of a more general result obtained in [5] (see also [9, 10] and [16]).

According to [23], given a hyperbolic polynomial $P(t; \tau, \xi)$ of degree N, with roots $\tau_j(t; \xi)$, $j = 1, \ldots, \mathrm{N}$, we say that a polynomial $R(t; \tau, \xi)$ of degree $\le \mathrm{N} - 1$ has a *proper decomposition w.r.t.* P if there exists $c_k \in L^\infty$ such that

$$R(t; \tau, \xi) = \sum_{k=1}^{\mathrm{N}} c_k(t) P_{\widehat{k}}(t; \tau, \xi), \tag{16.31}$$

where

$$P_{\widehat{k}}(t;\tau,\xi) := \prod_{\substack{j=1,\dots,\mathrm{N}\\ j\neq k}} \big(\tau - \tau_j(t)\xi\big).$$

We say that

$$P = P_0 + R_1 + \cdots + R_{\mathrm{N}},$$

where P_0 is the homogeneous part of degree N and R_j is the homogeneous part of degree $\mathrm{N} - j$, is *properly hyperbolic* if P_0 is hyperbolic and R_j has a proper decomposition w.r.t. $\partial_\tau^j P$.

Properly hyperbolic operators have been introduced by Peyser in [23], where an energy estimate has been established for constant coefficients operator. Svensson [26] was then able to show that an operator with constant coefficients is properly hyperbolic if, and only if, it is hyperbolic in the sense of Gårding [14].

Dunn [12] extended Peyser's method to operators whose lower order terms have variable coefficients, and successively Wakabayashi [28] proved that proper hyperbolicity is a necessary condition for the $\mathcal{C}^\infty$ well-posedness for this kind of operators.

We can prove the following result for operators with time dependent coefficients.

Theorem 4 *Let L be a hyperbolic operator of order N, whose principal symbol $P(t;\tau,\xi)$ has analytic coefficients and verifies* (16.23).

We assume that the homogeneous part of degree $\mathrm{N} - j$ has a decomposition w.r.t. $\partial_\tau^{j-1} P$ as in (16.31) *with coefficients verifying*

$$\left(\frac{\Delta(t)}{\widetilde{\Delta}(t)}\right)^{(1/2)(1+j\alpha)} c_k(t) \in L^\infty \quad \textit{for some } \alpha \geq 0. \tag{16.32}$$

Then the following statements hold:

- *if $\alpha = 0$, then the Cauchy problem is well-posed in $\mathcal{C}^\infty$*;
- *if $\alpha > 0$, then the Cauchy problem is well-posed in γ^s for $s < 1 + \frac{1}{\alpha}$.*

Here we give the proof in a very special case: a third order operator and we assume that $\Delta = Ct^\nu$ as in (16.13). We assume, moreover, that the term of order 0 (which has no influence on the well-posedness) is absent. Thus let

$$L(t;\partial_t,\partial_x)u = \partial_t^3 u - \sum_{j=1}^{3} a_j(t)\partial_t^{3-j}\partial_x^j u - \sum_{1\leq j+k\leq 2} b_{j,k}(t)\partial_t^j\partial_x^k u,$$

and let

$$\begin{aligned} P(t;\tau,\xi) &= \tau^3 - a_1(t)\tau^2\xi - a_2(t)\tau\xi^2 - a_3(t)\xi^3 \\ &= \big(\tau - \tau_1(t)\xi\big)\big(\tau - \tau_2(t)\xi\big)\big(\tau - \tau_3(t)\xi\big) \end{aligned}$$

be the principal symbol of $L(t; \partial_t, \partial_x)$. As remarked above (16.23) is equivalent to

$$|t|\left|\tau_j'(t)\right| \le C\left|\tau_j(t) - \tau_k(t)\right| \quad \text{if } j \ne k.$$

We transform as before the equation in u into a system in V: we get

$$V' = i|\xi|AV + BV, \tag{16.33}$$

instead of (16.9), where A and B are given in (16.2) and (16.3).

For $\varepsilon \in]0, 1]$ we define the *approximated energies*

$$E_{\varepsilon,\mathrm{kov}}(t;\xi) := \left|V(t;\xi)\right|^2, \quad \text{if } |t| < \varepsilon,$$
$$E_{\varepsilon,\mathrm{hyp}}(t;\xi) := \left|\left\langle \mathcal{Q}_\varepsilon(t;\xi)V(t;\xi), V(t;\xi)\right\rangle\right|^2, \quad \text{if } |t| \ge \varepsilon.$$

Concerning $E_{\varepsilon,\mathrm{kov}}(t)$ since V is a solution of (16.33) we have

$$\begin{aligned} E'_{\varepsilon,\mathrm{kov}}(t) &:= 2\,\mathrm{Re}\left\langle V(t), V'(t)\right\rangle \\ &= 2\,\mathrm{Re}\left\langle V(t), i|\xi|A(t)V(t)\right\rangle + 2\,\mathrm{Re}\left\langle V(t), i|\xi|B(t)V(t)\right\rangle \\ &\le C|\xi|E_{\varepsilon,\mathrm{kov}}(t), \end{aligned}$$

since $|\xi| \ge 1$. Thus by Gronwall's Lemma we get

$$E_{\mathrm{kov}}(t_2) \le e^{2C|\xi||t_2-t_1|}E_{\mathrm{kov}}(t_1) \le e^{4C\varepsilon|\xi|}E_{\mathrm{kov}}(t_1) \tag{16.34}$$

if $|t_1|, |t_2| < \varepsilon$.

Concerning $E_{\varepsilon,\mathrm{hyp}}(t)$ since V is a solution of (16.33) we have

$$\begin{aligned} E'_{\varepsilon,\mathrm{hyp}}(t) &= \left\langle \mathcal{Q}_\varepsilon' V, V\right\rangle + \left\langle \mathcal{Q}_\varepsilon V', V\right\rangle + \left\langle \mathcal{Q}_\varepsilon V, V'\right\rangle \\ &= \left\langle \mathcal{Q}_\varepsilon' V, V\right\rangle + |\xi|\langle i\mathcal{Q}_\varepsilon AV, V\rangle + |\xi|\langle \mathcal{Q}_\varepsilon V, iAV\rangle \\ &\quad + \langle \mathcal{Q}_\varepsilon BV, V\rangle + \langle \mathcal{Q}_\varepsilon V, BV\rangle \\ &\le \left[\frac{|\langle \mathcal{Q}_\varepsilon' V, V\rangle|}{\langle \mathcal{Q}_\varepsilon V, V\rangle} + |\xi|\frac{|\langle(\mathcal{Q}_\varepsilon A - (\mathcal{Q}_\varepsilon A)^*)V, V\rangle|}{\langle \mathcal{Q}_\varepsilon V, V\rangle} \right. \\ &\quad \left. + \frac{|2\,\mathrm{Re}\langle \mathcal{Q}_\varepsilon BV, V\rangle|}{\langle \mathcal{Q}_\varepsilon V, V\rangle}\right] E_{\varepsilon,\mathrm{hyp}}(t) \end{aligned}$$

hence, by Gronwall's Lemma, we get

$$E_{\varepsilon,\mathrm{hyp}}(t_2) \le \exp\left[\int_{-T}^{T} \frac{|\langle \mathcal{Q}_\varepsilon' V, V\rangle|}{\langle \mathcal{Q}_\varepsilon V, V\rangle}dt + \varepsilon|\xi| + \frac{|2\,\mathrm{Re}\langle \mathcal{Q}_\varepsilon BV, V\rangle|}{\langle \mathcal{Q}_\varepsilon V, V\rangle}dt\right]E_{\varepsilon,\mathrm{hyp}}(t_1).$$

The first term can be estimated as in the homogeneous case. Thus we have to estimate the last term.

Using Schwarz inequality for $\mathcal{Q}_\varepsilon$ we have

$$\left|\mathrm{Re}\langle \mathcal{Q}_\varepsilon BV, V\rangle\right| \le \langle \mathcal{Q}_\varepsilon BV, BV\rangle^{1/2}\langle \mathcal{Q}_\varepsilon V, V\rangle^{1/2} \le \|BV\|\sqrt{E_{\varepsilon,\mathrm{hyp}}}.$$

Note that

$$BV = (b_{0,2}, b_{1,1}, b_{2,0})\cdot V + |\xi|^{-1}(b_{0,1}, b_{1,0}, 0)\cdot V.$$

The homogeneous part of degree 2 of L has a proper decomposition w.r.t. P

$$\begin{aligned} b_{2,0}\tau^2 + b_{1,1}\tau\xi + b_{0,2}\xi^2 = {} & c_1(t)\big(\tau - \tau_2(t)\xi\big)\big(\tau - \tau_3(t)\xi\big) \\ & + c_2(t)\big(\tau - \tau_3(t)\xi\big)\big(\tau - \tau_1(t)\xi\big) \\ & + c_3(t)\big(\tau - \tau_1(t)\xi\big)\big(\tau - \tau_2(t)\xi\big), \end{aligned} \tag{16.35}$$

where, according to (16.32), the functions c_j satisfy

$$|t|^{1+\alpha}\big|c_1(t)\big|, |t|^{1+\alpha}\big|c_2(t)\big|, |t|^{1+\alpha}\big|c_3(t)\big| \in L^\infty.$$

Condition (16.35) is equivalent to

$$(b_{0,2}, b_{1,1}, b_{2,0}) = c_1 w_1 + c_2 w_2 + c_3 w_3,$$

where the w_j are the rows of $\mathcal{W}$. Thus

$$\begin{aligned} &\big|(b_{0,2}, b_{1,1}, b_{2,0})\cdot V\big| \\ &\quad\le |c_1||w_1\cdot V| + |c_2||w_2\cdot V| + |c_3||w_3\cdot V| \\ &\quad\le \big(c_1^2 + c_2^2 + c_3^2\big)^{1/2}\big(|w_1\cdot V|^2 + |w_2\cdot V|^2 + |w_3\cdot V|^2\big)^{1/2} \\ &\quad\le \frac{C}{t^{1+\alpha}}\sqrt{E_{\varepsilon,\mathrm{hyp}}} \le \frac{1}{\varepsilon^\alpha}\frac{C}{t}\sqrt{E_{\varepsilon,\mathrm{hyp}}} \end{aligned}$$

since $t \ge \varepsilon$.

The homogeneous part of degree 1 has a proper decomposition w.r.t. $\partial_\tau P$:

$$b_{1,0}(t)\tau + b_{0,1}(t)\xi = c_{1,1}(t)\big(\tau - \lambda_2(t)\xi\big) + c_{1,2}(t)\big(\tau - \lambda_1(t)\xi\big)$$

with, by (16.32),

$$t^{1+2\alpha}\big|c_{1,1}(t)\big|, t^{1+2\alpha}\big|c_{1,2}(t)\big| \in L^\infty,$$

where $\lambda_1(t)$ and $\lambda_2(t)$ are the roots of the normalized derivative of P:

$$\begin{aligned} P^{(1)}(t;\tau,\xi) &= \frac{1}{3}\partial_\tau P(t;\tau,\xi) = \tau^2 - \frac{2}{3}a_1(t)\tau\xi - \frac{1}{3}a_2(t)\xi^2 \\ &= \big(\tau - \lambda_1(t)\xi\big)\big(\tau - \lambda_2(t)\xi\big). \end{aligned}$$

Now, since

$$\tau_1(t) \le \lambda_1(t) \le \tau_2(t) \le \lambda_2(t) \le \tau_3(t)$$

we see that $\tau - \lambda_1(t)\xi$ and $\tau - \lambda_2(t)\xi$ are convex combinations of the $\tau - \tau_j(t)\xi$. Thus

$$b_{1,0}(t)\tau + b_{0,1}(t)\xi = \widetilde{c_1}(t)\big(\tau - \tau_1(t)\xi\big) + \widetilde{c_2}(t)\big(\tau - \tau_2(t)\xi\big) + \widetilde{c_3}(t)\big(\tau - \tau_3(t)\xi\big)$$

with

$$t^{1+2\alpha}\big|\widetilde{c_1}(t)\big|, t^{1+2\alpha}\big|\widetilde{c_2}(t)\big|, t^{1+2\alpha}\big|\widetilde{c_3}(t)\big| \in L^\infty.$$

This gives

$$(b_{1,0}, b_{0,1}, 0) = \widetilde{c_1} w_{1,1} + \widetilde{c_2} w_{1,2} + \widetilde{c_3} w_{1,3},$$

where $w_{1,j}$ is the j-th row of $\mathcal{W}_1$. Thus

$$\begin{aligned}
&|\xi|^{-1}\big|(b_{1,0}, b_{0,1}, 0)\cdot V\big| \\
&\quad\le |\xi|^{-1}\big(|\widetilde{c_1}||w_{1,1}\cdot V| + |\widetilde{c_2}||w_{1,2}\cdot V| + |\widetilde{c_3}||w_{1,3}\cdot V|\big) \\
&\quad\le |\xi|^{-1}\big(\widetilde{c_1}^2 + \widetilde{c_2}^2 + \widetilde{c_3}^2\big)^{1/2}\big(|w_{1,1}\cdot V|^2 + |w_{1,2}\cdot V|^2 + |w_{1,3}\cdot V|^2\big)^{1/2} \\
&\quad\le \frac{C}{\varepsilon|\xi|t^{1+2\alpha}}\sqrt{E_{\varepsilon,\mathrm{hyp}}} \le \frac{1}{\varepsilon^{1+2\alpha}|\xi|}\frac{C}{t}\sqrt{E_{\varepsilon,\mathrm{hyp}}}
\end{aligned}$$

since $t \ge \varepsilon$.

Finally, we get

$$E_{\varepsilon,\mathrm{hyp}}(t_2) \le \exp\left[\left(\frac{1}{\varepsilon^\alpha} + \frac{1}{\varepsilon^{1+2\alpha}|\xi|}\right)\int_{-T}^{T}\frac{1}{t}dt + \varepsilon|\xi|\right]E_{\varepsilon,\mathrm{hyp}}(t_1).$$

Choosing

$$\varepsilon := |\xi|^{-1/(\alpha+1)}$$

we get

$$E_{\varepsilon,\mathrm{hyp}}(t_2) \le \exp\big(|\xi|^{\alpha/(\alpha+1)}\log|\xi|\big)E_{\varepsilon,\mathrm{hyp}}(t_1),$$

which gives, together with (16.34), the estimates (16.5).

Appendix: Maple® Code Used to Check Condition (16.23)

Here is a simple Maple® code which is used to check condition (16.23) for the operator L_1 (the lines ending with a backslash are broken to fit the page size).

```
with(linalg,LinearAlgebra):

m:=10;
a[1]:=0 ; a[2]:=10 ; a[3]:=0 ;
a[4]:=-33 + 4*t^2 ; a[5]:=2 + 2*t^2 ;
a[6]:=40 - 20*t^2 ; a[7]:=-10 - 8*t^2 ;
a[8]:=-16 + 20*t^2 ; a[9]:=8 + t^2 ; a[10]:=-1 ;

P := tau -> tau^m-sum('a[k]*tau^(10-k)','k'=1..m);

Q := -BezoutMatrix(P(tau),diff(P(tau),tau),tau, \
          method=symmetric, \
          methodoptions=increasing_degree));

for k from 1 to m do
      det(submatrix(Q,k..m,k..m));
end do;

psi := 1/2*trace(evalm(map(D,Q) \
                    &* map(D,adjoint(Q))));
```

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Chapter 17
Thermo-elasticity for Anisotropic Media in Higher Dimensions

Jens Wirth

Abstract In this paper we develop tools to study the Cauchy problem for the system of thermo-elasticity in higher dimensions. The theory is developed for general homogeneous anisotropic media under non-degeneracy conditions. For degenerate cases a method of treatment is sketched and for the cases of cubic media and hexagonal media detailed studies are provided.

Mathematics Subject Classification 35B40 · 35B45 · 35Q72 · 74F05 · 74E10

17.1 Introduction

While isotropic thermo-elasticity is a well-known and well-established subject (see, e.g., the book of Jiang–Racke [7] and references therein) only very few results are available for the case of anisotropic media. Among them are the theses of Borkenstein [2] for cubic media and Doll [4] for the case of rhombic media together with the authors treatments [16, 23], all in two space dimensions.

In this paper the system of anisotropic thermo-elasticity in three (and more) dimensions, i.e.,

$$U_{tt} + A(\mathrm{D})U + \gamma \nabla\theta = 0, \tag{1a}$$

$$\theta_t - \kappa \Delta\theta + \gamma \nabla \cdot U_t = 0 \tag{1b}$$

for the elastic displacement $U(t,\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ and temperature difference $\theta(t,\cdot) : \mathbb{R}^n \to \mathbb{R}$ to the equilibrium state, will be considered. The system (1a) and (1b) couples the hyperbolic elasticity equation with the parabolic heat equation. The operator $A(\mathrm{D})$ describes the elastic properties of the underlying medium, while κ denotes its thermal conductibility. The constant γ describes the thermo-elastic coupling. Basic assumptions of our theory are $\kappa > 0$, $\gamma^2 > 0$ together with

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- $A(\xi) = |\xi|^2 A(\eta)$, $\eta = \xi/|\xi|$, is a 2-homogeneous matrix-valued symbol;
- $A : \mathbb{S}^{n-1} \to \mathbb{R}^{n\times n}$ is a real-analytic function of $\eta \in \mathbb{S}^{n-1}$, $n \geq 3$;
- $A(\eta) = A^*(\eta) > 0$ is self-adjoint and positive.

In general we can *not* assume that $A(\eta)$ is non-degenerate in the sense that $\#\operatorname{spec} A(\eta) = n$ for all $\eta \in \mathbb{S}^{n-1}$ (as done for the two-dimensional case in [16]). All basic examples show degeneracies in dimensions $n \geq 3$.

Example 1 Isotropic media

$$A(\eta) = \mu I + (\lambda + \mu)\eta \otimes \eta \tag{2}$$

with Lamé constants λ and μ. The matrix $A(\eta)$ is positive as long as $\mu > 0$ and $\lambda > -2\mu$. The eigenvectors of $A(\eta)$ are multiples of η and $\eta^\perp$ and thus invariant under rotations of frequency space.

Example 2 Cubic media

$$A(\eta) = \begin{pmatrix} (\tau-\mu)\eta_1^2 + \mu & (\lambda+\mu)\eta_1\eta_2 & \cdots & (\lambda+\mu)\eta_1\eta_n \\ (\lambda+\mu)\eta_1\eta_2 & (\tau-\mu)\eta_2^2 + \mu & & \vdots \\ \vdots & & \ddots & \vdots \\ (\lambda+\mu)\eta_1\eta_n & \cdots & \cdots & (\tau-\mu)\eta_n^2 + \mu \end{pmatrix} \tag{3}$$

described by parameters λ, μ and τ. Later we will describe the assumptions made on these parameters and the resulting spectral properties of the matrix function $A(\eta)$ more precisely. In the case of three space dimensions, the matrix $A(\eta)$ is positive if and only if $\mu > 0$, $\tau > 0$ together with $-2\mu - \tau/2 < \lambda < \tau$. In three space dimensions this will be one of our main examples.

Example 3 We can replace the constant τ on the diagonal by $\tau_1, \ldots, \tau_n$ in (3). This yields so-called *rhombic media*. The behaviour of rhombic media is close to that of cubic media if the parameters are of similar size, in general there will appear exceptional situations. See, e.g., [23] or [25] for a discussion of this effect in two space dimensions.

Example 4 Hexagonal media are another particularly interesting case for three space dimensions. Since we want to come back to them later on we introduce the corresponding operator. It is given by

$$A(\eta) = \mathbb{D}(\eta)^T \mathcal{C} \mathbb{D}(\eta), \tag{4}$$

where $\mathcal{C}$ contains the 5 structure constants τ_1, τ_2, λ_1, λ_2 and μ and $\mathbb{D}(\eta)$ is of a particular form,

$$\mathcal{C} = \begin{pmatrix} \tau_1 & \lambda_1 & \lambda_2 & & & \\ \lambda_1 & \tau_1 & \lambda_2 & & & \\ \lambda_2 & \lambda_2 & \tau_2 & & & \\ & & & \mu & & \\ & & & & \mu & \\ & & & & & \frac{\tau_1-\lambda_1}{2} \end{pmatrix}, \qquad \mathbb{D}(\eta) = \begin{pmatrix} \eta_1 & & \\ & \eta_2 & \\ & & \eta_3 \\ & \eta_3 & \eta_2 \\ \eta_3 & & \eta_1 \\ \eta_2 & \eta_1 & \end{pmatrix}. \tag{5}$$

Even the first (non-trivial) anisotropic example, the case of cubic media in three space dimensions, has degenerate directions in which $A(\eta)$ has double eigenvalues. Later on we will analyse this example in detail.

Definition 1 We call a direction $\eta \in \mathbb{S}^{n-1}$ (elastically) *non-degenerate* if

$$\#\operatorname{spec} A(\eta) = n \tag{6}$$

holds true for this direction η.

The set of non-degenerate directions is an open subset of $\mathbb{S}^{n-1}$. For non-degenerate directions the treatment of [16] transfers almost immediately and gives a representation of solutions. We will sketch the results in Sect. 17.2. In Sect. 17.3 we consider special degenerate directions and discuss the examples of cubic and hexagonal media. Dispersive estimates for solutions are given in Sect. 17.4. In the neighbourhood of degenerate directions they are essentially based on estimates developed by Liess [1, 10, 12] for the treatment of anisotropic acoustic equations.

Hyperbolic equations of higher order and systems were treated by Ruzhansky and Smith, [19]. Their micro-local decay estimates are related to our results, although the thermo-elastic coupling helps to simplify the proofs of decay estimates in our case.

17.2 Treatment of Non-degenerate Directions

For the following we consider a simply connected open subset $\mathcal{U}$ of $\mathbb{S}^{n-1}$, where the symbol $A(\eta)$ has n distinct (and real) eigenvalues. We denote these eigenvalues in ascending order as

$$0 < \varkappa_1(\eta) < \varkappa_2(\eta) < \cdots < \varkappa_n(\eta). \tag{7}$$

By analytic perturbation theory, see [8], we know that these eigenvalues are real-analytic and that we find corresponding normalised eigenvectors

$$r_1(\eta), \ldots, r_n(\eta) \in C^\infty\big(\mathcal{U}, \mathbb{S}^{n-1}\big) \tag{8}$$

depending analytically on $\eta \in \mathcal{U}$. Collecting them in the unitary matrix

$$M(\eta) = \big(r_1(\eta)|r_2(\eta)|\cdots|r_n(\eta)\big), \tag{9}$$

$$M^*(\eta)M(\eta) = I = M(\eta)M^*(\eta), \tag{10}$$

we can diagonalise the matrix $A(\eta)$

$$A(\eta)M(\eta) = M(\eta)\mathcal{D}(\eta), \tag{11}$$

$$\mathcal{D}(\eta) = \operatorname{diag}\big(\varkappa_1(\eta), \varkappa_2(\eta), \dots, \varkappa_n(\eta)\big). \tag{12}$$

In our treatment we will not make use of analyticity directly, instead our use of perturbation theory will be based on [6] und [24] and uses only smooth dependence. This will be of interest for generalisations later on. Therefore, whenever we use analyticity, we will explicitly state that.

We use $M(\eta)$ to reduce the thermo-elastic system to a system of first order. For this we denote by $\hat{U}$ and $\hat{\theta}$ the partial Fourier transforms of U and θ with respect to the spatial variables and consider

$$V = \begin{pmatrix} (\mathrm{D}_t + \mathcal{D}^{1/2}(\xi))M^*(\eta)\hat{U} \\ (\mathrm{D}_t - \mathcal{D}^{1/2}(\xi))M^*(\eta)\hat{U} \\ \hat{\theta} \end{pmatrix} \in \mathbb{C}^{2n+1}, \tag{13}$$

as usual $\mathrm{D}_t = -\mathrm{i}\partial_t$ and $\eta = \xi/|\xi|$. Then V satisfies a first order system of ordinary differential equations, which has an apparently simple structure. Straightforward calculation shows that

$$\mathrm{D}_t V = B(\xi)V \tag{14}$$

holds true with coefficient matrix

$$B(t,\xi) = \begin{pmatrix} \omega_1(\xi) & & & & & & \mathrm{i}\gamma a_1(\xi) \\ & \omega_2(\xi) & & & & & \mathrm{i}\gamma a_2(\xi) \\ & & \ddots & & & & \vdots \\ & & & -\omega_1(\xi) & & & \mathrm{i}\gamma a_1(\xi) \\ & & & & -\omega_2(\xi) & & \mathrm{i}\gamma a_2(\xi) \\ & & & & & \ddots & \vdots \\ \frac{\mathrm{i}\gamma}{2}a_1(\xi) & \frac{\mathrm{i}\gamma}{2}a_2(\xi) & \cdots & \frac{\mathrm{i}\gamma}{2}a_1(\xi) & \frac{\mathrm{i}\gamma}{2}a_2(\xi) & \cdots & \mathrm{i}\kappa|\xi|^2 \end{pmatrix}, \tag{15}$$

where $\omega_j(\xi) = \sqrt{\varkappa_j(\xi)} \in C^\infty(\mathcal{U}, \mathbb{R}_+)$ and

$$a_j(\xi) = r_j(\eta)\cdot\xi. \tag{16}$$

Following the conventions of [16] we denote these functions $a_j(\xi)$ as the *coupling functions* of the thermo-elastic system associated to the elastic operator $A(\mathrm{D})$. They play a prominent rôle for the description of the time-asymptotic behaviour of solutions. This reflects the fact that they couple the homogeneous first order entries in

$B(\xi)$ with the second order lower right corner entry. Note, that

$$\sum_{j=1}^{n} a_j^2(\eta) = 1. \tag{17}$$

Zeros of the coupling functions are of particular importance. Following Definition 1 in [16] we define:

Definition 2 A non-degenerate direction $\eta \in \mathbb{S}^{n-1}$ is called

- *hyperbolic* if one of the coupling functions vanishes; more precisely, it is called *hyperbolic with respect to the eigenvalue* $\varkappa_j(\eta)$ if $a_j(\eta) = 0$;
- *parabolic* if all coupling functions are non-zero.

In the anisotropic case the set of hyperbolic directions is (generically[1]) a lower dimensional subset of $\mathbb{S}^{n-1}$. In order to decide whether a direction is hyperbolic or parabolic we can employ the following proposition. We denote for a matrix A and a vector η by

$$\mathcal{Z}(A, \eta) = \operatorname{span}\{A^k \eta | k = 0, 1, \dots\} \tag{18}$$

the corresponding cyclic subspace, i.e. the span of the trajectory of η under the action of the matrix A.

Proposition 1 *The following statements are equivalent*:

1. *The cyclic subspace of η has dimension $n-k$, i.e.*, $\dim \mathcal{Z}(A(\eta), \eta) = n - k$.
2. *Exactly k of the coupling functions vanish in η.*

Hence, a non-degenerate direction $\eta \in \mathbb{S}^{n-1}$ is parabolic if and only if $\mathcal{Z}(A(\eta), \eta) = \mathbb{R}^n$ and therefore

$$\det\big(\eta | A(\eta)\eta | \cdots | A^{n-1}(\eta)\eta\big) \neq 0. \tag{19}$$

Proof If we represent η in the eigenbasis of $A(\eta)$ we obtain

$$\eta = a_1(\eta) r_1(\eta) + \cdots + a_n(\eta) r_n(\eta) \tag{20}$$

and therefore

$$A^\ell(\eta)\eta = \varkappa_1^\ell(\eta) a_1(\eta) r_1(\eta) + \cdots + \varkappa_n^\ell(\eta) a_n(\eta) r_n(\eta). \tag{21}$$

If k of the coupling functions vanish, then $A^{n-k}(\eta)\eta$ must be in the span of the $A^\ell(\eta)\eta$ with $\ell = 0, 1, \dots, n-k-1$ and thus the cyclic subspace is at most of

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[1] If not, by analyticity it follows that one coupling function vanishes on $\mathcal{U}$ and the system is therefore decoupled. This case is reduced to the study of the lower dimensional blocks, one is a hyperbolic system the other one a thermo-elastic system of lower dimension. This is, e.g., the case for hexagonal media, see Sect. 17.3.4.

dimension $n-k$. On the other hand, the first $n-k$ vectors in the trajectory are linearly independent since the corresponding matrix in the basis representation with respect to $a_1(\eta)r_1(\eta),\dots,a_n(\eta)r_n(\eta)$ is just the van der Monde matrix associated to the eigenvalues of $A(\eta)$ for non-vanishing coupling functions and therefore regular. □

17.2.1 On the Characteristic Polynomial of the Full Symbol

At first we collect some of the spectral properties of the matrix $B(\xi)$ which are directly related to the characteristic polynomial of $B(\xi)$.

Proposition 2 *The following identities hold true*:

$$\operatorname{tr} B(\xi) = \mathrm{i}\kappa|\xi|^2, \tag{22}$$

$$\det B(\xi) = \mathrm{i}\kappa|\xi|^2 \det A(\xi), \tag{23}$$

$$\det\bigl(\nu - B(\xi)\bigr) = \bigl(\nu - \mathrm{i}\kappa|\xi|^2\bigr)\prod_{j=1}^{n}\bigl(\nu^2 - \varkappa_j(\xi)\bigr) - \nu\gamma^2\sum_{j=1}^{n} a_j^2(\xi)\prod_{k\neq j}\bigl(\nu^2 - \varkappa_k(\xi)\bigr). \tag{24}$$

Furthermore, the matrix $B(\xi)$ has a purely real eigenvalue for $\xi \neq 0$ if and only if the direction $\eta = \xi/|\xi|$ is hyperbolic. If it is j-hyperbolic, then $\pm\omega_j(\xi) \in \operatorname{spec} B(\xi)$.

The proof of the last fact is fairly straightforward and consists of separating real and imaginary parts of the characteristic polynomial. Note that for all parabolic directions we can divide the characteristic polynomial by $\nu\prod_j(\nu^2 - \varkappa_j(\xi))$ to obtain

$$1 = \frac{\mathrm{i}\kappa|\xi|^2}{\nu} + \gamma^2\sum_{j=1}^{n}\frac{a_j^2(\xi)}{\nu^2 - \varkappa_j(\xi)}. \tag{25}$$

This formulation allows to consider the neighbourhoods of hyperbolic directions. Assume for this that the set of hyperbolic directions with respect to $\varkappa_j(\eta)$

$$M_j = \bigl\{\eta \in \mathcal{U} | a_j(\eta) = r_j(\eta)\cdot\eta = 0\bigr\} \tag{26}$$

is a regular submanifold of $\mathcal{U}$. If we consider the corresponding *hyperbolic* eigenvalues $\nu_j^{\pm}(\xi)$ of $B(\xi)$ in a neighbourhood of M_j, i.e. the eigenvalues which satisfy

$$\lim_{\eta\to M_j}\nu_j^{\pm}(\xi) = \pm\omega_j(\xi) \tag{27}$$

for fixed $|\xi|$, (25) gives a precise description of the behaviour of the imaginary part of these eigenvalues. The proof is a straightforward generalisation from Proposition 2.2 in [16].

Proposition 3 *The non-tangential limit*

$$\lim_{\eta\to M_j}\frac{a_j^2(\xi)}{\nu_j^{\pm}(\xi)^2-\varkappa_j(\xi)}=1\mp\frac{\mathrm{i}\kappa|\xi|^2}{\omega_j(\xi)}-\gamma^2\sum_{k\neq j}\frac{a_k^2(\xi)}{\varkappa_j(\xi)-\varkappa_k(\xi)}$$
$$=\gamma^2\big(C_{\bar\eta}\mp\mathrm{i}D_{\bar\eta}|\xi|\big) \tag{28}$$

exists and is non-zero for all $\xi\neq 0$. *Furthermore*,

$$\lim_{\eta\to M_j}\frac{\operatorname{Im}\nu_j^{\pm}(\xi)}{a_j^2(\eta)}=\frac{D_{\bar\eta}|\xi|^2}{2\omega_j(\bar\eta)(C_{\bar\eta}^2+|\xi|^2\mathrm{D}_{\bar\eta}^2)}>0. \tag{29}$$

17.2.2 Asymptotic Expansion of the Eigenvalues as $|\xi|\to 0$

We decompose $B(\xi)$ into homogeneous components $B(\xi)=B_1(\xi)+B_2(\xi)$ of degree 1 and 2, respectively. For sufficiently small $|\xi|$ we expect the eigenvalues of $B(\xi)$ to be close to the eigenvalues of $B_1(\xi)$. For parabolic directions the (non-zero) eigenvalues of $B_1(\eta)$ can be determined from the equation

$$\frac{1}{\gamma^2}=\sum_{j=1}^{n}\frac{a_j^2(\eta)}{\tilde\nu^2-\varkappa_j(\eta)}, \tag{30}$$

which follows directly from (25) with $\kappa=0$. It can be solved (e.g. graphically, see Fig. 17.1 for $n=3$) to obtain the distinct eigenvalues $0,\pm\tilde\nu_1(\eta),\dots,\pm\tilde\nu_n(\eta)$ ordered as

$$0<\omega_1(\eta)<\tilde\nu_1(\eta)<\omega_2(\eta)<\tilde\nu_2(\eta)<\cdots<\omega_n(\eta)<\tilde\nu_n(\eta). \tag{31}$$

For hyperbolic directions a similar result holds true. In the case of hyperbolic directions w.r.to $\varkappa_j(\eta)$ eigenvalues move to $\omega_j(\eta)$. According to the choice of the coupling constant γ different cases occur:

1. if $\frac{1}{\gamma^2}$ is large then $\tilde\nu_j(\eta)=\omega_j(\eta)$, the other inequalities are unchanged;
2. if $\frac{1}{\gamma^2}$ is small then $\tilde\nu_{j-1}(\eta)=\omega_j(\eta)$ and the other inequalities remain true.

The critical threshold between these two cases is

$$\frac{1}{\gamma^2}=\sum_{k\neq j}\frac{a_k^2(\eta)}{\varkappa_j(\eta)-\varkappa_k(\eta)}, \tag{32}$$

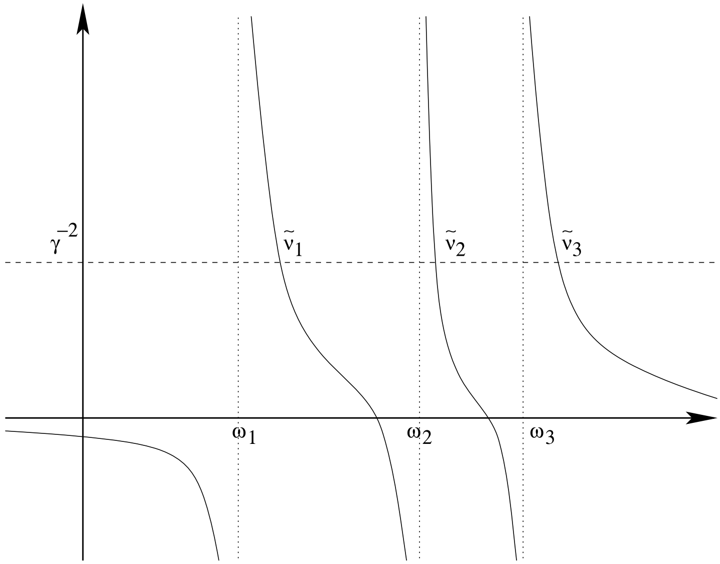

Fig. 17.1 Non-zero eigenvalues of $B_1(\xi)$ for parabolic directions

where $B_1(\eta)$ has the *double* eigenvalue $\tilde{\nu}_{j-1}(\eta) = \omega_j(\eta) = \tilde{\nu}_j(\eta)$. Following the conventions from [16] we define:

Definition 3 We denote a hyperbolic direction w.r.to $\varkappa_j(\eta)$ as γ-degenerate if (32) holds true.

For the following treatment we exclude γ-degenerate hyperbolic directions and assume instead that for all hyperbolic directions in $\mathcal{U}$ condition (32) is not satisfied for the corresponding index j. Then the following statement is apparent.

Proposition 4 *Let η be not γ-degenerate. Then the matrix $B_1(\eta)$ has $2n+1$ distinct real eigenvalues $0, \pm\tilde{\nu}_1, \dots, \pm\tilde{\nu}_n$ for all $\eta \in \mathcal{U}$.*

Proposition 4 allows to apply the standard diagonalisation scheme (see Sect. 2.1 in [6]) to $B(\xi) = B_1(\xi) + B_2(\xi)$ as $\xi \to 0$. Hence, eigenvalues, eigenprojections and all their derivatives have full asymptotic expansions as $\xi \to 0$. The proof is almost identical to that from Proposition 2.5 in [16] and is omitted.

Proposition 5 *For all not γ-degenerate directions $\eta = \xi/|\xi| \in \mathcal{U}$ the eigenvalues and eigenprojections of $B(\xi)$ have full asymptotic expansions as $\xi \to 0$. The main terms are given by*

$$\nu_0(\xi) = \mathrm{i}\kappa|\xi|^2 b_0(\eta) + \mathcal{O}\big(|\xi|^3\big) \tag{33a}$$

$$\nu_j^{\pm}(\xi) = \pm|\xi|\tilde{\nu}_j(\eta) + \mathrm{i}\kappa|\xi|^2 b_j(\eta) + \mathcal{O}\big(|\xi|^3\big) \tag{33b}$$

with

$$b_0(\eta) = \left(1 + \gamma^2 \sum_{k=1}^{n} \frac{a_k^2(\eta)}{\varkappa_k(\eta)}\right)^{-1} > 0 \tag{34a}$$

and

$$b_j(\eta) = \left(1 + \gamma^2 \sum_{k=1}^{n} a_k^2(\eta) \frac{\tilde{\nu}_j^2(\eta) + \varkappa_k(\eta)}{(\tilde{\nu}_j^2(\eta) - \varkappa_k(\eta))^2}\right)^{-1} \geq 0. \tag{34b}$$

Furthermore, $b_j(\eta) = 0$ *if and only if* η *is hyperbolic with respect to the eigenvalue* $\varkappa_j(\eta)$*.*

Remark 1 Note, that $\operatorname{tr} B(\xi) = \mathrm{i}\kappa|\xi|^2$ implies

$$b_0(\eta) + 2\sum_{j=1}^{n} b_j(\eta) = 1. \tag{35}$$

Recall that by Proposition 2 eigenvalues of $B(\xi)$ can only be real along hyperbolic directions (and then they are exactly the 'trivial' real eigenvalues). In combination with the fact that eigenvalues of $B(\xi)$ are continuous in ξ we obtain:

Corollary 1 *For all parabolic directions* $\eta = \xi/|\xi| \in \mathcal{U}$ *we have* $\operatorname{Im} \nu_j^{\pm}(\xi) > 0$. *The same is true as long as* η *is* not *hyperbolic w.r.to* $\varkappa_j(\eta)$*.*

17.2.3 Asymptotic Expansion of the Eigenvalues as $|\xi| \to \infty$

In this case the two-step procedure developed in Sect. 2.2 in [6], Proposition 2.6 in [16] applies in analogy. Essential assumption is the non-degeneracy of $A(\eta)$. We omit the proof and cite the corresponding result only.

Proposition 6 *For all non-degenerate directions the eigenvalues and eigenprojections of the matrix* $B(\xi)$ *have full asymptotic expansions as* $|\xi| \to \infty$*. The first terms are given by*

$$\nu_0(\xi) = \mathrm{i}\kappa|\xi|^2 - \frac{\mathrm{i}\gamma}{\kappa} + \mathcal{O}\big(|\xi|^{-1}\big), \tag{36a}$$

$$\nu_j^{\pm}(\xi) = \pm|\xi|\omega_j(\eta) + \frac{\mathrm{i}\gamma^2}{2\kappa} a_j^2(\eta) + \mathcal{O}\big(|\xi|^{-1}\big). \tag{36b}$$

Remark 2 We make a short notational remark. We use the same notation for descriptions of eigenvalues as $\xi \to 0$ and $|\xi| \to \infty$; this notation is consistent in a neighbourhood of infinity and also of 0. There might be obstructions to define such functions globally in $\xi \in \mathbb{R}^n$. As we will only use (micro-) localised arguments later on we will not make this more precise and prefer a more suggestive notation.

Corollary 2 *For all parabolic directions $\eta = \xi/|\xi|$ the eigenvalues of $B(\xi)$ satisfy* $\operatorname{Im}\nu(\eta) \geq C_\eta > 0$ *for $|\xi| \geq c$. The same is true for parabolic eigenvalues in hyperbolic directions.*

Remark 3 In particular, we see by the asymptotic expansions that the eigenvalues of $B(\xi)$ are simple for large and also for small values of $|\xi|$. Furthermore, we see that the hyperbolic eigenvalues are always separated (i.e. if multiplicities occur in hyperbolic directions, they involve only parabolic eigenvalues).

17.2.4 Behaviour of the Imaginary Part

The asymptotic expansions of Propositions 5 and 6 allow to draw conclusions for the behaviour of the imaginary part. We collect them for later use. The first result is apparent.

Proposition 7 *On any compact set of parabolic directions we have the uniform estimates*

$$\operatorname{Im}\nu_j^{(\pm)}(\xi) \geq C_\epsilon \quad \textit{for all } |\xi| \geq \epsilon, \tag{37}$$

$$\operatorname{Im}\nu_j^{(\pm)}(\xi) \sim b_j(\eta)|\xi|^2 \quad \textit{for all } |\xi| \leq \epsilon \tag{38}$$

for all eigenvalues of $B(\xi)$ and arbitrary $\epsilon > 0$.

The next statement is concerned with a tubular neighbourhood of a compact subset of a regular submanifold M_j of hyperbolic eigenvalues w.r.to $\varkappa_j(\eta)$. It is only of interest how the corresponding hyperbolic eigenvalues $\nu_j^{\pm}(\xi)$ behave, the others still satisfy Proposition 7.

Proposition 8 *Uniformly on any tubular neighbourhood of a compact subset of M_j of not γ-degenerate directions the corresponding hyperbolic eigenvalues $\nu_j^{\pm}(\xi)$ satisfy the estimates*

$$\operatorname{Im}\nu_j^{\pm}(\xi) \sim a_j^2(\eta) \quad \textit{for all } |\xi| \geq \epsilon, \tag{39}$$

$$\operatorname{Im}\nu_j^{\pm}(\xi) \sim b_j(\eta)|\xi|^2 \quad \textit{for all } |\xi| \leq \epsilon. \tag{40}$$

Proof By Proposition 3 we know that

$$\operatorname{Im}\nu_j^{\pm}(\xi) = a_j^2(\xi)K(\xi) \tag{41}$$

for some function $K(\xi)$. Our aim is to estimate $K(\xi)$. The left hand of this formula has a full asymptotic expansion as $|\xi| \to 0$ and $|\xi| \to \infty$. Therefore, also the right hand side has one and it follows that

$$K(\xi) = \frac{\gamma^2}{2\kappa} + \mathcal{O}\big(|\xi|^{-1}\big), \quad |\xi| \to \infty, \tag{42a}$$

$$K(\xi) = \kappa|\xi|^2 \frac{b_j(\eta)}{a_j^2(\eta)} + \mathcal{O}\big(|\xi|^3\big), \quad |\xi| \to 0. \tag{42b}$$

Thus, the desired estimate follows by a compactness argument as soon as we have a uniform lower/upper bound for $b_j(\eta)/a_j^2(\eta)$. The representation of $b_j(\eta)$ in Proposition 5 in combination with (30) implies

$$\begin{aligned}\lim_{\eta\to M_j} \frac{a_j^2(\eta)}{b_j(\eta)} &= \lim_{\eta\to M_j} \gamma^2\big(\tilde{\nu}_j^2 + \varkappa_j(\eta)\big)\frac{a_j^4(\eta)}{(\tilde{\nu}_j^2 - \varkappa_j(\eta))^2} \\ &\quad + \lim_{\eta\to M_j} a_j^2(\eta)\left(1 + \gamma^2 \sum_{k\neq j} a_k^2(\eta)\frac{\tilde{\nu}_j^2 + \varkappa_k(\eta)}{(\tilde{\nu}_j^2 - \varkappa_k(\eta))^2}\right) \\ &= 2\gamma^2\varkappa_j(\bar{\eta})\left(1 - \gamma^2\sum_{j\neq k}\frac{a_k^2(\bar{\eta})}{\varkappa_j(\bar{\eta}) - \varkappa_k(\bar{\eta})}\right)^2,\end{aligned} \tag{43}$$

which is clearly bounded and (uniformly) positive on any compact subset of M_j (where we have to use that $\bar{\eta} \in M_j$ is not γ-degenerate). □

17.2.5 Conclusions

We will draw several conclusions from what we have obtained so far and on what we still have to consider in the remaining part of this paper.

17.2.5.1 Cubic Media in 3D

If we consider the special case of cubic media in three space dimensions degenerate directions are given by $\bar{\eta} = (\bar{\eta}_1, \bar{\eta}_2, \bar{\eta}_3)^T$ with $\bar{\eta}_1^2 = \bar{\eta}_2^2 = \bar{\eta}_3^2$ (eight directions, corresponding to the corners of a cube) or $\bar{\eta}_i^2 = 1$ for some i (six directions, corresponding to its faces). This can be calculated directly, corresponding eigenspaces are span$\{\bar{\eta}\}$ and $\bar{\eta}^\perp = \{\xi \in \mathbb{R}^n | \bar{\eta}\cdot\xi = 0\}$, or concluded by the cubic symmetry[2] of $A(\xi)$ in this particular case. See Fig. 17.2.

[2] $A(\xi)$ is invariant under the hexaeder group, i.e. the symmetry group of a cube. Thus, eigenspaces must be transferred in an appropriate way, which implies that symmetries of order 3 or 4 can only be realised by higher dimensional eigenspaces.

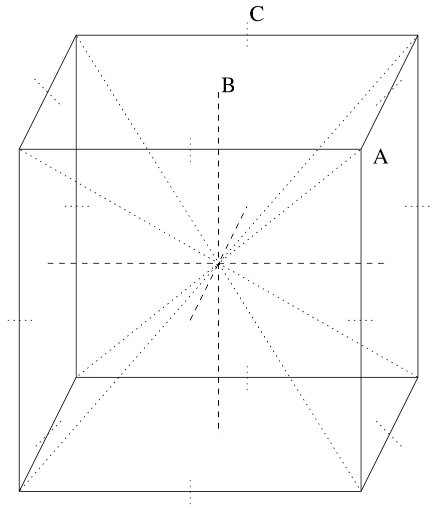

Fig. 17.2 Degenerate points for cubic media correspond to symmetries of a cube. *Corner points A* are conic singularities, *midpoints of faces B* uniplanar singularities of spec $A(\eta)$. The *midpoints of edges C* are non-degenerate, but hyperbolic with respect to two different eigenvalues

To obtain the hyperbolic directions we apply Proposition 1 and look for the action $A(\eta)$ on η. We obtain that

1. a direction η is hyperbolic if and only if

$$\det\bigl(\eta|A(\eta)\eta|A^2(\eta)\eta\bigr) = (\tau-\lambda-2\mu)^3\eta_1\eta_2\eta_3\bigl(\eta_1^2-\eta_2^2\bigr)\bigl(\eta_1^2-\eta_3^2\bigr)\bigl(\eta_2^2-\eta_3^2\bigr)=0, \tag{44}$$

   thus the set of hyperbolic directions is the union of nine great circles on $\mathbb{S}^2$;
2. $\eta||A(\eta)\eta$ for all 26 intersection points of these great circles, 14 of them are excluded as being degenerate.

Except for these 14 points on $\mathbb{S}^2$ we obtained an almost complete description of the spectrum of $B(\xi)$. We know full asymptotic expansions of eigenvalues for small and large frequencies $|\xi|$, estimates for the imaginary part of them and similar statements for eigenprojections. This information allows to draw conclusions on the large time behaviour of solutions, e.g. energy and dispersive estimates. This can be done similarly to the treatment of [16], see Sect. 17.4. The remaining degenerate directions appear in two types, which can be interchanged by the action of the symmetry group. The study of these degenerate directions is what is left open so far and will be the main point of Sect. 17.3.

17.2.5.2 Isotropic Media

If we consider the special case of isotropic media, $A(\eta) = \mu I + (\lambda+\mu)\eta\otimes\eta$, we see that spec $A(\eta) = \{\mu, \lambda+2\mu\}$ and corresponding eigenspaces are span$\{\eta\}$

(corresponding to $\lambda + \mu$) and $\eta^\perp$ (corresponding to μ). All directions are (elastically) degenerate. However, we still find locally smooth systems of eigenvectors. All directions are hyperbolic and the hyperbolic eigenvalue μ has multiplicity $n-1$. Therefore the system $D_t V = B(\xi)V$ decouples into a diagonal part of size $2n-2$ and a full 3×3 block and is given after a rearrangement of the entries as

$$B(t,\xi) = \begin{pmatrix} \sqrt{\mu}|\xi| & & & & & & \\ & \ddots & & & & & \\ & & -\sqrt{\mu}|\xi| & & & & \\ & & & \ddots & & & \\ & & & & \sqrt{\lambda+2\mu}|\xi| & & \mathrm{i}\gamma|\xi| \\ & & & & & -\sqrt{\lambda+2\mu}|\xi| & \mathrm{i}\gamma|\xi| \\ & & & & -\frac{\mathrm{i}\gamma}{2}|\xi| & -\frac{\mathrm{i}\gamma}{2}|\xi| & \mathrm{i}\kappa|\xi|^2 \end{pmatrix}. \tag{45}$$

This block structure corresponds to the Helmholtz decomposition of vector fields applied to the elastic displacement. If $\nabla \cdot U(t,\cdot) = 0$ the lower block gives no non-trivial contribution and we obtain wave equations with speed $\sqrt{\mu}$ for the components of U. Otherwise, if we cancel the upper block we obtain the 3×3 system corresponding to one-dimensional thermo-elasticity with its well-known properties.

17.2.5.3 One-dimensional Thermo-elasticity

For completeness we mention some results on the one-dimensional system

$$u_{tt} - \tau^2 u_{xx} + \gamma\theta_x = 0, \tag{46a}$$

$$\theta_t - \kappa\theta_{xx} + \gamma u_{tx} = 0. \tag{46b}$$

We assume $\gamma, \kappa, \tau > 0$. Following our strategy we can rewrite this problem as a first order system. The corresponding symbol $B(\xi)$ is given by

$$B(\xi) = \begin{pmatrix} \tau\xi & & \mathrm{i}\gamma\xi \\ & -\tau\xi & \mathrm{i}\gamma\xi \\ -\frac{\mathrm{i}}{2}\gamma\xi & -\frac{\mathrm{i}}{2}\gamma\xi & \mathrm{i}\kappa\xi^2 \end{pmatrix}. \tag{47}$$

Its eigenvalues satisfy asymptotic expansions for $\xi \to 0$ and $\xi \to \pm\infty$. Propositions 5 and 6 apply with $\tilde{\nu}^\pm = \pm\sqrt{\tau^2+\gamma^2}$ and

$$b_0 = \frac{\tau^2}{\tau^2+\gamma^2}, b_1 = \frac{1}{2}\frac{\gamma^2}{\tau^2+\gamma^2}. \tag{48}$$

Therefore, by Proposition 5

$$\nu_0(\xi) = \mathrm{i}\frac{\kappa\tau^2}{\tau^2+\gamma^2}\xi^2 + \mathcal{O}(\xi^3), \tag{49a}$$

$$\nu_1^{\pm}(\xi) = \pm\sqrt{\tau^2+\gamma^2}\xi + \mathrm{i}\frac{\kappa\gamma^2}{2(\tau^2+\gamma^2)}\xi^2 + \mathcal{O}(\xi^3), \tag{49b}$$

as $\xi \to 0$ and by Proposition 6

$$\nu_0(\xi) = \mathrm{i}\kappa\xi^2 - \mathrm{i}\frac{\gamma}{\kappa} + \mathcal{O}(\xi^{-1}), \tag{49c}$$

$$\nu_1^{\pm}(\xi) = \pm\tau\xi + \mathrm{i}\frac{\gamma^2}{2\kappa} + \mathcal{O}(\xi^{-1}), \tag{49d}$$

as $\xi \to \infty$. The essential information for large time estimates is given by the behaviour of the imaginary part. It follows that $\operatorname{Im}\nu(\xi) > C_\epsilon$ for $|\xi| \geq \epsilon$ for certain constants and

$$\operatorname{Im}\nu_0(\xi) \sim \frac{\kappa\tau^2}{\tau^2+\gamma^2}\xi^2, \qquad \operatorname{Im}\nu_1^{\pm}(\xi) \sim \frac{\kappa\gamma^2}{2(\tau^2+\gamma^2)}\xi^2, \quad \xi \to 0. \tag{50}$$

17.2.5.4 Hexagonal Media in 3D

For hexagonal media in three space dimensions the situation is (surprisingly) simpler than for cubic media. The elastic operator defined by (4)–(5) is invariant under rotation around the x_3-axis (taking into account a corresponding rotation of the reference frame for vectors) and therefore it suffices to understand its cross sections in the x_1–x_2 plane. We will sketch some of the properties of the corresponding symbol $A(\eta)$.

Following Proposition 1 we obtain

1. that

$$\det\big(\eta|A(\eta)\eta|A^2(\eta)\eta\big) = 0, \tag{51}$$

   such that *all* directions $\eta \in \mathbb{S}^2$ are hyperbolic. The corresponding eigenspace is (generically) given by multiples of $(\eta_2, -\eta_1, 0)$ such that the hyperbolic eigenvalue is

$$\frac{\tau_1-\lambda_1}{2}\big(\eta_1^2+\eta_2^2\big) + \mu\eta_3^2. \tag{52}$$

2. It remains to look for directions with two hyperbolic eigenvalues. They satisfy $\eta || A(\eta)$. This is true, if $\eta_3 = 0$ or if $\eta_1 = \eta_2 = 0$ or if

$$\eta_3^2 = \frac{\lambda_2+2\mu-\tau_1}{2\lambda_2+4\mu+\tau_1-\tau_2}, \tag{53}$$

provided the latter expression is non-negative. Except in the limiting case $\tau_1 = \lambda_2 + 2\mu$, the coupling functions vanish to first order along the corresponding circle. If $\tau_1 = \tau_2 = \lambda_2 + 2\mu$ all directions are hyperbolic with two hyperbolic eigenvalues and if $\tau_1 = \lambda_2 + 2\mu \neq \tau_2$ coupling functions vanish to third order.

3. The matrices $A(\eta)$ are invariant under rotation. Introducing spherical coordinates on $\mathbb{S}^2$

$$\eta = \begin{pmatrix}1\\0\\0\end{pmatrix}\cos\phi\cos\psi + \begin{pmatrix}0\\1\\0\end{pmatrix}\sin\phi\cos\psi + \begin{pmatrix}0\\0\\1\end{pmatrix}\sin\psi \tag{54}$$

and using a corresponding (moving) basis for vectors given by

$$\frac{\pm 1}{\sqrt{\eta_1^2+\eta_2^2}}\begin{pmatrix}\eta_2\\-\eta_1\\0\end{pmatrix}, \qquad \eta, \qquad \frac{\pm\eta_3}{\sqrt{1-\eta_3^2}}\begin{pmatrix}\eta_1\\ \eta_2\\ -\frac{\eta_1^2+\eta_2^2}{\eta_3}\end{pmatrix} \tag{55}$$

(sign chosen to make them smoothly dependent on $\eta \neq \pm(0,0,1)^\top$) decomposes $A(\eta)$ into $(1,2)$-block-diagonal structure (independent of the co-ordinate ϕ). The scalar block corresponds to the eigenvalue (52), while the 2×2 block has trace $\mu + \tau_1\cos^2\psi + \tau_2\sin^2\psi$ and determinant $\mu\tau_1\cos^4\psi + \mu\tau_2\sin^4\psi + \frac{\tau_1\tau_2-2\lambda_2-\lambda_2^2}{4}\sin^2 2\psi$.

If $(\tau_1-\mu)(\tau_2-\mu)\neq 0$, the 2×2 block has distinct eigenvalues for all ψ and therefore the only degenerate directions are directions where this block has (52) as one of its eigenvalues. This happens if and only if the right hand side of (53) is non-negative and on the circle defined by that equation.

Thus, the previously developed theory is applicable for all directions except the degenerate ones $\eta_1 = \eta_2 = 0$ or (53). The always existent hyperbolic eigenvalue (52) leads to a decoupling of the thermo-elastic system into two scalar blocks and a (at least formally) 2D thermo-elastic system.

Due to rotational invariance, it suffices to treat the cut $\eta_1 = 0$ for handling of degenerate directions. This will be sketched later.

17.3 Some Special Degenerate Directions

We want to study neighbourhoods of degenerate directions for some particular cases. To study degenerate directions in full generality is beyond the scope of this paper. We relate our approach to the type of singularity of the corresponding *Fresnel surface*

$$\mathcal{S} = \left\{\xi \in \mathbb{R}^n \,|\, 1 \in \operatorname{spec} A(\xi)\right\}. \tag{56}$$

This surface is in general n-sheeted and for all non-degenerate directions these sheets are given by

$$\begin{aligned}\mathcal{S}_j &= \{\xi \in \mathbb{R}^n \text{ non-degenerate} | \omega_j(\xi) = 1\} \\ &= \{\omega_j^{-1}(\eta)\eta | \eta \in \mathbb{S}^{n-1} \text{ non-degenerate}\},\end{aligned} \tag{57}$$

while in degenerate points the surface is self-intersecting. For the importance of these surfaces in elasticity theory and some interesting properties of them we refer to Duff [5] or the investigations from Musgrave [14, 15] and Miller–Musgrave [13].

We remark only one of the general properties of $\mathcal{S}$ here. If $A(\xi)$ is polynomial in ξ then the surface $\mathcal{S}$ is algebraic of degree $2n$ and therefore any straight line intersecting $\mathcal{S}$ has at most $2n$ intersection points with $\mathcal{S}$. In particular, if the inner sheet $\mathcal{S}_n$ does not touch any of the outer sheets, it has to be strictly convex.

17.3.1 General Strategy

If we investigate isolated degenerate directions or regular manifolds of degenerate directions of codimension greater than one we are faced with two major obstacles. Generically, eigenvectors of $A(\eta)$ can not be chosen continuously in a neighbourhood of the degenerate direction and therefore a reformulation as system of first order as in (13) is problematic. This problem is related to higher-dimensional perturbation theory of matrices. It is well-known that in the one-dimensional situation eigenspaces are continuous (see, e.g., the book of Kato, [8]) and it can be resolved by introducing polar co-ordinates/normal co-ordinates around the degenerate directions and a system related to (13) can be formulated on a corresponding blown-up space (see, e.g., (65) below). A second obstacle are the multiplicities itself. Eigenvalues and eigenvectors of the constructed system of first order do not possess asymptotic expansions in powers of $|\xi|$ as $|\xi|$ tends to 0 or ∞. However, especially in the three-dimensional setting we can write full asymptotic expansions in the distance to the degeneracy uniform in all remaining co-ordinates.

We will discuss the application of this general strategy in detail for conic singularities of the Fresnel surface appearing for the case of cubic media and give the corresponding results for uniplanar singularities afterwards. Finally we will consider hexagonal media and show that they are much simpler in their analytical structure. See Fig. 17.3 to get an impression of the generic shape of cubic and hexagonal Fresnel surfaces. Cuts of cubic Fresnel surfaces are also depicted in Fig. 17.4.

17.3.2 Cubic Media, Conic Singularities

The Fresnel surface for cubic media has eight conic singularities which are related by the symmetries of the underlying medium. It suffices to consider one of them

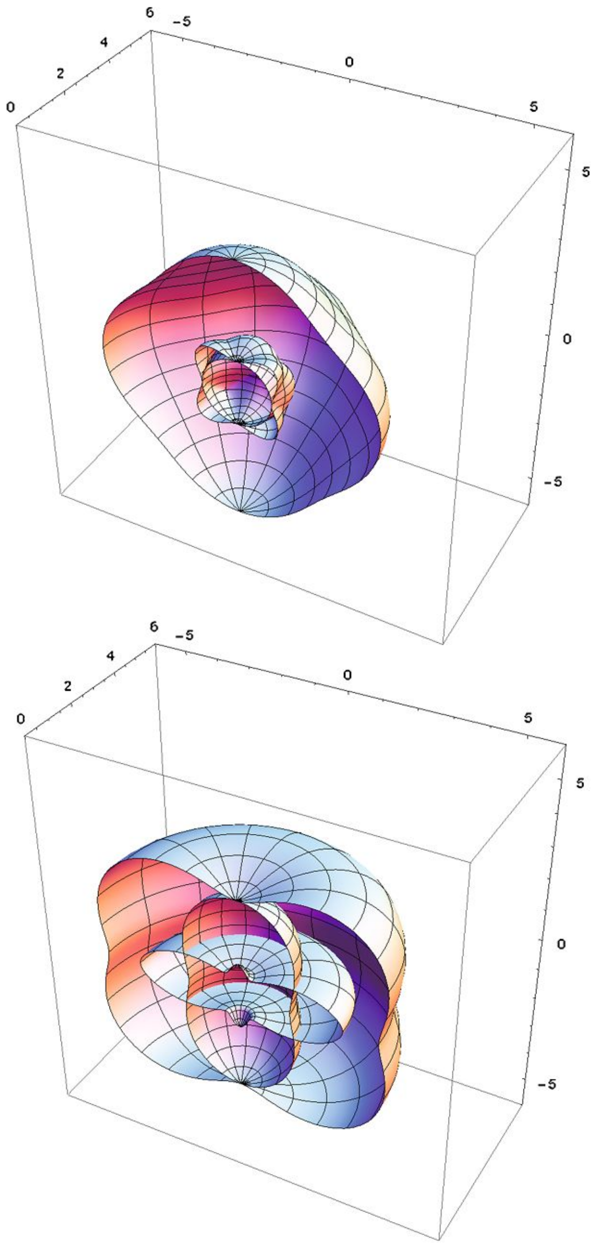

Fig. 17.3 A cut through the Fresnel surfaces for examples of a cubic and a hexagonal medium. The material parameter are $\lambda = 1$, $\tau = 4$ and $\mu = 1$ for the *picture on the top* (cubic) and $\lambda_1 = 1$, $\lambda_2 = \frac{1}{5}$, $\tau_1 = 4$, $\tau_2 = 1$ and $\mu = 3$ for the *picture on the bottom* (hexagonal)

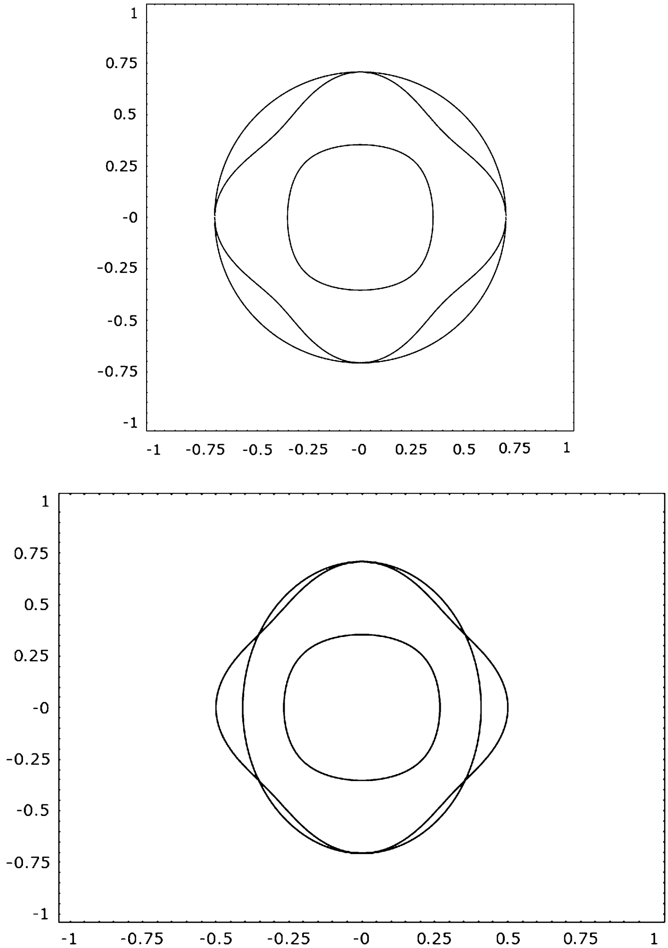

Fig. 17.4 Cuts of the Fresnel surface for cubic media; *on the top* in the plane $\eta_3 = 0$, *on the bottom* for $\eta_1 = \eta_2$. The parameters are chosen as $\tau = 8$, $\lambda = 2$ and $\mu = 2$

and we choose $\bar\eta = \frac{1}{\sqrt{3}}(1,1,1)^T \in \mathbb{S}^2$. Near this direction we introduce polar coordinates (ϵ, ϕ) on the sphere $\mathbb{S}^2$ by

$$\eta = \begin{pmatrix}\eta_1\\ \eta_2\\ \eta_3\end{pmatrix} = \sqrt{1-\epsilon^2}\frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix} + \epsilon\frac{1}{\sqrt{6}}\begin{pmatrix}-1\\-1\\2\end{pmatrix}\cos\phi + \epsilon\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\\0\end{pmatrix}\sin\phi. \quad (58)$$

They allow to blow up the singularity by looking at $[0,\infty)\times\mathbb{S}^1$ instead of $\mathbb{R}^2$ as local model of $\mathbb{S}^2$. In order to simplify notation, we apply a diagonaliser $\tilde M$ of $A(\bar\eta)$

to our matrices. For this we choose the unitary matrix

$$\tilde{M} = \frac{1}{\sqrt{6}} \begin{pmatrix} \sqrt{2} & -1 & \sqrt{3} \\ \sqrt{2} & -1 & -\sqrt{3} \\ \sqrt{2} & 2 & 0 \end{pmatrix} \tag{59}$$

(corresponding to the vectors chosen already in (58)). The matrix $\tilde{M}^{-1}A(\eta)\tilde{M}$ has a full asymptotic expansion as $\epsilon \to 0$ and can be written as

$$\tilde{M}^{-1}A(\epsilon,\phi)\tilde{M} = A_0 + \epsilon A_1(\phi) + \mathcal{O}(\epsilon^2), \quad \epsilon \to 0 \tag{60}$$

with matrices

$$A_0 = \operatorname{diag}\left(\frac{\tau+2\lambda+4\mu}{3}, \frac{\tau+\mu-\lambda}{3}, \frac{\tau+\mu-\lambda}{3}\right), \tag{61a}$$

$$A_1(\phi) = \frac{2\tau-\mu+\lambda}{3}\begin{pmatrix} & \cos\phi & \sin\phi \\ \cos\phi & & \\ \sin\phi & & \end{pmatrix} + \frac{\sqrt{2}(-\tau+2\mu+\lambda)}{3}\begin{pmatrix} 0 & & \\ & -\cos\phi & \sin\phi \\ & \sin\phi & \cos\phi \end{pmatrix}. \tag{61b}$$

Now we can apply the block-diagonalisation procedure (again following Sect. 2.2 in [6]) to obtain the behaviour of eigenvalues and eigenprojections of $\tilde{M}^{-1}A(\epsilon,\phi)\tilde{M}$ as $\epsilon \to 0$ for all ϕ. We restrict consideration to the case where $\lambda+\mu \neq 0$, such that A_0 has two different eigenvalues.

Proposition 9 *The eigenvalues $\varkappa_j(\epsilon,\phi)$ and the corresponding eigenprojections of $A(\epsilon,\phi)$ have uniformly in ϕ full asymptotic expansions as $\epsilon \to 0$. The main terms are given by*

$$\varkappa_1(\epsilon,\phi) = \frac{\tau+2\lambda+4\mu}{3} + \mathcal{O}(\epsilon^2), \tag{62a}$$

$$\varkappa_2(\epsilon,\phi) = \frac{\tau+\mu-\lambda}{3} + \frac{\sqrt{2}(-\tau+2\mu+\lambda)}{3}\epsilon + \mathcal{O}(\epsilon^2), \tag{62b}$$

$$\varkappa_3(\epsilon,\phi) = \frac{\tau+\mu-\lambda}{3} - \frac{\sqrt{2}(-\tau+2\mu+\lambda)}{3}\epsilon + \mathcal{O}(\epsilon^2). \tag{62c}$$

Remark 4 The exceptional case when $\tau = \lambda + 2\mu$ corresponds to isotropic media and is therefore excluded. In all other cases the two sheets $\omega_2(\eta) = \sqrt{\varkappa_2(\eta)}$ and $\omega_3(\eta) = \sqrt{\varkappa_3(\eta)}$ form a double cone on the Fresnel surface S. Hence, the statement explains the notion of conical singularity. Note, that the linear terms are independent of ϕ and therefore the cone is approximately a spherical cone near the conic point.

Proof of Proposition 9 We will only shortly review the main steps. First we $(1,2)$-block-diagonalise $\tilde{M}^{-1}A(\epsilon,\phi)\tilde{M}$ (modulo $\mathcal{O}(\epsilon^N)$ for any N we like). The diagonaliser we are going to construct has the form $I+\epsilon N_1^{(1)}(\phi)+\cdots+\epsilon^{N-1}N_1^{(N-1)}(\phi)$ and as in Sect. 2.2 in [6] its terms are given by recursion formulae. For $N_1^{(1)}$ we divide the off-(block-)diagonal terms of A_1 by the difference of the corresponding diagonal entries of A_0. This gives as first term

$$N_1^{(1)}(\phi)=\frac{2\tau-\mu+\lambda}{3(\lambda+\mu)}\begin{pmatrix} & \cos\phi & \sin\phi \\ -\cos\phi & & \\ -\sin\phi & & \end{pmatrix} \tag{63}$$

and allows to cancel the off-(block-)diagonal entries of A_1. We skip the further construction and move to the next step. Since the lower 2×2 block of $\text{b-diag}_{1,2}\,A_1$ has distinct eigenvalues (namely ± 1) we can now perform a diagonalisation scheme in the subspace corresponding to this block (modulo $\mathcal{O}(\epsilon^N)$). Again we restrict ourselves to the main terms. A unitary diagonaliser of the 2×2-block can be chosen as the unitary matrix

$$\tilde{M}_2(\phi)=\begin{pmatrix} 1 & & \\ & \sin\frac{\phi}{2} & \cos\frac{\phi}{2} \\ & \cos\frac{\phi}{2} & -\sin\frac{\phi}{2} \end{pmatrix}. \tag{64}$$

After transforming with that matrix we apply the recursive scheme to diagonalise further. Note that after applying $\tilde{M}_2(\phi)$ the matrix is diagonal modulo $\mathcal{O}(\epsilon^2)$ and therefore, $\tilde{M}(I+\epsilon N_1^{(1)}(\phi))\tilde{M}_2(\phi)=M_0(\phi)+\epsilon M_1(\phi)+\mathcal{O}(\epsilon^2)$ determines the main terms of a diagonaliser of the matrix $A(\epsilon,\phi)$ and we can deduce the statements about the eigenvalue asymptotics. □

17.3.2.1 System Formulation

Let $M(\epsilon,\phi)$ be the diagonaliser of $A(\epsilon,\phi)$ constructed in Proposition 9. Then we consider

$$V\big(t,\epsilon,\phi,|\xi|\big)=\begin{pmatrix} (\mathrm{D}_t+|\xi|\mathcal{D}^{1/2}(\epsilon,\phi))M^{-1}(\epsilon,\phi)\hat{U}(t,\xi) \\ (\mathrm{D}_t-|\xi|\mathcal{D}^{1/2}(\epsilon,\phi))M^{-1}(\epsilon,\phi)\hat{U}(t,\xi) \\ \hat{\theta} \end{pmatrix}\in\mathbb{C}^7, \tag{65}$$

with $\xi=|\xi|\eta(\epsilon,\phi)$ and $\mathcal{D}^{1/2}(\epsilon,\phi)=\operatorname{diag}(\omega_1(\epsilon,\phi),\dots)$ the diagonal matrix containing the square roots $\omega_j(\epsilon,\phi)=\sqrt{\varkappa_j(\epsilon,\phi)}$ of the eigenvalues of $A(\epsilon,\phi)$. The vector V satisfies the first order system $\mathrm{D}_t V=B(\epsilon,\phi,|\xi|)V$ with $B(\epsilon,\phi,|\xi|)=$

$B_1(\epsilon,\phi)|\xi| + B_2|\xi|^2$ given by

$$B_1(\epsilon,\phi) = \begin{pmatrix} \omega_1(\epsilon,\phi) & & & & \mathrm{i}\gamma a_1(\epsilon,\phi) \\ & \omega_2(\epsilon,\phi) & & & \mathrm{i}\gamma a_2(\epsilon,\phi) \\ & & \ddots & & \vdots \\ & & & -\omega_3(\epsilon,\phi) & \mathrm{i}\gamma a_3(\epsilon,\phi) \\ -\frac{\mathrm{i}}{2}\gamma a_1(\epsilon,\phi) & -\frac{\mathrm{i}}{2}\gamma a_2(\epsilon,\phi) & \cdots & -\frac{\mathrm{i}}{2}\gamma a_3(\epsilon,\phi) & 0 \end{pmatrix} \tag{66}$$

and $B_2 = \mathrm{diag}(0,\dots,0,\mathrm{i}\kappa)$. The coupling functions $a_j(\epsilon,\phi)$ are the components of the vector $M^{-1}(\epsilon,\phi)\eta$. From Proposition 9 we know that they have asymptotic expansions as $\epsilon \to 0$.

Remark 5

1. Since $M^{-1}(\epsilon,\phi) = \tilde{M}_2^*(\phi)(I - \epsilon N_1^{(1)}(\phi))\tilde{M}^* + \mathcal{O}(\epsilon^2)$ by our construction it follows that

$$a_1(\epsilon,\phi) = 1 + \mathcal{O}\big(\epsilon^2\big), \tag{67a}$$

$$a_2(\epsilon,\phi) = \epsilon\frac{2(\mu+2\lambda-\tau)}{3(\lambda+\mu)}\sin\frac{3\phi}{2} + \mathcal{O}\big(\epsilon^2\big), \tag{67b}$$

$$a_3(\epsilon,\phi) = \epsilon\frac{2(\mu+2\lambda-\tau)}{3(\lambda+\mu)}\cos\frac{3\phi}{2} + \mathcal{O}\big(\epsilon^2\big). \tag{67c}$$

   We know that the coupling functions vanish along three great circles through $\bar{\eta}$. We see that the numbering of the eigenprojections is not consistent along the circles. The coupling functions a_2 and a_3 vanish both in the degenerate direction.
2. Since we do not assume that $M(\epsilon,\phi)$ is unitary the relation $\sum_j a_j^2 = 1$ does not hold for these coupling functions. However, $M_0(\phi)$ is unitary and therefore $\sum_j a_j^2 = 1 + \mathcal{O}(\epsilon)$ as already observed.

17.3.2.2 Real and Imaginary Parts of Eigenvalues

The coefficient matrix $B(\epsilon,\phi,|\xi|)$ has the same structure as $B(\xi)$ in Sect. 17.2. Therefore, we can conclude similar statements on eigenvalues and their behaviour by (a) investigating the characteristic polynomial and (b) block-diagonalising for small and large $|\xi|$, respectively.

Proposition 10

1. $\mathrm{tr}\, B(\epsilon,\phi,|\xi|) = \mathrm{i}\kappa|\xi|^2$ *and* $\det B(\epsilon,\phi,|\xi|) = \mathrm{i}\kappa|\xi|^2 \det A(\xi)$.
2. $B(\epsilon,\phi,|\xi|)$ *has purely real eigenvalues for* $|\xi| \neq 0$ *if and only if the product* $a_2(\epsilon,\phi)a_3(\epsilon,\phi) = 0$ *vanishes, i.e.,* $\epsilon = 0$ *or* $\phi \in \frac{\pi}{3}\mathbb{Z}$.
3. $B(0,\phi,|\xi|)$ *has the real eigenvalues* $\pm\omega_{2,3}(0,\phi) = \frac{\sqrt{3}}{3}(\tau+\mu-\lambda)$ *and three eigenvalues satisfying* $\mathrm{Im}\,\nu \geq C$ *if* $|\xi| \geq c$ *and* $\mathrm{Im}\,\nu \sim |\xi|^2$ *if* $|\xi| < c$.

4. *The quotient*

$$\begin{aligned}&\big(a_2^2(\epsilon,\phi)\big(\nu_{2,3}^2(\epsilon,\phi,|\xi|)-\varkappa_3(\epsilon,\phi)|\xi|^2\big)\\&\quad+a_3^2(\epsilon,\phi)\big(\nu_{2,3}^2(\epsilon,\phi,|\xi|)-\varkappa_2(\epsilon,\phi)|\xi|^2\big)\big)\\&\quad/\big(\big(\nu_{2,3}^2(\epsilon,\phi,|\xi|)-\varkappa_2(\epsilon,\phi)|\xi|^2\big)\big(\nu_{2,3}^2(\epsilon,\phi)-\varkappa_3(\epsilon,\phi)|\xi|^2\big)\big)\end{aligned} \tag{68}$$

involving the hyperbolic eigenvalues $\nu_{2,3}^{\pm}$ *of* $B(\epsilon,\phi,|\xi|)$ *is smooth and non-vanishing for fixed values of* $|\xi|$.

Proof We consider only part (2) to (4). The characteristic polynomial of B is given by an expression like (24). If we assume that eigenvalues are purely real we can split the expression into real and imaginary part. We consider the imaginary part first, which leads to

$$\kappa|\xi|^2\prod_{j=1}^{3}\big(\nu^2-\varkappa_j(\epsilon,\phi)|\xi|^2\big)=0. \tag{69}$$

Therefore, real eigenvalues have to coincide with the square roots of eigenvalues of $A(\xi)$. If we assume $\nu^2=\varkappa_j(\epsilon,\phi)|\xi|^2$ is a root of the characteristic equation, we can divide by the corresponding factor and obtain if $\epsilon\neq 0$ (and therefore A is non-degenerate)

$$a_j^2(\epsilon,\phi)=0. \tag{70}$$

If $\epsilon=0$ the characteristic polynomial factors as

$$\big(\big(\nu-\mathrm{i}\kappa|\xi|^2\big)\big(\nu^2-\bar\varkappa_1|\xi|^2\big)-\nu\gamma^2|\xi|^2\big)\big(\nu-\bar\varkappa_{2,3}|\xi|^2\big)^2=0 \tag{71}$$

with $\bar\varkappa_1=\frac{1}{3}(\tau+2\lambda+4\mu)$ and $\bar\varkappa_{2,3}=\frac{1}{3}(\tau+\mu-\lambda)$. The first factor resembles one-dimensional thermo-elasticity (with $\tau^2=\bar\varkappa_1$) and gives three roots with positive imaginary parts subject to (49a)–(49d) and (50). Finally (4) follows by collecting the two related terms in the characteristic equation of form (25). The imaginary part of the quotient is given by $-\kappa|\xi|^2/\nu_{2,3}^{\pm}$ in hyperbolic/degenerate directions and therefore non-zero. □

The quotient (68) may be used to determine asymptotic expansions of the hyperbolic eigenvalue and its imaginary part as $\epsilon\to 0$ for fixed $|\xi|$ and $\phi\notin\frac{\pi}{3}\mathbb{Z}$. We will follow a different strategy and diagonalise as $\epsilon\to 0$ uniform on bounded ξ.

17.3.2.3 Asymptotic Expansion as $\epsilon\to 0$ Uniform in $|\xi|$

Note first, that $B(|\xi|,0,\phi)$ is independent of ϕ and just the system of one-dimensional thermo-elasticity (47) extended by four additional diagonal entries. Since we need to understand this system first, we are going to recall some facts about the one-dimensional theory. As $|\xi|$ becomes small/large we already gave asymptotic expansions of eigenvalues in Sect. 17.2.5.3. The bit of information which is still missing is contained in the following lemma.

Lemma 1 *The coefficient matrix $B(\xi)$ of the one-dimensional thermo-elastic system given in (47) has for $\xi \neq 0$ and under the natural assumptions $\gamma, \kappa, \tau > 0$ only simple eigenvalues.*

Proof Note that the characteristic polynomial of this matrix $B(\xi)$ is given by

$$\nu^3 - \mathrm{i}\kappa|\xi|^2\nu^2 + \tau^2|\xi|^2\nu + \mathrm{i}\tau^2\kappa|\xi|^4,$$

which is invariant under the transform $\nu \mapsto \overline{-\nu}$ and has alternating real and imaginary coefficients. From that we conclude that the only possible solutions are of the form $\mathrm{i}a$, $b + \mathrm{i}c$ and $-b + \mathrm{i}c$ for certain real a, b, c. Furthermore, from the general theory of Sect. 17.2 it is clear that $a, c > 0$. Thus, the only possibility for multiplicities to occur is if $b = 0$. Plugging in $b = 0$ and multiplying the corresponding linear factors gives

$$\nu^3 - \nu^2(\mathrm{i}a + 2\mathrm{i}c) - \nu\big(c^2 + 2ac\big) + \mathrm{i}c^2 a.$$

Comparing coefficients with the above polynomial implies that $\kappa|\xi|^2 = -ca/(c + 2a)$, which contradicts to the positivity of all quantities involved. □

We write the coefficient matrix $B(|\xi|, \epsilon, \phi)$ as sum of homogeneous components in ϵ

$$|\xi|^{-1} B\big(|\xi|, \epsilon, \phi\big) = B^{(0)}\big(|\xi|, \phi\big) + \epsilon B^{(1)}\big(|\xi|, \phi\big) + \mathcal{O}\big(\epsilon^2\big), \tag{72}$$

where

$$B^{(0)}\big(|\xi|, \phi\big) = \begin{pmatrix} \bar{\omega}_1 & & & & & & \mathrm{i}\gamma \\ & \bar{\omega}_2 & & & & & \\ & & \bar{\omega}_2 & & & & \\ & & & -\bar{\omega}_1 & & & \mathrm{i}\gamma \\ & & & & -\bar{\omega}_2 & & \\ & & & & & -\bar{\omega}_2 & \\ -\frac{\mathrm{i}}{2}\gamma & & & -\frac{\mathrm{i}}{2}\gamma & & & \mathrm{i}\kappa|\xi| \end{pmatrix}, \tag{73}$$

$$B^{(1)}\big(|\xi|, \phi\big)$$
$$= \begin{pmatrix} 0 & & & & & & 0 \\ & \delta_1 & & & & & \mathrm{i}\gamma\delta_2 \sin\frac{3\phi}{2} \\ & & -\delta_1 & & & & \mathrm{i}\gamma\delta_2 \cos\frac{3\phi}{2} \\ & & & 0 & & & 0 \\ & & & & \delta_1 & & \mathrm{i}\gamma\delta_2 \sin\frac{3\phi}{2} \\ & & & & & -\delta_1 & \mathrm{i}\gamma\delta_2 \cos\frac{3\phi}{2} \\ 0 & -\frac{\mathrm{i}\gamma\delta_2}{2}\sin\frac{3\phi}{2} & -\frac{\mathrm{i}\gamma\delta_2}{\cos}\frac{3\phi}{2} & 0 & -\frac{\mathrm{i}\gamma\delta_2}{2}\sin\frac{3\phi}{2} & -\frac{\mathrm{i}\gamma\delta_2}{2}\cos\frac{3\phi}{2} & 0 \end{pmatrix}, \tag{74}$$

and $\bar{\omega}_1 = \sqrt{\frac{\tau+2\lambda+4\mu}{3}}$, $\bar{\omega}_2 = \sqrt{\frac{\tau+\mu-\lambda}{3}}$, $\delta_1 = \frac{1}{\sqrt{6}} \frac{-\tau+2\mu+\lambda}{\sqrt{\tau+\mu-\lambda}}$ and $\delta_2 = \frac{2(\mu+2\lambda-\tau)}{3(\lambda+\mu)}$.

As a direct consequence of the previous lemma in combination with the asymptotic expansions of Sect. 17.2.5.3 we obtain

Proposition 11 *Assume, that $\lambda+\mu \neq 0$ and $\gamma^2+\lambda+\mu \neq 0$. Then the matrix $B^{(0)}(|\xi|, \phi)$ has uniformly separated eigenvalues in $|\xi| \in \mathbb{R}$, $\phi \in \mathbb{S}^1$ (where $\pm\bar{\omega}_2$ are of constant multiplicity two).*

Now we can apply several steps of diagonalisation based on the scheme of Sect. 2 in [6]. At first we apply the diagonaliser of the main part. This has only effects on the two entries related to $\bar{\omega}_1$ and the last row/column and determines the eigenvalues $\nu_0(|\xi|, \epsilon, \phi)$ and $\nu_1^{\pm}(|\xi|, \epsilon, \phi)$ modulo ϵ^2. Furthermore, Proposition 11 allows to $(1, 2, 1, 2, 1)$-block-diagonalise modulo $\mathcal{O}(\epsilon^N)$, N arbitrary.

Finally we can investigate the remaining 2×2-blocks and diagonalise again because the ϵ-homogeneous entries $\pm\delta_1\epsilon$ are distinct (trivially uniform in $|\xi|$ and ϕ).

Proposition 12 *Assume, that $\lambda+\mu \neq 0$ and $\gamma^2+\lambda+\mu \neq 0$. The eigenvalues of $B(|\xi|, \epsilon, \phi)$ have uniformly in $|\xi|$ and ϕ full asymptotic expansions as $\epsilon \to 0$. The first main terms are given as*

$$\nu_0\big(|\xi|, \epsilon, \phi\big) = \check{\nu}_0\big(|\xi|\big) + |\xi|\mathcal{O}\big(\epsilon^2\big), \tag{75a}$$

$$\nu_1^{\pm}\big(|\xi|, \epsilon, \phi\big) = \check{\nu}_1^{\pm}\big(|\xi|\big) + |\xi|\mathcal{O}\big(\epsilon^2\big), \tag{75b}$$

$$\nu_{2/3}^{\pm_1,\pm_2}\big(|\xi|, \epsilon, \phi\big) = \pm_1\bar{\omega}_2|\xi| \pm_2 \delta_1|\xi|\epsilon + |\xi|\mathcal{O}\big(\epsilon^2\big) \tag{75c}$$

where $\check{\nu}_0(|\xi|)$ and $\check{\nu}_1^{\pm}(|\xi|)$ are the eigenvalues of the one-dimensional thermo-elastic system with propagation speed $\bar{\omega}_1$ and the signs $\pm_1$ and $\pm_2$ are independent of each other.

Remark 6 The statement holds true uniformly in $|\xi|$. However, it is only of use as long as the error terms $|\xi|\epsilon^N$ are smaller than the size of the eigenvalues. For $|\xi| \to 0$ the eigenvalues of the one-dimensional thermo-elastic system behave like $\check{\nu}_0(|\xi|) \sim |\xi|^2$ and $\check{\nu}_1^{\pm}(|\xi|) \sim \pm|\xi|$. Hence, the statement of (75a) is only of use if $|\xi|\epsilon^2 \ll |\xi|^2$, i.e. if $\epsilon^2 \ll |\xi|$. For $|\xi| \to \infty$ we know similarly $\check{\nu}_0(|\xi|) \sim |\xi|^2$ and $\check{\nu}_1^{\pm}(|\xi|) \sim \pm|\xi|$, which in turn implies that the expansion determines the behaviour of the eigenvalues.

This restriction is by no means a severe one; the expansion is only of interest for the 'degenerate' eigenvalues $\nu_{2/3}^{\pm_1,\pm_2}(|\xi|, \epsilon, \phi)$ (for which no such restriction appears).

17.3.2.4 Diagonalisation for Small and Large $|\xi|$

To complete the picture we want to give some comments on expansions for small and large values of $|\xi|$ under the same assumptions as in Proposition 12. Using the ideas from [24] we can employ the (block) diagonalisation scheme to separate the three non-degenerate eigenvalues from the two degenerate ones asymptotically and give full asymptotic expansions for them as $|\xi|$ tends to zero or infinity. The obtained expressions coincide with the formulae from Propositions 5 and 6. It remains to understand the behaviour of the remaining 2×2-blocks. This can be done directly by solving the characteristic polynomial as in Proposition 2.7 in [16] or by a second diagonalisation scheme.

We focus on the latter idea and consider the case of small $|\xi|$ first. The 2×2-blocks have the form

$$f_0\big(|\xi|, \epsilon, \phi\big) I + \begin{pmatrix} \delta_0(|\xi|, \epsilon, \phi) & \\ & -\delta_0(|\xi|, \epsilon, \phi) \end{pmatrix} + \mathcal{O}\big(|\xi|^2\big) \tag{76}$$

with a function $\delta_0(|\xi|, \epsilon, \phi) \sim \epsilon|\xi|$. If we restrict the consideration to the zone

$$\mathcal{Z}_0(c) = \big\{\big(|\xi|, \epsilon, \phi\big) : |\xi| \le c\epsilon, \epsilon \ll 1\big\}, \tag{77}$$

the remainder can be written as $\epsilon|\xi|\mathcal{O}(\epsilon^{-1}|\xi|)$ and the standard diagonalisation scheme applied to the last two terms gives full asymptotic expansions in powers of $\epsilon^{-1}|\xi|$ as $\epsilon^{-1}|\xi| \to 0$,

$$f_0\big(|\xi|, \epsilon, \phi\big) \pm \delta_0\big(|\xi|, \epsilon, \phi\big) + \cdots + \epsilon|\xi|\mathcal{O}\big(\epsilon^{-N}|\xi|^N\big). \tag{78}$$

A similar idea applies for large $|\xi|$ in the zone

$$\mathcal{Z}_\infty(N) = \big\{\big(|\xi|, \epsilon, \phi\big) : \epsilon|\xi| \ge N, \epsilon \ll 1\big\}. \tag{79}$$

Based on

$$f_\infty\big(|\xi|, \epsilon, \phi\big) I + \begin{pmatrix} \delta_\infty(|\xi|, \epsilon, \phi) & \\ & -\delta_\infty(|\xi|, \epsilon, \phi) \end{pmatrix} + \mathcal{O}(1) \tag{80}$$

with a function $\delta_\infty(|\xi|, \epsilon, \phi) \sim \epsilon|\xi|$ it gives asymptotic expansions in powers of $\epsilon|\xi|$ as $\epsilon|\xi| \to \infty$.

17.3.3 Cubic Media, Uniplanar Singularities

The Fresnel surface for cubic media has six uniplanar singularities. Again they are equivalent and it suffices to consider the neighbourhood of $\bar{\eta} = (1, 0, 0)^T \in \mathbb{S}^2$.

We introduce polar co-ordinates near $\bar{\eta}$. Let $\epsilon \ge 0$ and $\phi \in [-\pi, \pi)$. Then we set

$$\eta = \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix} = \sqrt{1-\epsilon^2} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \epsilon \cos\phi \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \epsilon \sin\phi \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \tag{81}$$

and use an asymptotic expansion of $A(\eta)$ as $\epsilon \to 0$

$$A(\eta) = A_0 + \epsilon A_1(\phi) + \epsilon^2 A_2(\phi) + \mathcal{O}\big(\epsilon^3\big) \tag{82}$$

with coefficients

$$A_0 = \operatorname{diag}(\tau, \mu, \mu) \tag{83a}$$

$$A_1(\phi) = (\lambda + \mu) \begin{pmatrix} & \cos\phi & \sin\phi \\ \cos\phi & & \\ \sin\phi & & \end{pmatrix} \tag{83b}$$

$$A_2(\phi) = (\tau - \mu) \begin{pmatrix} -1 & & \\ & \cos^2\phi & \\ & & \sin^2\phi \end{pmatrix} + \frac{\lambda+\mu}{2} \begin{pmatrix} 0 & & \\ & & \sin 2\phi \\ & \sin 2\phi & \end{pmatrix} \tag{83c}$$

to deduce properties of the eigenvalues and eigenprojections of $A(\eta)$ near $\bar\eta$. We restrict considerations to the case when $\tau \neq \mu$. Then the following statement follows again by the two-step diagonalisation procedure (like in the conical case and as developed in [6, 16]).

Proposition 13 *Assume $\lambda + \mu \neq 0$, $\tau \neq \mu$ and $\tau \neq \lambda + 2\mu$. Then the eigenvalues $\varkappa_j(\eta)$ and the corresponding eigenprojections have uniformly in ϕ full asymptotic expansions as $\epsilon \to 0$. The main terms are given by*

$$\varkappa_1(\eta) = \tau - C\epsilon^2 + \mathcal{O}\big(\epsilon^3\big), \tag{84a}$$

$$\varkappa_2(\eta) = \mu + \frac{1}{2}\Big(C + \sqrt{C^2 - \big(C^2 - D^2\big)\sin^2(2\phi)}\Big)\epsilon^2 + \mathcal{O}\big(\epsilon^3\big), \tag{84b}$$

$$\varkappa_3(\eta) = \mu + \frac{1}{2}\Big(C - \sqrt{C^2 - \big(C^2 - D^2\big)\sin^2(2\phi)}\Big)\epsilon^2 + \mathcal{O}\big(\epsilon^3\big), \tag{84c}$$

where

$$C = \frac{(\tau-\mu)^2 - (\lambda+\mu)^2}{\tau - \mu}, \qquad D = \lambda + \mu \tag{85}$$

Remark 7 This statement reflects what we mean by an uniplanar singularity. Two of the eigenvalues coincide up to second order.

Proof of Proposition 13 We follow the diagonalisation scheme. A_0 is already diagonal, A_1 does not contain $(1, 2)$-block diagonal entries. To get expansions for the eigenvalues we have to apply two steps of block-diagonalisation. First we treat A_1

by the aid of

$$N_1^{(1)}(\phi) = \frac{\lambda+\mu}{\tau-\mu}\begin{pmatrix} & -\cos\phi & -\sin\phi \\ \cos\phi & & \\ \sin\phi & & \end{pmatrix}, \tag{86}$$

such that $I + \epsilon N_1^{(1)}(\phi)$ block-diagonalises the matrix modulo ϵ^2. This preserves A_0 and $0 = \text{b-diag}_{1,2} A_1$ and gives the new 2-homogeneous component

$$A_2 + A_1 N_1^{(1)}, \qquad A_1 N_1^{(1)} = \frac{(\lambda+\mu)^2}{\tau-\mu}\operatorname{diag}\big(1, -\cos^2\phi, -\sin^2\phi\big). \tag{87}$$

The starting terms of the expansion of the first eigenvalue can be read directly from these matrices. For the remaining two we have to diagonalise the lower 2×2 block. This block has the form

$$\begin{pmatrix} C\cos^2\phi & D\sin\phi\cos\phi \\ D\sin\phi\cos\phi & C\sin^2\phi \end{pmatrix} \tag{88}$$

with C, D from (85). The eigenvalues of this matrix are uniformly separated if the condition

$$C^2 > \big(C^2 - D^2\big)\sin^2(2\phi), \quad \text{i.e. } C \neq 0, D \neq 0 \tag{89}$$

is satisfied. Under this assumption the full diagonalisation scheme works through and the main terms can be calculated directly and give (84a)–(84c). For completeness we also give a unitary diagonaliser of the matrix (88), namely

$$\begin{aligned} M_2(\phi) &= \frac{1}{\sqrt{2D^2\sin^2(2\phi) + 2C^2\cos^2(2\phi) + 2C\cos 2\phi\sqrt{:}}} \\ &\quad \times \begin{pmatrix} C\cos 2\phi + \sqrt{:} & -D\sin 2\phi \\ D\sin 2\phi & C\cos 2\phi + \sqrt{:} \end{pmatrix} \\ &= \begin{pmatrix} m_1(\phi) & m_2(\phi) \\ -m_2(\phi) & m_1(\phi) \end{pmatrix} \end{aligned} \tag{90}$$

where $\sqrt{:} = \sqrt{C^2 - (C^2 - D^2)\sin^2(2\phi)}$, $\phi \neq \frac{\pi}{2}, \frac{3\pi}{2}$. Expressions are extended by continuity. □

17.3.3.1 System Form

Again we use the diagonaliser $M(\epsilon,\phi)$ of $A(\epsilon,\phi)$ constructed in Proposition 13 to reformulate the thermo-elastic system as a system of first order. Formulae (65) and (66) give the corresponding representation.

Remark 8

1. Since $M^{-1}(\epsilon,\phi) = \operatorname{diag}(1, M_2^*(\phi))(I - \epsilon N_1^{(1)}(\phi)) + \mathcal{O}(\epsilon^2)$ in the notation of the proof of Proposition 13 it follows that the coupling functions satisfy

$$a_1(\epsilon,\phi) = 1 + \mathcal{O}\big(\epsilon^2\big) \tag{91a}$$

$$a_2(\epsilon,\phi) = \epsilon\frac{\tau-\lambda-2\mu}{\tau-\mu}\big(m_1(\phi)\cos\phi + m_2(\phi)\sin\phi\big) + \mathcal{O}\big(\epsilon^2\big) \tag{91b}$$

$$a_3(\epsilon,\phi) = \epsilon\frac{\tau-\lambda-2\mu}{\tau-\mu}\big(m_1(\phi)\sin\phi - m_2(\phi)\cos\phi\big) + \mathcal{O}\big(\epsilon^2\big) \tag{91c}$$

   Since $\tau \neq \lambda + 2\mu$ the function $a_2(\phi)$ vanishes only for $\phi = k\frac{\pi}{2}$, $k \in \mathbb{Z}$, while $a_3(\phi)$ vanishes for $\phi = (2k+1)\frac{\pi}{4}$, $k \in \mathbb{Z}$.

2. Note that in contrast to the conic situation the eigenvalues coincide to second order in the degenerate direction, while the coupling functions still vanish to first order (if we approach the degeneracy from parabolic directions).

17.3.3.2 Asymptotic Expansion of Eigenvalues as $\epsilon \to 0$

We write the coefficient matrix $B(|\xi|,\epsilon,\phi)$ as a sum of homogeneous components in ϵ, cf. (72). This gives

$$B^{(0)}\big(|\xi|,\phi\big) = \begin{pmatrix} \sqrt{\tau} & & & & & & \mathrm{i}\gamma \\ & \sqrt{\mu} & & & & & \\ & & \sqrt{\mu} & & & & \\ & & & -\sqrt{\tau} & & & \mathrm{i}\gamma \\ & & & & -\sqrt{\mu} & & \\ & & & & & -\sqrt{\mu} & \\ -\frac{\mathrm{i}}{2}\gamma & & & -\frac{\mathrm{i}}{2}\gamma & & & \mathrm{i}\kappa|\xi| \end{pmatrix} \tag{92}$$

$$B^{(1)}\big(|\xi|,\phi\big) = \begin{pmatrix} & & & & & & 0 \\ & & & & & & \mathrm{i}\gamma a_2^{(1)}(\phi) \\ & & & & & & \mathrm{i}\gamma a_3^{(1)}(\phi) \\ & & & & & & 0 \\ & & & & & & \mathrm{i}\gamma a_2^{(1)}(\phi) \\ & & & & & & \mathrm{i}\gamma a_3^{(1)}(\phi) \\ 0 & \frac{\mathrm{i}}{2}\gamma a_2^{(1)}(\phi) & \frac{\mathrm{i}}{2}\gamma a_3^{(1)}(\phi) & 0 & \frac{\mathrm{i}}{2}\gamma a_2^{(1)}(\phi) & \frac{\mathrm{i}}{2}\gamma a_3^{(1)}(\phi) & 0 \end{pmatrix} \tag{93}$$

and $B^{(2)}(|\xi|,\phi)$ has entries on the diagonal, in the last row and last column. In order to apply a block-diagonalisation as $\epsilon \to 0$ we assume that the matrix $B^{(0)}(|\xi|,\phi)$ has as many distinct eigenvalues as possible. This is ensured if $\mu \neq \tau$, $\mu \neq \tau + \gamma^2$ and we can $(1,2,1,2,1)$-block-diagonalise this matrix family. Note, that due to the structure of the last rows and columns, the system decouples modulo ϵ^2 into a one-dimensional thermo-elastic system and one containing the elastic eigenvalues. The coupling comes only into play for the ϵ^3 entries.

Proposition 14 *Assume $\mu \neq \tau$, $\mu \neq \tau + \gamma^2$. Then the eigenvalues and eigenprojections of $B(|\xi|, \epsilon, \phi)$ have full asymptotic expansions as $\epsilon \to 0$. The main terms are given by*

$$\nu_0\big(|\xi|, \epsilon, \phi\big) = \check{\nu}_0\big(|\xi|\big) + |\xi|\mathcal{O}\big(\epsilon^3\big), \tag{94a}$$

$$\nu_1^{\pm}\big(|\xi|, \epsilon, \phi\big) = \check{\nu}_1^{\pm}\big(|\xi|\big) + |\xi|\mathcal{O}\big(\epsilon^3\big), \tag{94b}$$

$$\nu_{2/3}^{\pm_1,\pm_2}\big(|\xi|, \epsilon, \phi\big) = \pm_1\sqrt{\mu}|\xi| + \frac{C \pm_2 \sqrt{C^2 - (C^2 - D^2)\sin^2(2\phi)}}{4\sqrt{\mu}}|\xi|\epsilon^2 + |\xi|\mathcal{O}\big(\epsilon^3\big) \tag{94c}$$

where $\check{\nu}_0(|\xi|)$ and $\check{\nu}_1^{\pm}(|\xi|)$ are eigenvalues of the one-dimensional thermo-elastic system with parameter $\sqrt{\tau}$. The signs $\pm_1$ and $\pm_2$ are independent and the parameters C and D are given by (85).

Remark 9 Similar to Proposition 12 this statement is uniform in $|\xi|$. It will be of particular importance for us that the hyperbolic eigenvalues $\nu_{2/3}^{\pm}$ coincide up to second order in ϵ with the corresponding (roots of) eigenvalues of the elastic operator. This will be the key observation to transfer stationary phase estimates from elastic systems to the thermo-elastic ones.

17.3.4 Hexagonal Media

Finally we want to discuss the case of hexagonal media. The elastic operator defined by (4)–(5) is invariant under rotations around the x_3-axis. We will make use of this fact and reduce considerations to a two-dimensional situation corresponding to cross-sections of the Fresnel surface, see Fig. 17.5.

As already pointed out in Sect. 17.2.5.4 degenerate directions are $\pm(0, 0, 1)^\top$, which are uniplanar. They could be handled similarly to the cubic case, but rotational invariance makes estimates simpler. There are further circles of degenerate directions if

$$\tau_2 - 2\tau_1 \geq \lambda_2 + 2\mu. \tag{95}$$

We exploit rotational symmetry and consider the system only in the frequency hyperplane $\eta_1 = 0$. Then it is possible to express the eigenvectors $r_j(\eta)$ corresponding to eigenvalues $\varkappa_j(\eta)$ smoothly and the thermo-elastic system can be reformulated as system of first order in full analogy to the general treatment in Sect. 17.2. The previously derived asymptotic expansions for eigenvalues and the description of

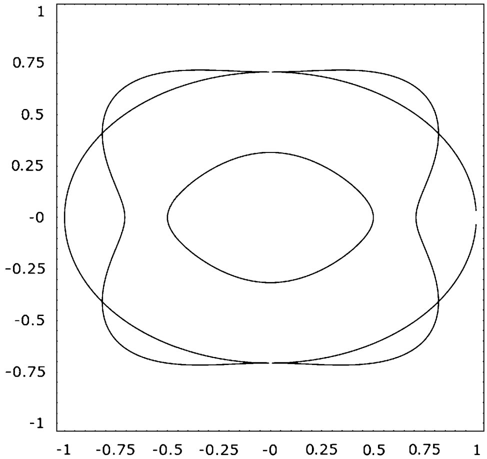

Fig. 17.5 Cut of the Fresnel surface for hexagonal media, $\eta_2 = 0$. The parameters are chosen as $\tau_1 = 4$, $\tau_2 = 10$, $\lambda_1 = 2$, $\lambda_2 = 4$, $\mu = 2$. The complete surface is obtained by rotation along the vertical axis

their behaviour in non-degenerate directions are valid. It remains to combine this information with an additional description near degenerate directions.

Away from the ξ_3-axis it is possible to find smooth families of eigenvectors $r_j(\eta)$ of $A(\eta)$. This follows directly from rotational invariance combined with one-dimensional perturbation theory of matrices, [8]. If we assume that the frequency support of initial data and therefore of the solution U, θ is conically separated from the uniplanar directions we can follow Sect. 17.2 and rewrite as first order system in $V(t, \xi)$ with coefficient matrix $B(\xi)$ given by (15) and of $(1, 1, 5)$-block structure. In what follows, we will ignore the scalar hyperbolic blocks and consider the remaining 5×5 matrix.

Based on the discussions from Sect. 17.2.5.4 we know that this 5×5 block is non-degenerate in the sense that its 1-homogeneous part has distinct eigenvalues if $(\tau_1 - \mu)(\tau_2 - \mu) \neq 0$. We assume this in the sequel. But this means that the theory of Sect. 17.2 is applicable and gives a full description of eigenvalues and eigenprojections and we are done.

Near the uniplanar directions, i.e., on the ξ_3-axis, we follow the same approach as for cubic media. We introduce polar co-ordinates around this direction and construct expressions for the corresponding asymptotics. There is one major simplification, due to rotational invariance the construction is independent of the angular variable.

17.4 Dispersive Estimates

We will show how to use the information obtained in Sects. 17.2 and 17.3 to derive L^p-L^q decay estimates for solutions to thermo-elastic systems. Some of the ideas we present are general in the sense that they can be applied to arbitrary space dimensions, however, our main focus will be the three-dimensional case and the examples considered in Sect. 17.3.

The estimates we have in mind are micro-localised to (a) non-degenerate parabolic, (b) non-degenerate hyperbolic or (c) degenerate directions. The first two situations generalise the consideration of [16, 23] taking also into account the estimates due to Sugimoto [21, 22], while the treatment of degenerate directions is inspired by the work of Liess [1, 9–12].

17.4.1 Non-degenerate Directions

We will consider two situations and micro-localise solutions to either open sets of parabolic directions or tubular neighbourhoods of compact parts of regular submanifolds of hyperbolic directions.

17.4.1.1 Estimates in Parabolic Directions and for Parabolic Modes

Let first $\psi, \tilde\psi \in C_0^\infty(\mathbb{S}^{n-1})$ be supported in $\mathcal{U}$ with $\tilde\psi = 1$ on $\operatorname{supp}\psi$ and $\chi \in C^\infty(\mathbb{R}_+)$ a cut-off function satisfying $\chi(s) = 0$ for $s \le \epsilon$ and $\chi(s) = 1$ for $s \ge 2\epsilon$. We extend both ψ and $\tilde\psi$ as 0-homogeneous functions to $\mathbb{R}^n$. Then we consider the solution to the first order system

$$\mathrm{D}_t V = B(D)V, \qquad V(0,\cdot) = \tilde\psi(D)V_0, \tag{96}$$

with data $V_0 \in \mathcal{S}(\mathbb{R}^n; \mathbb{C}^{2n+1})$. Note, that this is well-defined and $B(\xi)$ needs only to be defined on $\operatorname{supp}\tilde\psi$.

Lemma 2 (Parabolic estimate) *Assume that* $\operatorname{supp}\psi$ *is contained in the set of parabolic directions. Then the solutions to* (96) *satisfy the a-priori estimates*

$$\big\|\chi\big(|D|\big)\psi(D)V(t,\cdot)\big\|_q \lesssim \mathrm{e}^{-Ct}\|V_0\|_{p,r}, \tag{97a}$$

$$\big\|\big(1-\chi\big(|D|\big)\big)\psi(D)V(t,\cdot)\big\|_q \lesssim (1+t)^{-(n/2)(1/p-1/q)}\|V_0\|_p \tag{97b}$$

for all $1 \le p \le 2 \le q \le \infty$ *and with Sobolev regularity* $r > n(1/p - 1/q)$.

Proof (Sketch of proof) The proof of this estimate is straightforward from the two-dimensional situation considered in [16]. For small frequencies we write the solution V as sum

$$V(t,x) = \sum_{\nu(\xi)\in\operatorname{spec}B(\xi)} \mathrm{e}^{\mathrm{i}t\nu(\mathrm{D})} P_\nu(\mathrm{D})V_0, \tag{98}$$

P_ν the corresponding eigenprojections. We know that $\|P_\nu(\xi)\| \lesssim 1$ and $\operatorname{Im}\nu(\xi) \sim |\xi|^2$ by Proposition 5. Now each of the appearing terms can be estimated using the L^p-$L^{p'}$ boundedness of the Fourier transform (for $p \in [1,2]$) and Hölder inequality. Similarly, the representation (98) in combination with the bound $\operatorname{Im}\operatorname{spec}B(\xi) \ge$

C gives exponential decay of L^2 and H^s norms and this combined with Sobolev embedding yields the desired estimate.

For intermediate frequencies we may have to deal with multiplicities and resulting singularities of the spectral projections. Instead of (98) we use a spectral calculus representation which implies

$$\left|\hat{V}(t,\xi)\right| \le \mathrm{e}^{-Ct}\frac{1}{2\pi}\int_{\Gamma}\left\|\left(\zeta - B(\xi)\right)^{-1}\right\|\mathrm{d}\zeta \lesssim \mathrm{e}^{-Ct} \tag{99}$$

based on the compactness of the relevant set of frequencies ξ and the bound on the imaginary part due to Corollary 2/Proposition 7. Here, Γ is a smooth curve encircling the family of parabolic eigenvalues for the relevant ξ. □

If we consider hyperbolic directions we know that the parabolic eigenvalues are separated from the hyperbolic ones and we can use the spectral projection associated to the group of parabolic eigenvalues to separate them from the hyperbolic one(s). In this case the estimate of the above theorem is valid for the corresponding 'parabolic modes' of the solution. So we can restrict consideration to hyperbolic eigenvalues near hyperbolic directions.

17.4.1.2 Treatment of Non-degenerate Hyperbolic Directions

We consider *only* the for us interesting case when hyperbolic directions form part of a regular submanifold of $\mathbb{S}^{n-1}$ and coupling functions vanish to first order, i.e., we assume that the corresponding coupling function $a_j : \mathbb{S}^{n-1} \supset \mathcal{U} \to \mathbb{R}$ satisfies

$$\mathrm{d}a_j(\eta) \ne 0 \quad \text{when } a_j(\eta) = 0,\ \eta \in \mathcal{U}. \tag{100}$$

This implies that $M_j = \{\eta \in \mathcal{U} : a_j(\eta) = 0\}$ is regular of dimension $n-2$, the normal derivative $\partial_n a_j(\eta) \ne 0$ never vanishes and $a_j(\eta) \le \epsilon$ defines a tubular neighbourhood of M_j with a natural parameterisation. The desired dispersive estimate is related to geometric properties of the section $\mathcal{S}_{(M_j)}$ of the Fresnel surface lying directly over M_j,

$$\mathcal{S}_{(M_j)} = \left\{\omega_j^{-1}(\eta)\eta : \eta \in M_j\right\} = \mathcal{S}_j \cap \mathrm{co}M_j. \tag{101}$$

Here $\mathrm{co}\, M_j$ denotes the cone over M_j. For dimensions $n \ge 4$ we have to distinguish between different cases, depending on whether the cross-section $\mathcal{S}_{(M_j)}$ of the Fresnel surface satisfies a convexity assumption or not. By the latter we mean that any intersection of $\mathcal{S}_j$ with a hyperplane tangent to $\mathrm{co}\, M_j$ is convex in a neighbourhood of $\mathcal{S}_{(M_j)}$.

If this convexity assumption is satisfied (or if $n = 3$ and therefore $\dim M_j = 1$), we define the convex Sugimoto index of $\mathcal{S}_{(M_j)}$ as maximal order of contact of $\mathcal{S}_{(M_j)}$ with hyperplanes normal to $\mathrm{co}\, M_j$.

Theorem 1 (Hyperbolic estimate, convex case) *Assume that ψ is supported in a sufficiently small tubular neighbourhood of the regular hyperbolic submanifold M_j and that $\mathcal{S}_{(M_j)}$ satisfies the convexity assumption. Let further $\gamma_j = \gamma(\mathcal{S}_{(M_j)})$ be defined as above.*

Then the solutions to (96) *satisfy the a-priori estimate*

$$\left\| \psi(\mathrm{D}) P_{\nu_j}(\mathrm{D}) V(t,\cdot) \right\|_q \lesssim (1+t)^{-(1/2+(n-2)/\gamma_j)(1/p-1/q)} \|V_0\|_{p,r} \tag{102}$$

for all $p \in (1,2]$, $pq = p+q$ and with Sobolev regularity $r > n(1/p-1/q)$.

Proof First, we outline the strategy of the proof. We split variables in the tubular neighbourhood of the regular hyperbolic submanifold M_j, one coordinate being the defining function $a_j(\eta)$ and the other parameterising points on M_j. We have to combine a (simple) parabolic type estimate in normal directions taking care of the imaginary part of the phase with stationary phase estimates for the integration along M_j. The stationary phase estimate is done first and follows the lines of [21, 22] along with Sect. 5 in [20].

It is sufficient to show the estimate for $t \geq 1$. We follow the treatment of Brenner [3] and decompose the Fourier integral representing the corresponding hyperbolic modes of the solution V into dyadic pieces. For large and intermediate frequencies this amounts to estimate for all $k \in \mathbb{N}_0$

$$\mathcal{I}_k(t) = \sup_{z\in\mathbb{R}^n} \left| \int_{\tilde\eta=-\epsilon}^{\tilde\eta=\epsilon} \int_{\check\eta\in M_j} \int_{2^{k-1}\leq|\xi|\leq 2^{k+1}} \mathrm{e}^{\mathrm{i}t|\xi|(z\cdot\eta+|\xi|^{-1}\nu_j(\xi))} \right.$$
$$\left. \times\, p_j(\xi)\chi_k(\xi)|\xi|^{n-1-r}\,\mathrm{d}|\xi|\,\mathrm{d}\check\eta\,\mathrm{d}\tilde\eta \right| \tag{103}$$

with the notation $z = x/t$, $\xi = |\xi|\eta$, $\eta \simeq (\check\eta, \tilde\eta)$ with $\check\eta \in M_j$ and $\tilde\eta = a_j(\eta)$. The amplitude $p_j(\xi)$ arises from the spectral projector $P_{\nu_j}(\mathrm{D})$ and the phase $\nu_j(\xi)$ is complex-valued with $\operatorname{Im}\nu_j(\xi) \sim \tilde\eta^2$ uniform in $\xi \in \operatorname{supp}\chi_k$ and $k \in \mathbb{N}_0$.

If $z + \nabla_\xi \nu_j(\xi) \neq 0$, $\xi/|\xi| \in M_j$ or if z is not near a direction from M_j, the principle of non-stationary phase implies and gives a rapid decay. It suffices to restrict to z corresponding to stationary points. We use the method of stationary phase to estimate the integral over M_j, this can be done uniformly over ξ and $\tilde\eta$, provided ϵ is chosen small enough and yields an estimate of the form

$$\left| \int_{\check\eta\in M_j} \cdots \mathrm{d}\check\eta \right| \leq C t^{-(n-2)/\gamma_j} |\xi|^{n-1-r-(n-2)/\gamma_j} \mathrm{e}^{-c\tilde\eta^2 t} \tag{104}$$

uniform in k and $|\tilde\eta| \leq \epsilon$. In order to obtain this estimate we apply Ruzhansky's multi-dimensional van der Corput lemma, [17, 18], based on the uniformity of the Sugimoto index $\gamma(\mathcal{S}_j \cap \mathrm{co}\{\eta : a_j(\eta) = \tilde\eta, \eta \approx \check\eta\})$ for small $\tilde\eta$ and the uniform bounds on the appearing amplitude. Similar to [16] the imaginary part of the phase can be incorporated in the estimate for the amplitude. Integration over $\tilde\eta$ yields a further decay of $t^{-1/2}$, while integrating over ξ and using $|\xi| \sim 2^k$ yields

$$\mathcal{I}_k(t) \le C t^{-1/2-(n-2)/\gamma_j} 2^{k(n-r-(n-2)/\gamma_j)}. \tag{105}$$

Hence, we need $r \ge n - \frac{n-2}{\gamma_j}$ (compared to the elasticity or wave equation with $r \ge n - \frac{n-1}{\gamma}$) to apply Brenner's argument and obtain the desired estimate for the high frequency part. The required regularity follows from using Sobolev embedding for small t.

The treatment of small frequencies is somewhat simpler. We do not apply a dyadic decomposition, but still have to use a stationary phase argument along M_j combined with the behaviour of the imaginary part of the phase away from it,

$$\begin{aligned}\mathcal{I}(t) &= \sup_{z\in\mathbb{R}^n} \left| \int_{\tilde\eta=-\epsilon}^{\tilde\eta=\epsilon} \int_{\check\eta\in M_j} \int_{|\xi|\le 1} \mathrm{e}^{\mathrm{i}t|\xi|(z\cdot\eta+|\xi|^{-1}\nu_j(\xi))} p_j(\xi)\chi(\xi)|\xi|^{n-1}\mathrm{d}|\xi|\mathrm{d}\check\eta\mathrm{d}\tilde\eta \right| \\ &\le C t^{-(n-2)/\gamma_j} \int_{|\xi|\le 1} \int_{\tilde\eta=-\epsilon}^{\tilde\eta=\epsilon} \mathrm{e}^{-c\tilde\eta^2 t|\xi|^2} |\xi|\mathrm{d}\tilde\eta |\xi|^{n-2-(n-2)/\gamma_j} \mathrm{d}|\xi| \\ &\le C t^{-1/2-(n-2)/\gamma_j}. \end{aligned}$$

□

Without proof we comment on the non-convex situation. If the convexity assumption is violated we have to replace the convex Sugimoto index by a corresponding non-convex one, called $\gamma_0(\mathcal{S}_{(M_j)})$. The index $\gamma_0(\mathcal{S}_{(M_j)})$ is defined as the maximum over the minimal contact orders of $\mathcal{S}_{(M_j)}$ with hyperplanes normal to the cone co M_j, the maximum taken over all points of $\mathcal{S}_{(M_j)}$. The price we have to pay for non-convexity is a loss of decay.

Theorem 2 (Hyperbolic estimate, non-convex case) *Assume that ψ is supported in a sufficiently small tubular neighbourhood of the regular hyperbolic submanifold M_j and that $\mathcal{S}_{(M_j)}$ does not satisfy the convexity assumption. Let further $\widetilde{\gamma}_j = \gamma_0(\mathcal{S}_{(M_j)})$ be the non-convex Sugimoto index.*

Then the solutions to (96) satisfy the a-priori estimate

$$\left\| \psi(\mathrm{D})P_{\nu_j}(\mathrm{D})V(t,\cdot)\right\|_q \lesssim (1+t)^{-(1/2+1/\widetilde{\gamma}_j)(1/p-1/q)} \|V_0\|_{p,r} \tag{106}$$

for all $p \in (1,2]$, $pq = p+q$ and with Sobolev regularity $r > n(1/p-1/q)$.

17.4.1.3 Application to Cubic and Hexagonal Media

Because of its importance later on we remark that in our applications to three-dimensional thermo-elasticity the manifolds M_j are parts of circles on $\mathbb{S}^2$, i.e. can be seen as intersections of $\mathbb{S}^2$ with a cone. So we have to look at the corresponding sections of the Fresnel surface. In this case γ_j is just the maximal order of tangency between the curve $\mathcal{S}_{(M_j)}$and its tangent lines. If the curvature of this curve is nowhere vanishing, then $\gamma_j = 2$. Furthermore, algebraicity of $\mathcal{S}$ of order 6 implies that the highest order of contact is 6 and therefore $\gamma_j \in \{2, \dots, 6\}$ is the admissible range of these indices.

For cubic media there are two types of regular hyperbolic submanifolds. One is up to symmetry given by the circle $\eta_3 = 0$ on $\mathbb{S}^2$ and the corresponding eigenvalue is equal to μ. Thus the section of the Fresnel surface is just a circle and therefore its curvature is nowhere vanishing. Similarly, for intersections of the Fresnel surface with the plane $\eta_2 = \eta_3$ we obtain the hyperbolic eigenvalue $\varkappa = \eta_2^2(\tau - \lambda) + \eta_1^2\mu$. It is a simple calculation[3] to show that the curvature of the corresponding section of the Fresnel surface is nowhere vanishing as soon as $\lambda \neq \tau$ and $\mu \neq 0$. Hence, $\gamma_j = 2$ in both cases.

For hexagonal media regular hyperbolic submanifolds correspond to circles on the Fresnel surface. Again, $\gamma_j = 2$.

17.4.2 Cubic Media in 3D

We want to discuss the derivation for estimates near degenerate directions by the example of cubic media in three-dimensional space and combine them with the general estimates from Sect. 17.4.1.

17.4.2.1 Conic Points

The following statement resembles Theorem 1.5 in [10]. In Sect. 3 in [1] a stronger decay rate is obtained for some conic degenerations, but they require a sufficiently bent cone and we can not guarantee that in our case.

Theorem 3 (Conic degeneration) *Assume U_1, U_2 and θ_0 are micro-locally supported in a sufficiently small conical neighbourhood of a conically degenerate point on $\widehat{\mathbb{S}^2}$. Then the corresponding solution to the thermo-elastic system for cubic media satisfies the a-priori estimate*

$$\left\| \sqrt{A(\mathrm{D})}U(t,\cdot), U_t(t,\cdot), \theta(t,\cdot) \right\|_q \lesssim (1+t)^{-(1/2)(1/p-1/q)} \left\| \sqrt{A(\mathrm{D})}U_1, U_2, \theta_0 \right\|_{p,r} \tag{107}$$

for $p \in (1,2]$, $pq = p + q$ and $r > 3(1/p - 1/q)$.

Proof The main idea is that the proof of [9] uses polar co-ordinates around the singularities of the Fresnel surface similar to our treatment in Sect. 17.3. Stationary

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[3] Parametrising by the angle, the hyperbolic eigenvalue is given by $\varkappa(\phi) = \mu + \frac{\tau-\lambda-2\mu}{2}\sin^2\phi$ and it remains to check that $\partial_\phi^2\sqrt{\varkappa(\phi)} + \sqrt{\varkappa(\phi)} \neq 0$, see [23] for such a calculation.

phase arguments are applied in tangential direction and are uniform for small radii, while the final estimate follows after integration over the remaining variables.

It suffices to prove the statement for $t \geq 1$, the small time estimate is a direct consequence of Sobolev embedding theorem in combination with the obvious energy estimate. Similar to the hyperbolic estimate discussed before, we apply a dyadic decomposition of frequency space (localised to a small conic neighbourhood of the degenerate direction). The estimate for single dyadic components follows [9] resp. Theorem 1.5 in [10]; the only thing we have to check is that the necessary assumptions are satisfied uniform with respect to $|\xi|$ and $k \in \mathbb{N}$. We consider

$$\mathcal{I}_k(t) = \sup_{z\in\mathbb{R}^3} \bigg| \int_0^{\tilde{\epsilon}} \int_0^{2\pi} \int_{2^{k-1}\leq|\xi|\leq 2^{k+1}} \mathrm{e}^{\mathrm{i}t|\xi|(z\cdot\eta+|\xi|^{-1}\nu_j(|\xi|,\epsilon,\phi))} \times p_j\big(|\xi|,\epsilon,\phi\big)\chi_k\big(|\xi|\big)|\xi|^{2-r}\mathrm{d}|\xi|\mathrm{d}\phi\epsilon\mathrm{d}\epsilon \bigg|, \tag{108}$$

where $\eta \in \mathbb{S}^2$ denotes the point with polar co-ordinates (ϵ, ϕ) near the conic degenerate direction and $\xi = |\xi|\eta$. The amplitude $p_j(|\xi|, \epsilon, \phi)$ arises from the spectral projector (given in terms of the diagonaliser) constructed in the blown-up polar co-ordinates and $\chi_k(\xi)$ corresponds to the dyadic decomposition. The complex phase $\nu_j(|\xi|, \epsilon, \phi)$ is described in Proposition 12. Its imaginary part is non-negative and vanishes to second order in $\epsilon = 0$ as well as for three hyperbolic manifolds emanating from the conic degenerate point. Again we may treat this imaginary part as part of the amplitude and apply stationary phase estimates for the integral with respect to ϕ. As the approximation of the phase modulo $\mathcal{O}(\epsilon^2)$ is independent of ϕ and uniform in $|\xi|$ this yields

$$\bigg| \int_0^{2\pi} \cdots \mathrm{d}\phi \bigg| \lesssim t^{-1/2}|\xi|^{3/2-r}\epsilon^{1/2} \tag{109}$$

uniform in $|\xi|$, k and $0 \leq \epsilon \leq \tilde{\epsilon}$. There is no further benefit from the imaginary part (as there can not be a lower bound with respect to ϵ) and integrating with respect to $|\xi|$ and ϵ concludes the estimate for $\mathcal{I}_k(t)$. Similarly, we estimate the small frequency part

$$\begin{aligned}\mathcal{I}(t) = \sup_{z\in\mathbb{R}^3} \bigg| \int_0^{\tilde{\epsilon}} \int_0^{2\pi} \int_{|\xi|\leq 1} &\mathrm{e}^{\mathrm{i}t|\xi|(z\cdot\eta+|\xi|^{-1}\nu_j(|\xi|,\epsilon,\phi))} \\ &\times p_j\big(|\xi|,\epsilon,\phi\big)\chi\big(|\xi|\big)|\xi|^2\mathrm{d}|\xi|\mathrm{d}\phi\epsilon\mathrm{d}\epsilon \bigg| \\ \leq Ct^{-1/2},\end{aligned} \tag{110}$$

such that Brenner's method again yields the desired decay estimate. □

17.4.2.2 Uniplanar Points

The treatment of uniplanar degeneracies follows [10]. We have to make one further additional assumption related to the shape of certain curves on the Fresnel surface near the degenerate point. To be precise, we either require that

$$\varOmega \cap \mathcal{S} \cap \varPi \text{ has non-vanishing curvature} \tag{111}$$

for $\varOmega \subset \mathbb{R}^n$ an open neighbourhood of the uniplanarly degenerate point and for any plane $\varPi$ sufficiently close and parallel to the common tangent plane at the unode. This condition is equivalent to the technical assumption (1.12) made in [1]. If (111) is violated, we need to consider Sugimoto indices $\gamma_u = \gamma(\varOmega \cap \mathcal{S} \cap \varPi \subset \varPi)$, i.e., contact orders of these planar curves with its tangent planes combined with a uniformity assumption. Under assumption (111) the index is given by $\gamma_u = 2$.

For cubic media we have to use the statement of Proposition 13 to determine the index γ_u. Using the notation of (85), it suffices to calculate the indices of the indicator curves determined by

$$\epsilon^2\big(\mu + C \pm \sqrt{C^2\cos^2(2\phi) + D^2\sin^2(2\phi)}\big) = 1. \tag{112}$$

This yields

$$\gamma_u \in \{2, 3, 4\} \tag{113}$$

In the nearly isotropic case we have $\gamma_u = 2$, away from it $\gamma_u = 3$. Both are generic, while the borderline case with $\gamma_u = 4$ is not. The asymptotic construction of the eigenvalues and eigenprojections near the uniplanarly degenerate point of Proposition 14 yields that the assumption is satisfied uniformly for the phase functions appearing in all dyadic components of the operator.

Theorem 4 (Uniplanar degeneration) *Assume U_1, U_2 and θ_0 are micro-locally supported in a sufficiently small conical neighbourhood of a uniplanarly degenerate point on $\widehat{\mathbb{S}^2}$. Let further γ_u be the index of the uniplanar point. Then the corresponding solution to the thermo-elastic system for cubic media satisfies the a-priori estimate*

$$\begin{aligned} &\big\|\sqrt{A(\mathrm{D})}U(t,\cdot), U_t(t,\cdot), \theta(t,\cdot)\big\|_q \\ &\quad\lesssim (1+t)^{-(1/2+1/\gamma_u)(1/p-1/q)}\big\|\sqrt{A(\mathrm{D})}U_1, U_2, \theta_0\big\|_{p,r} \end{aligned} \tag{114}$$

for $p \in (1, 2]$, $pq = p + q$ and $r > 3(1/p - 1/q)$.

Proof (Sketch of proof) We will sketch the major differences to the treatment of conic degeneracies. We will again use polar co-ordinates and estimate corresponding dyadic components (108), where now $\nu_j(|\xi|, \epsilon, \phi)$ is determined by Proposition 14. The imaginary part of $\nu_j(|\xi|, \epsilon, \phi)$ vanishes to third order and is of no benefit, while the real part coincides to third order with the corresponding elastic

Table 17.1 Contributions to the dispersive decay rate for cubic media

	Small frequencies	Large frequencies
Parabolic directions	$(1+t)^{-3/2}$	e^{-Ct}
Hyperbolic directions	$(1+t)^{-1}$	$(1+t)^{-1}$
Conic degeneracies	$(1+t)^{-1/2}$	$(1+t)^{-1/2}$
Uniplanar degeneracies	$(1+t)^{-1/2-1/\gamma}$ $\gamma \in \{2,3,4\}$	$(1+t)^{-1/2-1/\gamma}$ $\gamma \in \{2,3,4\}$

eigenvalue. This allows to use estimates from [10] and Sect. 4 in [1], the main difference to the previous situation is that we now use stationary phase estimates for both, the angular and the radial integral. The proof itself then coincides with the corresponding proof for cubic elasticity, cf. [12].

Using a change of variables the integral is written in the new variables $\omega_j(\eta)|\xi|$ (i.e., roughly $\operatorname{Re}\nu_j$) and $\eta/\omega_j(\eta) \in \mathcal{S}_j$. In this form the phase splits and the crucial estimate is just a Fourier transform of a density carried by the sheet of the Fresnel surface (with possible singularity in the unode). This is calculated by introducing *distorted* polar co-ordinates on the surface. As level sets we use cuts of the surface by planes parallel to the common tangent plane. Then we will at first apply the method of stationary phase to the radial variable in these co-ordinates. These stationary points are non-degenerate and we use the obtained first terms in the asymptotics for a second stationary phase argument in the angular variables. The condition (111) would imply again that stationary points are non-degenerate and we are done, while if (111) is violated we use the Lemma of van der Corput instead to prove the estimate. □

17.4.2.3 Collecting the Estimates

It remains to collect all the estimates into a final statement for cubic media. Parabolic directions are treated by Lemma 2; hyperbolic manifolds away from degenerate points are covered by Theorem 1. The remaining 24 degenerate directions fall into either of the previously discussed categories and estimates follow from Theorem 3 and 4. The resulting estimates are collected in Table 17.1.

Corollary 3 (Cubic decay rates) *Cubic media in three space dimensions satisfy the dispersive type estimate*

$$\begin{aligned} &\left\|\sqrt{A(\mathrm{D})}U(t,\cdot), U_t(t,\cdot), \theta(t,\cdot)\right\|_{L^q(\mathbb{R}^n)} \\ &\quad\lesssim (1+t)^{-(1/2)(1/p-1/q)}\left\|\sqrt{A(\mathrm{D})}U_1, U_2, \theta_0\right\|_{p,r} \end{aligned} \tag{115}$$

for all data $U_1 \in W^{p,r+1}(\mathbb{R}^3;\mathbb{C}^3)$, $U_2 \in W^{p,r}(\mathbb{R}^3;\mathbb{C}^3)$ *and* $\theta \in W^{p,r}(\mathbb{R}^3)$, *provided* $p \in (1,2]$, $pq = p+q$ *and* $r > 3(1/p-1/q)$.

Table 17.2 Contributions to the dispersive decay rate for hexagonal media

	Small frequencies	Large frequencies
Genuine hyperbolic mode	$(1+t)^{-1}$	$(1+t)^{-1}$
Parabolic modes	$(1+t)^{-3/2}$	e^{-Ct}
Hyperbolic directions	$(1+t)^{-1}$	$(1+t)^{-1}$
Uniplanar degeneracies	$(1+t)^{-1}$	$(1+t)^{-1}$

Decay rates improve if the Fourier transform of the initial data vanishes at the conically degenerate directions. This could be achieved by posing particular symmetry conditions.

17.4.3 Hexagonal Media

The treatment of hexagonal media is somewhat simpler. The uniplanar degenerations trivially satisfy the assumption (111) and therefore yield the decay rates specified by the above theorem. The additionally appearing manifolds of degenerate directions are easily treated as there are smooth families of eigenprojections associated to both eigenvalues (as we stay away from the uniplanar points) and we can therefore treat the modes separately.

One of them is hyperbolic for all directions, we refer to it as the genuine hyperbolic mode. The sheet of the Fresnel surface corresponding to this mode, i.e., to the eigenvalue $\varkappa(\eta) = \frac{\tau_1 - \lambda_1}{2}(\eta_1^2 + \eta_2^2) + \mu\eta_3^2$ is easily seen to be strictly convex for all choices of the parameter and gives a decay order of t^{-1}. The proof is similar to that for the wave equation, see [3].

The parabolic modes away from the degenerate hyperbolic directions are treated as before, while the remaining degenerate hyperbolic manifold is treated by the estimate of Theorem 1 with $\gamma = 2$ due to rotational invariance. The resulting estimates are collected in Table 17.2.

Corollary 4 (Hexagonal decay rates) *Hexagonal media in three space dimensions satisfy the dispersive type estimate*

$$\begin{aligned} &\left\| \sqrt{A(\mathrm{D})}U(t,\cdot), U_t(t,\cdot), \theta(t,\cdot) \right\|_{L^q(\mathbb{R}^n)} \\ &\quad \lesssim (1+t)^{-(1/p-1/q)} \left\| \sqrt{A(\mathrm{D})}U_1, U_2, \theta_0 \right\|_{p,r} \end{aligned} \tag{116}$$

for all data $U_1 \in W^{p,r+1}(\mathbb{R}^3; \mathbb{C}^3)$, $U_2 \in W^{p,r}(\mathbb{R}^3; \mathbb{C}^3)$ *and* $\theta \in W^{p,r}(\mathbb{R}^3)$, *provided* $p \in (1, 2]$, $pq = p + q$ *and* $r > 3(1/p - 1/q)$.

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Chapter 18
Global Solutions of Semilinear System of Klein-Gordon Equations in de Sitter Spacetime

Karen Yagdjian

Abstract In this article we prove global existence of small data solutions of the Cauchy problem for a system of semilinear Klein-Gordon equations in the de Sitter spacetime. The mass matrix is assumed to be diagonalizable with positive eigenvalues. The existence is proved under the assumption that the eigenvalues are outside of some open bounded interval.

Mathematics Subject Classification Primary 35L52 · 35L71 · Secondary 81T20 · 35C15

18.1 Introduction and Statement of Results

In this article we prove global existence of small data solutions of the Cauchy problem for the semilinear system of Klein-Gordon equations in the de Sitter spacetime. Unlike the same problem in the Minkowski spacetime, we have no restriction on the order of nonlinearity and structure of the nonlinear term, provided that the spectrum of the mass matrix of the fields is in the positive half-line and has no intersection with some open bounded interval.

A large amount of work has been devoted to the Cauchy problem for the scalar semilinear Klein-Gordon equation in the Minkowski spacetime. The existence of global weak solutions has been obtained by Jörgens [18], Segal [26, 27], Pecher [22], Brenner [6], Strauss [28], Ginibre and Velo [12, 13] for the equation

$$u_{tt} - \Delta u + m^2 u = |u|^\alpha u.$$

For global solvability, the exact relation between n and $\alpha > 0$ was finally established. More precisely, consider the Cauchy problem for the nonlinear Klein-Gordon equation

$$u_{tt} - \Delta u = -V'(u),$$

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where Δ is the Laplace operator in R^n and $V' = V'(u)$ is a nonlinear function, a typical form of which is the sum of two powers

$$V'(u) = \lambda_0 u + \lambda |u|^\alpha u$$

with $\alpha \geq 0$ and $\lambda \geq 0$. For this equation, a conservation of energy is valid. For finite energy solutions scaling arguments suggest the assumption $\alpha < 4/(n-1)$. In [13], by a contraction method, the existence and uniqueness of strong global solutions in the energy space $H_{(1)} \oplus L^2$ are proved for arbitrary space dimension n under assumptions on V' that cover the case of sum of powers $\lambda|u|^\alpha u$ with $0 \leq \alpha < 4/(n-1)$, $n \geq 2$, and $\lambda > 0$ for the highest α. Some of the results can be extended to the case $\alpha = 4/(n-1)$ (see, e.g. [12], Sect. 4 in [13], Sect. 142 in [29]). Nakamura and Ozawa studied in [20] the global well-posedness in the Sobolev space $H_{(s)}$ with $s \geq n/2$ for the Cauchy problem for semilinear Klein-Gordon equations with a nonlinearity, which behaves as a power $|u|^{1+4/n}$ near zero, and has at infinity an arbitrary growth rate.

The Klein-Gordon equation arising in relativistic physics and, in particular, general relativity and cosmology, as well as, in more recent quantum field theories, is a covariant equation that is considered in curved pseudo-Riemannian manifolds. (See, e.g., Birrell and Davies [5], Parker and Toms [21], Weinberg [31].) Moreover, the latest astronomical observational discovery that the expansion of the universe is speeding supports the model of the expanding universe that is mathematically described by a manifold with metric tensor depending on time and spatial variables. In this paper we restrict ourselves to a manifold arising in the so-called de Sitter model of the universe, which is a curved manifold due to the cosmological constant. Thus, there is a need to study partial differential equations related to such models and, in particular, to investigate the question of the global solvability of the semilinear hyperbolic equations with variable coefficients. The lack of results for the global solvability of such semilinear hyperbolic equations can be explained, among other reasons, by the fact that there are only very few known examples of linearized equations with explicit formulas for the fundamental solutions.

The line element in the de Sitter spacetime has the form

$$ds^2 = -\left(1 - \frac{r^2}{R^2}\right)c^2 dt^2 + \left(1 - \frac{r^2}{R^2}\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta \, d\phi^2\right).$$

The Lamaître-Robertson transformation leads to the following form for the line element (Sect. 134 in [19], Sect. 142 in [29]):

$$ds^2 = -c^2 dt'^2 + e^{2ct'/R}\left(dx'^2 + dy'^2 + dz'^2\right).$$

Here R is the "radius" of the universe. In fact, the de Sitter model belongs to the family of the Friedmann-Lemaître-Robertson-Walker spacetimes (FLRW spacetimes). In the FLRW spacetime [14], one can choose coordinates so that the metric has the form $ds^2 = -dt^2 + S^2(t)d\sigma^2$. In particular, the metric in the de Sitter spacetime in the Lamaître-Robertson coordinates [19] has this form with the cosmic scale factor $S(t) = e^t$.

The homogeneous and isotropic cosmological models possess the highest degree of symmetry that makes them more amenable to rigorous study. Among them we mention FLRW models. The simplest class of cosmological models can be obtained if we assume, additionally, that the metric of the slices of constant time is flat and that the spacetime metric can be written in the form $ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2)$ with an appropriate scale factor $a(t)$. The assumption that the universe is expanding leads to the positivity of the time derivative $\frac{d}{dt}a(t)$. A further assumption that the universe obeys the accelerated expansion suggests that the second derivative $\frac{d^2}{dt^2}a(t)$ is positive. Under the assumption of the FLRW symmetry the equation of motion in the case of a positive cosmological constant Λ leads to the solution $a(t) = a(0)e^{t\sqrt{\Lambda/3}}$, which produces models with exponentially accelerated expansion, which is referred to as the *de Sitter model*.

In general the matter fields described by the function ϕ must satisfy equations of motion and, in the case of the massive scalar field, the equation of motion is that ϕ should satisfy the Klein-Gordon equation generated by the metric g,

$$\frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x^i}\left(\sqrt{|g|}g^{ik}\frac{\partial\psi}{\partial x^k}\right) = m^2\psi + V'(\psi).$$

In physical terms this equation describes a local self-interaction for a scalar particle. In the de Sitter universe the equation for the scalar field with mass m and potential function V written out explicitly in coordinates is (see, e.g., Sect. 5.4 in [11] and [24].)

$$\phi_{tt} + nH\phi_t - e^{-2Ht} \bigtriangleup \phi + m^2\phi = -V'(\phi). \tag{1}$$

Here $x \in R^n$, $t \in R$, and $\bigtriangleup$ is the Laplace operator on the flat metric, $\bigtriangleup := \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}$, while $H = \sqrt{\Lambda/3}$ is the Hubble constant. For the sake of simplicity, from now on, we set $H = 1$. A typical example of a potential function would be $V(\phi) = \phi^4$.

In this paper we consider the model of interacting fields, which can be described by the system of Klein-Gordon equations with different masses, containing interaction via mass matrix and the semilinear term. The model obeys the following system

$$\Phi_{tt} + n\Phi_t - e^{-2t} \bigtriangleup \Phi + \mathbf{M}\Phi = F(\Phi). \tag{2}$$

Here F is a vector-valued function of the vector-valued function Φ. We assume that the mass matrix $\mathbf{M}$ is real-valued, diagonalizable, and it has eigenvalues $m_1^2, \dots, m_l^2$, $i = 1, 2, \dots, l$. By the similarity transformation with the real-valued matrix $\mathbf{O}$ (the diagonalizer of $\mathbf{M}$), the mass matrix can be diagonalized, therefore, we use the change of the unknown function as follows:

$$\Psi = e^{(n/2)t}\mathbf{O}\Phi, \quad \Phi = e^{-(n/2)t}\mathbf{O}^{-1}\Psi.$$

Then the system (2) of the semilinear Klein-Gordon equations for Ψ in the de Sitter spacetime takes the form

$$\Psi_{tt} - e^{-2t} \bigtriangleup \Psi + \mathcal{M}^2\Psi = e^{(n/2)t}\mathbf{O}F\left(e^{-(n/2)t}\mathbf{O}^{-1}\Psi\right), \tag{3}$$

where the diagonal matrix $\mathcal{M}$, with nonnegative real part $\Re\mathcal{M} \geq 0$, is

$$\mathcal{M}^2 := \mathbf{O}\mathbf{M}\mathbf{O}^{-1} - \frac{n^2}{4}\mathbf{I}, \quad \mathbf{I} \text{ is the identity matrix.}$$

The matrix $\mathcal{M}^2$ will be called the *curved mass matrix* of the particles, which is also sometimes referred to as the *effective mass matrix*.

It is convenient to use the diagonal matrix $M = \operatorname{diag}(|m_i^2 - \frac{n^2}{4}|^{1/2})$. We distinguish the following three cases: the case of large mass matrix $\mathbf{M}$ that is $\mathcal{M}^2 \geq 0$ ($m_i^2 \geq \frac{n^2}{4}$, $i = 1, 2, \ldots, l$); the case of dimensional mass matrix $\mathbf{M}$ that is $\mathcal{M}^2 = 0$ ($m_i^2 = \frac{n^2}{4}$, $i = 1, 2, \ldots, l$); and the case of small mass matrix $\mathbf{M}$ that is $\mathcal{M}^2 < 0$ ($m_i^2 < \frac{n^2}{4}$, $i = 1, 2, \ldots, l$). They lead to three different equations: the Klein-Gordon equation with the *real curved mass matrix* $\mathcal{M}$,

$$\Psi_{tt} - e^{-2t} \bigtriangleup \Psi + M^2\Psi = e^{(n/2)t}\mathbf{O}F\big(e^{-(n/2)t}\mathbf{O}^{-1}\Psi\big);$$

the wave equation with the *zero curved mass matrix*

$$\Psi_{tt} - e^{-2t} \bigtriangleup \Psi = e^{(n/2)t}\mathbf{O}F\big(e^{-(n/2)t}\mathbf{O}^{-1}\Psi\big);$$

and the Klein-Gordon equation with the *imaginary curved mass matrix* $\mathcal{M}$,

$$\Psi_{tt} - e^{-2t} \bigtriangleup \Psi - M^2\Psi = e^{(n/2)t}\mathbf{O}F\big(e^{-(n/2)t}\mathbf{O}^{-1}\Psi\big).$$

We also call the mass matrix $\mathbf{M}$ *critical* if $\mathcal{M}^2 = -\frac{1}{4}I$.

Let $W^{l,p}(R^n)$ be the Sobolev space and $H_{(s)}(R^n) = W^{s,2}(R^n)$. We use the notation $\|\cdot\|_{H_{(s)}(R^n)}$ for both the norm of vector valued function and for the norm of its components. To estimate the nonlinear term $F = F(\Phi)$ we use the following Lipschitz condition:

Condition ($\mathcal{L}$) *The function F is said to be Lipschitz continuous in the space $H_{(s)}(R^n)$ with the norm $\|\cdot\|_{H_{(s)}(R^n)}$ if there is a constant $\alpha \geq 0$ such that*

$$\big\|F(\Phi_1) - F(\Phi_2)\big\|_{H_{(s)}(R^n)} \leq C\|\Phi_1 - \Phi_2\|_{H_{(s)}(R^n)}\big(\|\Phi_1\|^{\alpha}_{H_{(s)}(R^n)} + \|\Phi_2\|^{\alpha}_{H_{(s)}(R^n)}\big) \tag{4}$$

for all $\Phi_1, \Phi_2 \in H_{(s)}(R^n)$.

Define the complete metric space

$$X(R, s, \gamma) := \Big\{\Phi \in C\big([0,\infty); H_{(s)}\big(R^n\big)\big) \,|\, \|\Phi\|_X := \sup_{t\in[0,\infty)} e^{\gamma t}\big\|\Phi(x,t)\big\|_{H_{(s)}(R^n)} \leq R\Big\}$$

with the metric

$$d(\Phi_1, \Phi_2) := \sup_{t\in[0,\infty)} e^{\gamma t} \left\| \Phi_1(x,t) - \Phi_2(x,t) \right\|_{H_{(s)}(R^n)}.$$

The first result of the present paper is the following theorem.

Theorem 1 *Assume that the nonlinear term $F(\Phi)$ is Lipschitz continuous in the space $H_{(s)}(R^n)$, $s > n/2 \geq 1$, $\alpha > 0$, and $F(0) = 0$. Assume also that the system has a large mass matrix. Then, there exists $\varepsilon_0 > 0$ such that, for every given vector-valued functions $\varphi_0, \varphi_1 \in H_{(s)}(R^n)$, such that*

$$\|\varphi_0\|_{H_{(s)}(R^n)} + \|\varphi_1\|_{H_{(s)}(R^n)} \leq \varepsilon, \quad \varepsilon < \varepsilon_0,$$

there exists a global solution $\Phi \in C^1([0,\infty); H_{(s)}(R^n))$ of the Cauchy problem

$$\Phi_{tt} + n\Phi_t - e^{-2t}\Delta\Phi + \mathbf{M}\Phi = F(\Phi), \tag{5}$$

$$\Phi(x,0) = \varphi_0(x), \qquad \Phi_t(x,0) = \varphi_1(x). \tag{6}$$

The solution $\Phi(x,t)$ belongs to the space $X(2\varepsilon, s, 0)$, that is,

$$\sup_{t\in[0,\infty)} \left\| \Phi(x,t) \right\|_{H_{(s)}(R^n)} < 2\varepsilon.$$

For the scalar equation this result implies Theorem 0.1 in [35]. In fact, for the scalar equation if

$$F(\Phi) = \pm|\Phi|^{\alpha}\Phi \quad \text{or} \quad F(\Phi) = \pm|\Phi|^{\alpha+1},$$

then, according to Theorem 0.1 in [35], the small data Cauchy problem is globally solvable for every $\alpha \in (0,\infty)$ if $m \in (0, \sqrt{n^2-1}/2) \cup [n/2, \infty)$ and Condition $(\mathcal{L})$ is fulfilled. It is conjectured in [35] that $(\sqrt{n^2-1}/2, n/2)$ is a forbidden physical mass interval for the small data global solvability of the Cauchy problem for all $\alpha \in (0,\infty)$.

Consider the particular case of the scalar equation with the spatial dimension $n = 3$. In this case the interval $(\sqrt{n^2-1}/2, n/2)$ for the physical mass is reduced to $(\sqrt{2}, 3/2)$, which corresponds to the interval $(0, 1/2)$ for the curved mass. For the physical mass in the physical variables the interval $(0, \sqrt{2})$ implies $0 < m^2 < 2H^2h^2/c^4 = 2\Lambda/3$, which means for the curved mass $M = \sqrt{9-4m^2}/2$ the interval $1/2 < M < 3/2$. It turns out that the interval $(0, \sqrt{2})$ (with the right endpoint in the physical variables $2\Lambda/3$) plays a significant role in the linear quantum field theory [16], in a completely different context than the explicit representation of the solutions of the Cauchy problem. More precisely, it is the so-called Higuchi bound [1, 9, 16], which arises in the quantization of free massive fields with spin 2 in the de Sitter spacetime. It is the forbidden mass range for spin-2 field theory in de Sitter spacetime because of the appearance of negative norm states. Thus, the point $m = \sqrt{2}$ is exceptional for the quantum fields theory.

This is why in the present paper we pay special attention to the system of equations with the mass matrix $\mathbf{M} = \frac{n^2-1}{4}\mathbf{I}$. We call such mass matrix $\mathbf{M}$ critical, while the square root of its eigenvalue will be called a critical mass. Thus, for the critical matrix $\mathbf{M}$ the curved mass matrix is $\mathcal{M}^2 = -\frac{1}{4}\mathbf{I}$. Then, we prove in Sect. 18.3.1 (see Theorem 3 and Corollary 1) that for all n the endpoint $m = \sqrt{n^2-1}/2$ (the critical mass) of the forbidden mass interval, is the only value of the eigenvalues of the physical mass matrix $\mathbf{M}$, such that the solutions of the linear system of equations

$$\Phi_{tt} + n\Phi_t - e^{-2t}\Delta\Phi + \mathbf{M}\Phi = 0$$

obey the strong Huygens' Principle, whenever the wave equation in the Minkowski spacetime does. Moreover, in Sect. 18.3.1 we give also a complete asymptotic expansion for the solution of the scalar linear equation without source. Unlike to the result by Vasy [30] it does not have the logarithmic term.

We also call the mass matrix $\mathbf{M}$ *semi-critical mass matrix* if the spectrum $\sigma(\mathbf{M})$ of the mass matrix $\mathbf{M}$ is a subset of $(0, (n^2-1)/4]$. For the system with the semi-critical mass matrix $\mathbf{M}$ we prove the following global existence theorem, which is new in the critical case even for the scalar equation.

Theorem 2 *Assume that the nonlinear term $F(\Phi)$ is Lipschitz continuous in the space $H_{(s)}(R^n)$, $s > n/2 \geq 1$, $\alpha > 0$, and $F(0) = 0$. Assume also that the mass matrix $\mathbf{M}$ is semi-critical, that is $\sigma(\mathbf{M}) \subset (0, (n^2-1)/4]$.*

Then, there exists $\varepsilon_0 > 0$ such that, for every given vector-valued functions $\varphi_0, \varphi_1 \in H_{(s)}(R^n)$, such that

$$\|\varphi_0\|_{H_{(s)}(R^n)} + \|\varphi_1\|_{H_{(s)}(R^n)} \leq \varepsilon, \quad \varepsilon < \varepsilon_0,$$

there exists a global solution $\Phi \in C^1([0,\infty); H_{(s)}(R^n))$ of the Cauchy problem

$$\Phi_{tt} + n\Phi_t - e^{-2t}\Delta\Phi + \mathbf{M}\Phi = F(\Phi),$$
$$\Phi(x,0) = \varphi_0(x), \qquad \Phi_t(x,0) = \varphi_1(x).$$

The solution $\Phi(x,t)$ belongs to the space $X(2\varepsilon, s, \gamma)$, where

$$\gamma < \frac{1}{\alpha+1}\left(\frac{n}{2} - \max\left\{\sqrt{\frac{n^2}{4} - \lambda}; \lambda \in \sigma(\mathbf{M})\right\}\right),$$

that is,

$$\sup_{t\in[0,\infty)} e^{\gamma t}\left\|\Phi(x,t)\right\|_{H_{(s)}(R^n)} < 2\varepsilon.$$

We note here that, due to the dependence of the coefficient on time, there is no conservation of energy, and that, for the general nonlinearity $F(\Phi)$, the decay of the energy cannot be established although the equation contains a dissipative term. We also note that the evident combinations of Theorems 1–2 give some generalizations, which we do not formulate here.

Baskin [3] discussed small data global solutions for the scalar Klein-Gordon equation on asymptotically de Sitter spaces, which is a compact manifold with boundary. More precisely, in [3] the Cauchy problem is considered for the semilinear equation

$$\Box_g u + m^2 u = f(u), \quad u(x,t_0) = \varphi_0(x) \in H_{(1)}\big(R^n\big), u_t(x,t_0) = \varphi_1(x) \in L^2\big(R^n\big),$$

where mass is large, $m^2 > n^2/4$, F is a smooth function and satisfies conditions $|f(u)| \le c|u|^{\alpha+1}$, $|u| \cdot |f'(u)| \sim |f(u)|$, $f(u) - f'(u) \cdot u \le 0$, $\int_0^u f(v)dv \ge 0$, and $\int_0^u f(v)dv \sim |u|^{\alpha+2}$ for large $|u|$. It is also assumed that $\alpha = \frac{4}{n-1}$. In Theorem 1.3 in [3] the existence of the global solution for small energy data is stated. (For more references on the asymptotically de Sitter spaces, see the bibliography in [2, 30].)

D'Ancona [7] considered the Cauchy problem for the equation

$$u_{tt} - a(t)\Delta u = -f(u), \quad t \in [0,T], x \in R^3,$$

with the nonnegative real-analytic function $a(t)$, which has a locally finite number of zeros and those zeros are of finite order only. It was supposed in [7] that the nonlinear term obeys conditions $f(u)u \ge 0$, $f(0) = 0$,

$$\big|f(s)\big|^{1+1/p} \le c\left(1 + \int_0^s f(\sigma)d\sigma\right), \qquad \big|f'(s)\big| \le c\big(1+|s|\big)^{p-1},$$
$$f^{(3)}(s)f'(s) + \beta\big(f^{(2)}(s)\big)^2 \ge 0,$$

with some $p \ge 1$ and $\beta < 1$. Then, assuming that the possible zeros of $a(t)$ are of order not greater than 2λ, $\lambda = 1, 2, \dots$, the existence of the solution $u \in C^\infty([0,T] \times R^3)$ without restriction on the size of the initial data is proved, provided that $p < (3\lambda + 5)/(3\lambda + 1)$. In [8] this result was extended to the case of 1 and 2 spatial dimensions.

The remaining part of this paper is organized as follows. In Sect. 18.2 we give integral representations for the solutions of the Cauchy problem for the linear equation with large physical mass. Then, we quote from [32], the L^p-L^q estimates for the solutions of that equation with and without a source term. In Sect. 18.3 we introduce similar representations for the cases of small real mass and of the imaginary mass. These representations are used in the Sects. 18.3.2–18.3.3 for the derivation of asymptotic expansions and the L^p-L^q estimates for the linear equation with and without a source term. The last section, Sect. 18.4, is devoted to the solvability of the associated integral equation and to the proof of Theorems 1–2.

18.2 The Scalar Equation. Case of Large Mass

Scalar fields play a fundamental role in the standard model of particle physics, as well as, its possible extensions. In particular, scalar fields generate spontaneous symmetry breaking and provide masses to gauge bosons and chiral fermions by the Brout-Englert-Higgs mechanism [10] using a Higgs-type potential [15].

The nonlinear equations (1) and (2) are those we would like to solve, but the linear problem is a natural first step. The exceptionally efficient tools for studying nonlinear equations are the fundamental solution of the associated linear operator and explicit representation formulas for the solutions of the linear equation. We extract a linear part of the system (3) as an initial model that must be treated first. That linear system is diagonal, which allows us to restrict ourselves to one scalar equation

$$u_{tt} - e^{-2t} \bigtriangleup u + \mathcal{M}^2 u = f, \tag{7}$$

where $\mathcal{M}$ is a non-negative number throughout this section. In this section we list the explicit formulas for the solution of the Cauchy problem for (7).

Equation (7) is strictly hyperbolic. That implies the well-posedness of the Cauchy problem for (7) in several function spaces. The coefficient of the equation is an analytic function and, consequently, Holmgren's theorem implies local uniqueness in the space of distributions. Moreover, the speed of propagation is finite, namely, it is equal to e^{-t} for every $t \in R$. The second-order strictly hyperbolic equation (7) possesses two fundamental solutions resolving the Cauchy problem. They can be written microlocally in terms of Fourier integral operators [17], which give a complete description of the wave front sets of the solutions. The distance between two characteristic roots $\lambda_1(t,\xi)$ and $\lambda_2(t,\xi)$ of (7) is $|\lambda_1(t,\xi) - \lambda_2(t,\xi)| = e^{-t}|\xi|$, $t \in R$, $\xi \in R^n$. It tends to zero as t approaches infinity. Thus, the operator is not uniformly strictly hyperbolic. Moreover, the finite integrability of the characteristic roots, $\int_0^\infty |\lambda_i(t,\xi)|dt < \infty$, leads to the existence of a so-called *horizon* for that equation. More precisely, any signal emitted from the spatial point $x_0 \in R^n$ at time $t_0 \in R$ remains inside the ball $|x - x_0| < e^{-t_0}$ for all time $t \in (t_0, \infty)$. Equation (7) is neither Lorentz invariant nor invariant with respect to usual scaling and that brings additional difficulties.

In this section we introduce some necessary notations, definitions, formulas, and results from [32], where the case of the large mass, that is, $m^2 \geq n^2/4$, is discussed. First, we define the *forward light cone* $D_+(x_0,t_0)$, $x_0 \in R^n$, $t_0 \in R$, and the *backward light cone* $D_-(x_0,t_0)$, $x_0 \in R^n$, $t_0 \in R$, as follows:

$$D_\pm(x_0,t_0) := \left\{(x,t) \in R^{n+1};\, |x - x_0| \leq \pm\left(e^{-t_0} - e^{-t}\right)\right\}.$$

In fact, any intersection of $D_-(x_0,t_0)$ with the hyperplane $t = const < t_0$ determines the so-called dependence domain for the point (x_0,t_0), while the intersection of $D_+(x_0,t_0)$ with the hyperplane $t = const > t_0$ is the so-called domain of influence of the point (x_0,t_0). Equation (7) is non-invariant with respect to time inversion. Moreover, the dependence domain is wider than any given ball if time $const > t_0$ is sufficiently large, while the domain of influence is permanently, for all time $const < t_0$, in the ball of the radius e^{t_0}.

Define for $t_0 \in R$ in the domain $D_+(x_0,t_0) \cup D_-(x_0,t_0)$ the function

$$\begin{aligned} E(x,t;x_0,t_0) &= \left(4e^{-t_0-t}\right)^{iM} \left(\left(e^{-t} + e^{-t_0}\right)^2 - (x-x_0)^2\right)^{-1/2-iM} \\ &\quad \times F\left(\frac{1}{2} + iM, \frac{1}{2} + iM; 1; \frac{(e^{-t_0} - e^{-t})^2 - (x-x_0)^2}{(e^{-t_0} + e^{-t})^2 - (x-x_0)^2}\right), \end{aligned} \tag{8}$$

where $F(a,b;c;\zeta)$ is the hypergeometric function. (For the definition of $F(a,b;c;\zeta)$ see, e.g., [4].) Here the notation $(x-x_0)^2=(x-x_0)\cdot(x-x_0)$ for the points $x,x_0\in R^n$ has been used. The kernels $K_0(z,t)$ and $K_1(z,t)$ are defined by

$$\begin{aligned}
K_0(z,t) &:= -\left[\frac{\partial}{\partial b}E(z,t;0,b)\right]_{b=0} \\
&= \big(4e^{-t}\big)^{iM}\big(\big(1+e^{-t}\big)^2-z^2\big)^{-iM}\frac{1}{[(1-e^{-t})^2-z^2]\sqrt{(1+e^{-t})^2-z^2}} \\
&\quad\times\left[\big(e^{-t}-1-iM\big(e^{-2t}-1-z^2\big)\big)\right. \\
&\quad\times F\left(\frac{1}{2}+iM,\frac{1}{2}+iM;1;\frac{(1-e^{-t})^2-z^2}{(1+e^{-t})^2-z^2}\right) \\
&\quad+\big(1-e^{-2t}+z^2\big)\left(\frac{1}{2}-iM\right) \\
&\quad\left.\times F\left(-\frac{1}{2}+iM,\frac{1}{2}+iM;1;\frac{(1-e^{-t})^2-z^2}{(1+e^{-t})^2-z^2}\right)\right]
\end{aligned}\tag{9}$$

and $K_1(z,t):=E(z,t;0,0)$, that is,

$$\begin{aligned}
K_1(z,t) &= \big(4e^{-t}\big)^{iM}\big(\big(1+e^{-t}\big)^2-z^2\big)^{-1/2-iM} \\
&\qquad\times F\left(\frac{1}{2}+iM,\frac{1}{2}+iM;1;\frac{(1-e^{-t})^2-z^2}{(1+e^{-t})^2-z^2}\right), \\
& 0\le z\le 1-e^{-t},
\end{aligned}\tag{10}$$

respectively. The main properties of $K_0(z,t)$ and $K_1(z,t)$ are listed and proved in Sect. 3 in [32].

We consider the equation with $n\ge 2$. The solution $u=u(x,t)$ to the Cauchy problem

$$u_{tt}-e^{-2t}\Delta u+M^2u=f,\quad u(x,0)=0, u_t(x,0)=0, \tag{11}$$

with $f\in C^\infty(R^{n+1})$ and with vanishing initial data is given by the next expression

$$u(x,t)=2\int_0^t db\int_0^{e^{-b}-e^{-t}} dr\, v(x,r;b)E(r,t;0,b),$$

where the function $v(x,t;b)$ is a solution to the Cauchy problem for the wave equation

$$v_{tt}-\triangle v=0,\quad v(x,0;b)=f(x,b), v_t(x,0;b)=0. \tag{12}$$

Thus, for the solution Φ of the equation

$$\Phi_{tt}+n\Phi_t-e^{-2t}\triangle\Phi+m^2\Phi=f, \tag{13}$$

due to the relation $u = e^{(n/2)t}\Phi$, we obtain

$$\Phi(x,t) = 2e^{-(n/2)t}\int_0^t db \int_0^{e^{-b}-e^{-t}} dr e^{(n/2)b} v(x,r;b)E(r,t;0,b), \quad (14)$$

where the function $v(x,t;b)$ is defined by (12).

The solution $u = u(x,t)$ to the Cauchy problem

$$u_{tt} - e^{-2t}\,\triangle\, u + M^2 u = 0, \quad u(x,0) = \varphi_0(x), u_t(x,0) = \varphi_1(x), \quad (15)$$

with $\varphi_0, \varphi_1 \in C_0^\infty(R^n)$, $n \geq 2$, can be represented as follows:

$$u(x,t) = e^{(t/2)} v_{\varphi_0}\big(x,\phi(t)\big) + 2\int_0^1 v_{\varphi_0}\big(x,\phi(t)s\big)K_0\big(\phi(t)s,t\big)\phi(t)ds$$
$$+ 2\int_0^1 v_{\varphi_1}\big(x,\phi(t)s\big)K_1\big(\phi(t)s,t\big)\phi(t)ds, \quad x \in R^n, t > 0,$$

where $\phi(t) := 1 - e^{-t}$. Here, for $\varphi \in C_0^\infty(R^n)$ and for $x \in R^n$, the function $v_\varphi(x,\phi(t)s)$ coincides with the value $v(x,\phi(t)s)$ of the solution $v(x,t)$ of the Cauchy problem

$$v_{tt} - \triangle v = 0, \quad v(x,0) = \varphi(x), v_t(x,0) = 0. \quad (16)$$

Thus, for the solution Φ of the Cauchy problem

$$\Phi_{tt} + n\Phi_t - e^{-2t}\,\triangle\,\Phi + m^2\Phi = 0, \quad \Phi(x,0) = \varphi_0(x), \Phi_t(x,0) = \varphi_1(x), \quad (17)$$

due to the relation $u = e^{(n/2)t}\Phi$, we obtain

$$\Phi(x,t) = e^{-((n-1)/2)t} v_{\varphi_0}\big(x,\phi(t)\big)$$
$$+ e^{-(n/2)t}\int_0^1 v_{\varphi_0}\big(x,\phi(t)s\big)\big(2K_0\big(\phi(t)s,t\big) + nK_1\big(\phi(t)s,t\big)\big)\phi(t)ds$$
$$+ 2e^{-\frac{n}{2}t}\int_0^1 v_{\varphi_1}\big(x,\phi(t)s\big)K_1\big(\phi(t)s,t\big)\phi(t)ds, \quad x \in R^n, t > 0.$$

18.2.1 L^p-L^q Estimates for Equation with Source

The Cauchy problem

$$v_{tt} - \triangle v = 0, \quad v(x,0) = \psi_0(x), v_t(x,0) = \psi_1(x),$$

with $\psi_0, \psi_1 \in C_0^\infty(R^n)$ for the linear wave equation has a unique solution that can be written as follows:

$$u_0(x,t) = V_1(t,D_x)\psi_0(x) + V_2(t,D_x)\psi_1(x).$$

The operators $V_1(t, D_x)$ and $V_2(t, D_x)$ are chosen in accordance with

$$V_1(0, D_x) = I \quad \text{(identity operator)}, \qquad \partial_t V_1(0, D_x) = 0,$$
$$V_2(0, D_x) = 0, \qquad \partial_t V_2(0, D_x) = I \quad \text{(identity operator)}.$$

The microlocal description of those operators by means of Fourier integral operators is known (see, e.g. [25]). Let $W^{l,p}(R^n)$, $B^{l,p}(R^n)$, and $\dot{B}^{l,p}(R^n)$ be Sobolev, Besov, and homogeneous Besov spaces, respectively. In what follows the space $\mathcal{M}^{s,q}$ can be each of the next spaces $L^q(R^n)$, $W^{s,q}(R^n)$, $\dot{W}^{s,q}(R^n)$, $B^{s,q}(R^n)$, or $\dot{B}^{s,q}(R^n)$. The following decay estimates for the linear operators $V_1(t, D_x)$ and $V_2(t, D_x)$ can be found, e.g., in [6, 22].

For all $\psi \in C_0^\infty(R^n)$, $n > 1$, one has the estimates

$$\left\|(-\Delta)^{-s} V_1(t, D_x)\psi(x)\right\|_{\mathcal{M}^{l,q}} \le C t^{2s-n(1/p-1/q)} \|\psi\|_{\mathcal{M}^{l,p}}, \quad t \in (0, \infty),$$

under the conditions $s \ge 0$, $1 < p \le 2$, $\frac{1}{p} + \frac{1}{q} = 1$, and $\frac{1}{2}(n+1)(\frac{1}{p} - \frac{1}{q}) \le 2s \le n(\frac{1}{p} - \frac{1}{q})$. Then, for all $g \in C_0^\infty(R^n)$ one has the estimate

$$\left\|(-\Delta)^{-s} V_2(t, D_x)g(x)\right\|_{\mathcal{M}^{l,q}} \le C t^{1+2s-n(1/p-1/q)} \|g\|_{\mathcal{M}^{l,p}}, \quad t \in (0, \infty),$$

under the conditions $s \ge 0$, $1 < p \le 2$, $\frac{1}{p} + \frac{1}{q} = 1$, $k \ge 0$, and $\frac{1}{2}(n+1)(\frac{1}{p} - \frac{1}{q}) - 1 \le 2s \le n(\frac{1}{p} - \frac{1}{q})$. Moreover, a standard interpolation argument implies that these estimates hold for s and r in some range (see for details [23]). Scaling arguments show that the time dependent factors are exact. The Duhamel's principle gives corresponding estimates for equations with source terms.

Let $u = u(x,t)$ be a solution of the Cauchy problem (11). Then according to Corollary 9.3 in [32][1] for $n \ge 2$ one has the following estimate

$$\begin{aligned}
&\left\|(-\triangle)^{-s} u(x,t)\right\|_{L^q(R^n)} \\
&\quad \le C_M \int_0^t \left\|f(x,b)\right\|_{L^p(R^n)} e^b \left(e^{-b} - e^{-t}\right)^{1+2s-n(1/p-1/q)} (1+t-b)^{1-\operatorname{sgn} M}\, db,
\end{aligned}$$

provided that $1 < p \le 2$, $\frac{1}{p} + \frac{1}{q} = 1$, $\frac{1}{2}(n+1)(\frac{1}{p} - \frac{1}{q}) \le 2s \le n(\frac{1}{p} - \frac{1}{q}) < 2s + 1$. Thus, for the solution Φ (14) of (13), due to the relation $u = e^{(n/2)t}\Phi$, we obtain

$$\begin{aligned}
&\left\|(-\triangle)^{-s} \Phi(x,t)\right\|_{L^q(R^n)} \\
&\quad \le C_M e^{-(n/2)t} \int_0^t e^{(n/2)b} \left\|f(x,b)\right\|_{L^p(R^n)} e^b \left(e^{-b} - e^{-t}\right)^{1+2s-n(1/p-1/q)} \\
&\qquad \times (1+t-b)^{1-\operatorname{sgn} M}\, db.
\end{aligned}$$

[1] There is a misprint in [32].

For $M > 0$ we obtain

$$\begin{aligned}&\left\|(-\triangle)^{-s}\Phi(x,t)\right\|_{L^q(R^n)}\\&\le C_M e^{-(n/2)t}\int_0^t e^{(n/2)b}\left\|f(x,b)\right\|_{L^p(R^n)} e^b\left(e^{-b}-e^{-t}\right)^{1+2s-n(1/p-1/q)}db.\end{aligned}$$

For $M = 0$ we obtain

$$\begin{aligned}&\left\|(-\triangle)^{-s}\Phi(x,t)\right\|_{L^q(R^n)}\\&\le C_M e^{-(n/2)t}\int_0^t e^{(n/2)b}\left\|f(x,b)\right\|_{L^p(R^n)} e^b\left(e^{-b}-e^{-t}\right)^{1+2s-n(1/p-1/q)}\\&\quad\times(1+t-b)db.\end{aligned}$$

In particular, for $s = 0$ and $p = q = 2$, we have

$$\left\|\Phi(x,t)\right\|_{L^2(R^n)} \le C_M e^{-(n/2)t}\int_0^t e^{(n/2)b}\left\|f(x,b)\right\|_{L^2(R^n)}(1+t-b)^{1-\operatorname{sgn} M}db.$$

Here the rates of exponential factors are independent of the curved mass $\mathcal{M}$ and, consequently, of the mass m.

18.2.2 L^p-L^q Estimates for Equations Without Source

According to Theorem 10.1 in [32] the solution $u = u(x,t)$ of the Cauchy problem (15) satisfies the following L^p-L^q estimate

$$\begin{aligned}\left\|(-\triangle)^{-s}u(x,t)\right\|_{L^q(R^n)} &\le C_M(1+t)^{1-\operatorname{sgn} M}\left(1-e^{-t}\right)^{2s-n(1/p-1/q)}\\&\quad\times\left\{e^{t/2}\|\varphi_0\|_{L^p(R^n)}+\left(1-e^{-t}\right)\|\varphi_1\|_{L^p(R^n)}\right\}\end{aligned}$$

for all $t \in (0,\infty)$, provided that $1 < p \le 2$, $\frac{1}{p}+\frac{1}{q}=1$, $\frac{1}{2}(n+1)(\frac{1}{p}-\frac{1}{q}) \le 2s \le n(\frac{1}{p}-\frac{1}{q}) < 2s+1$.

In particular, for large t we obtain the following *no decay* estimate

$$\left\|(-\triangle)^{-s}u(x,t)\right\|_{L^q(R^n)} \le C_M(1+t)^{1-\operatorname{sgn} M}\left\{e^{t/2}\|\varphi_0\|_{L^p(R^n)}+\|\varphi_1\|_{L^p(R^n)}\right\}.$$

Thus, for the solution Φ of the Cauchy problem (17), due to the relation $u = e^{(n/2)t}\Phi$, we obtain the decay estimate

$$\begin{aligned}\left\|(-\triangle)^{-s}\Phi(x,t)\right\|_{L^q(R^n)} &\le C_M e^{-(n/2)t}(1+t)^{1-\operatorname{sgn} M}\left(1-e^{-t}\right)^{2s-n(1/p-1/q)}\\&\quad\times\left\{e^{t/2}\|\varphi_0\|_{L^p(R^n)}+\left(1-e^{-t}\right)\|\varphi_1\|_{L^p(R^n)}\right\}\end{aligned}\tag{18}$$

for all $t > 0$, while

$$\begin{aligned}&\left\|(-\triangle)^{-s}\Phi(x,t)\right\|_{L^q(R^n)}\\&\quad\le C_M e^{-(n/2)t}(1+t)^{1-\operatorname{sgn}M}\left\{e^{t/2}\|\varphi_0\|_{L^p(R^n)}+\|\varphi_1\|_{L^p(R^n)}\right\}\end{aligned}$$

for large t. Here the rate of decay is essentially independent of the curved mass $\mathcal{M}$ and, consequently, of the mass m.

18.3 The Scalar Equation. Imaginary Curved Mass

In this section we consider the linear part of the scalar equation

$$u_{tt}-e^{-2t}\triangle u-M^2u=-e^{(n/2)t}V'\big(e^{-(n/2)t}u\big), \tag{19}$$

with $M\ge 0$. Equation (19) covers two important cases. The first one is the Higgs boson equation, which has $V'(\phi)=\lambda\phi^3$ and $M^2=\mu m^2+n^2/4$ with $\lambda>0$ and $\mu>0$, while $n=3$. The second case is the case of the small physical mass, that is $0\le m\le \frac{n}{2}$. For the last case $M^2=\frac{n^2}{4}-m^2$.

We introduce new functions $E(x,t;x_0,t_0;M)$, $K_0(z,t;M)$, and $K_1(z,t;M)$, which can be obtained by continuation in complex domain the ones introduced in [32] and which have been used in Sect. 18.2. First we define the function

$$\begin{aligned}E(x,t;x_0,t_0;M)&=4^{-M}e^{M(t_0+t)}\big((e^{-t}+e^{-t_0})^2-(x-x_0)^2\big)^{-1/2+M}\\&\quad\times F\left(\frac{1}{2}-M,\frac{1}{2}-M;1;\frac{(e^{-t_0}-e^{-t})^2-(x-x_0)^2}{(e^{-t_0}+e^{-t})^2-(x-x_0)^2}\right).\end{aligned} \tag{20}$$

Hence, it is related to the function $E(x,t;x_0,t_0)$ of (8) as follows:

$$E(x,t;x_0,t_0)=E(x,t;x_0,t_0;-iM).$$

Next we define also new kernels $K_0(z,t;M)$ and $K_1(z,t;M)$ by

$$\begin{aligned}K_0(z,t;M)&:=-\left[\frac{\partial}{\partial b}E(z,t;0,b;M)\right]_{b=0}\\&=4^{-M}e^{tM}\big((1+e^{-t})^2-z^2\big)^M\frac{1}{[(1-e^{-t})^2-z^2]\sqrt{(1+e^{-t})^2-z^2}}\\&\quad\times\Bigg[\big(e^{-t}-1+M\big(e^{-2t}-1-z^2\big)\big)\\&\quad\times F\left(\frac{1}{2}-M,\frac{1}{2}-M;1;\frac{(1-e^{-t})^2-z^2}{(1+e^{-t})^2-z^2}\right)\\&\quad+\big(1-e^{-2t}+z^2\big)\\&\quad\times\left(\frac{1}{2}+M\right)F\left(-\frac{1}{2}-M,\frac{1}{2}-M;1;\frac{(1-e^{-t})^2-z^2}{(1+e^{-t})^2-z^2}\right)\Bigg],\end{aligned} \tag{21}$$

and $K_1(z,t;M) := E(z,t;0,0;M)$, that is,

$$K_1(z,t;M)$$
$$= 4^{-M} e^{Mt} \big((1+e^{-t})^2 - z^2\big)^{-1/2+M} F\left(\frac{1}{2}-M, \frac{1}{2}-M; 1; \frac{(1-e^{-t})^2 - z^2}{(1+e^{-t})^2 - z^2}\right),$$
$$0 \le z \le 1 - e^{-t}, \tag{22}$$

respectively. These kernels will be used in the representation of the solutions of the Cauchy problem.

The solution $u = u(x,t)$ to the Cauchy problem

$$u_{tt} - e^{-2t}\Delta u - M^2 u = f, \quad u(x,0) = 0, u_t(x,0) = 0, \tag{23}$$

with $f \in C^\infty(R^{n+1})$ and with vanishing initial data is given in [33] by the next expression

$$u(x,t) = 2\int_0^t db \int_0^{e^{-b}-e^{-t}} dr\, v(x,r;b) E(r,t;0,b;M),$$

where the function $v(x,t;b)$ is a solution to the Cauchy problem for the wave equation (12).

The solution $u = u(x,t)$ to the Cauchy problem

$$u_{tt} - e^{-2t} \triangle u - M^2 u = 0, \quad u(x,0) = \varphi_0(x), u_t(x,0) = \varphi_1(x),$$

with $\varphi_0, \varphi_1 \in C_0^\infty(R^n)$, $n \ge 2$, can be represented (see [33]) as follows:

$$u(x,t) = e^{t/2} v_{\varphi_0}\big(x, \phi(t)\big) + 2\int_0^1 v_{\varphi_0}\big(x, \phi(t)s\big) K_0\big(\phi(t)s, t; M\big)\phi(t)\,ds$$
$$+ 2\int_0^1 v_{\varphi_1}\big(x, \phi(t)s\big) K_1\big(\phi(t)s, t; M\big)\phi(t)\,ds, \quad x \in R^n, t > 0,$$

where $\phi(t) := 1 - e^{-t}$. Here, for $\varphi \in C_0^\infty(R^n)$ and for $x \in R^n$, the function $v_\varphi(x, \phi(t)s)$ coincides with the value $v(x, \phi(t)s)$ of the solution $v(x,t)$ of the Cauchy problem (16).

Thus, for the solution Φ of the Cauchy problem

$$\Phi_{tt} + n\Phi_t - e^{-2t} \triangle \Phi + m^2 \Phi = f, \quad \Phi(x,0) = 0, \Phi_t(x,0) = 0,$$

due to the relation $u = e^{(n/2)t}\Phi$, we obtain with $f \in C^\infty(R^{n+1})$ and with vanishing initial data the next expression

$$\Phi(x,t) = 2e^{-(n/2)t} \int_0^t db \int_0^{e^{-b}-e^{-t}} dr\, e^{(n/2)b} v(x,r;b) E(r,t;0,b;M), \tag{24}$$

where the function $v(x,t;b)$ is a solution to the Cauchy problem for the wave equation (12).

Thus, for the solution Φ of the Cauchy problem (17), due to the relation $u = e^{(n/2)t}\Phi$, we obtain

$$\begin{aligned}
&\Phi(x,t)\\
&= e^{-((n-1)/2)t} v_{\varphi_0}\big(x,\phi(t)\big)\\
&\quad + e^{-(n/2)t}\int_0^1 v_{\varphi_0}\big(x,\phi(t)s\big)\big(2K_0\big(\phi(t)s,t;M\big)+nK_1\big(\phi(t)s,t;M\big)\big)\phi(t)ds\\
&\quad + 2e^{-(n/2)t}\int_0^1 v_{\varphi_1}\big(x,\phi(t)s\big)K_1\big(\phi(t)s,t;M\big)\phi(t)ds, \quad x\in R^n, t>0. \quad (25)
\end{aligned}$$

Here for $\varphi \in C_0^\infty(R^n)$ and for $x \in R^n$, the function $v_\varphi(x,\phi(t)s)$ coincides with the value $v(x,\phi(t)s)$ of the solution $v(x,t)$ of the Cauchy problem (16).

In fact, the representation formulas of this section have been used in [34] to establish some qualitative properties of the solutions of the Higgs boson equation.

18.3.1 The Critical Case of $m^2 = (n^2-1)/4$

Here we want to distinguish certain mass m. More precisely, looking for the simplest possible function $E(x,t;x_0,t_0;M)$, and consequently, $K_0(z,t;M)$ and $K_1(z,t;M)$, we set $M = 1/2$, that is, $m^2 = (n^2-1)/4$, which simplifies the hypergeometric functions, as well as, the kernels $K_0(z,t;M)$ and $K_1(z,t;M)$. Indeed, in that case we have

$$E\left(x,t;x_0,t_0;\frac{1}{2}\right) = \frac{1}{2}e^{(1/2)(t_0+t)}, \qquad E\left(z,t;0,b;\frac{1}{2}\right) = \frac{1}{2}e^{(1/2)(b+t)},$$

while

$$K_0\left(z,t;\frac{1}{2}\right) = -\frac{1}{4}e^{(1/2)t}, \qquad K_1\left(z,t;\frac{1}{2}\right) = \frac{1}{2}e^{(1/2)t}.$$

For the solution (14) of (13) with the source term it follows

$$\Phi(x,t) = e^{-((n-1)/2)t}\int_0^t e^{((n+1)/2)b}db\int_0^{e^{-b}-e^{-t}} v(x,r;b)dr,$$

where the function $v(x,r;b)$ is defined by (12). In fact, if we denote by $V_f(x,t;b)$ the solution of the problem

$$V_{tt} - \triangle V = 0, \quad V(x,0) = 0,\ V_t(x,0) = f(x,b),$$

then

$$v(x,t;b) = \frac{\partial}{\partial t} V_f(x,t;b).$$

Hence,

$$\Phi(x,t) = e^{-((n-1)/2)t} \int_0^t e^{((n+1)/2)b} V_f\big(x, e^{-b} - e^{-t}; b\big) db.$$

Further, for the solution Φ (25) of the equation without source term we have

$$\Phi(x,t) = e^{-((n-1)/2)t} v_{\varphi_0}\big(x, 1-e^{-t}\big) + \frac{n-1}{2} e^{-((n-1)/2)t} \int_0^{1-e^{-t}} v_{\varphi_0}(x,s) ds$$
$$+ e^{-((n-1)/2)t} \int_0^{1-e^{-t}} v_{\varphi_1}(x,s) ds, \quad x \in R^n, t > 0,$$

where the functions v_{φ_0} and v_{φ_1} are defined by (16). Now, if we denote by V_φ the solution of the problem

$$V_{tt} - \triangle V = 0, \quad V(x,0) = 0, V_t(x,0) = \varphi(x),$$

then

$$v_\varphi(x,t) = \frac{\partial}{\partial t} V_\varphi(x,t),$$

and

$$\Phi(x,t) = e^{-((n-1)/2)t} v_{\varphi_0}\big(x, 1-e^{-t}\big) + \frac{n-1}{2} e^{-((n-1)/2)t} V_{\varphi_0}\big(x, 1-e^{-t}\big)$$
$$+ e^{-((n-1)/2)t} V_{\varphi_1}\big(x, 1-e^{-t}\big), \quad x \in R^n, t > 0,$$

or, equivalently,

$$\Phi(x,t) = e^{-((n-1)/2)t} \left(\frac{\partial V_{\varphi_0}}{\partial t}\right)\big(x, 1-e^{-t}\big) + \frac{n-1}{2} e^{-((n-1)/2)t} V_{\varphi_0}\big(x, 1-e^{-t}\big)$$
$$+ e^{-((n-1)/2)t} V_{\varphi_1}\big(x, 1-e^{-t}\big), \quad x \in R^n, t > 0.$$

The last formula can be also verified by direct substitution.

Thus, in particular, we have proven the following theorem.

Theorem 3 *The value $m = \sqrt{n^2-1}/2$ is the only value of the physical mass m, such that the solutions of the equation*

$$\Phi_{tt} + n\Phi_t - e^{-2t} \triangle \Phi + m^2 \Phi = 0, \tag{26}$$

obey the strong Huygens' Principle, whenever the wave equation in the Minkowski spacetime does, that is $n \geq 3$ is an odd number.

Corollary 1 *The solutions of the equation*

$$\Phi_{tt} + n\Phi_t - e^{-2t}\Delta\Phi + \mathbf{M}\Phi = 0,$$

obey the strong Huygens' Principle, if and only if $n \geq 3$ is an odd number and the mass matrix $\mathbf{M}$ is the diagonal matrix $\frac{n^2-1}{4}\mathbf{I}$.

18.3.2 The Critical Case. Asymptotic Expansions of Solutions at Infinite Time

For $\varphi_1 \in C_0^\infty(R^n)$ the formula for the solution $u(x,t)$ of the Cauchy problem

$$u_{tt} - \Delta u = 0, \quad u(x,0) = 0, u_t(x,0) = \varphi(x),$$

is well-known. It can be written for odd and even n separately as follows. We have

$$V_\varphi(x,t) := \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{(n-3)/2} \frac{t^{n-2}}{\omega_{n-1}c_0^{(n)}} \int_{S^{n-1}} \varphi(x+ty)dS_y,$$

where $c_0^{(n)} = 1\cdot 3\cdot \ldots \cdot (n-2)$ if $n \geq 3$ is odd. For $x \in R^n$, and even n, we have

$$V_\varphi(x,t) := \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{(n-2)/2} \frac{2r^{n-1}}{\omega_{n-1}c_0^{(n)}} \int_{B_1^n(0)} \frac{1}{\sqrt{1-|y|^2}}\varphi(x+ty)dV_y,$$

where $c_0^{(n)} = 1\cdot 3\cdot \ldots \cdot (n-1)$. Similarly, for $\varphi_0 \in C_0^\infty(R^n)$ and for $x \in R^n$, if n is odd, the formula for the solution $u(x,t)$ of the Cauchy problem

$$u_{tt} - \Delta u = 0, \quad u(x,0) = \varphi_0(x), u_t(x,0) = 0,$$

implies

$$v_\varphi(x,t) := \frac{\partial}{\partial t}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{(n-3)/2} \frac{t^{n-2}}{\omega_{n-1}c_0^{(n)}} \int_{S^{n-1}} \varphi(x+ty)dS_y.$$

In the case of $x \in R^n$ and even n we have

$$v_\varphi(x,t) := \frac{\partial}{\partial t}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{(n-2)/2} \frac{2t^{n-1}}{\omega_{n-1}c_0^{(n)}} \int_{B_1^n(0)} \frac{1}{\sqrt{1-|y|^2}}\varphi(x+ty)dV_y.$$

The constant ω_{n-1} is the area of the unit sphere $S^{n-1} \subset R^n$. In particular,

$$v_\varphi(x,1) = \begin{cases} [\frac{\partial}{\partial t}(\frac{1}{t}\frac{\partial}{\partial t})^{(n-3)/2}\frac{t^{n-2}}{\omega_{n-1}c_0^{(n)}}\int_{S^{n-1}}\varphi(x+ty)dS_y]_{t=1}, & \text{if } n \text{ is odd,} \\ [\frac{\partial}{\partial t}(\frac{1}{t}\frac{\partial}{\partial t})^{(n-2)/2}\frac{2t^{n-1}}{\omega_{n-1}c_0^{(n)}}\int_{B_1^n(0)}\frac{1}{\sqrt{1-|y|^2}}\varphi(x+ty)dV_y]_{t=1}, & \\ & \text{if } n \text{ is even,} \end{cases}$$

and

$$V_\varphi(x,1)=\begin{cases}[(\frac{1}{t}\frac{\partial}{\partial t})^{(n-3)/2}\frac{t^{n-2}}{\omega_{n-1}c_0^{(n)}}\int_{S^{n-1}}\varphi(x+ty)dS_y]_{t=1}, & \text{if } n \text{ is odd,}\\ [(\frac{1}{t}\frac{\partial}{\partial t})^{(n-2)/2}\frac{2t^{n-1}}{\omega_{n-1}c_0^{(n)}}\int_{B_1^n(0)}\frac{1}{\sqrt{1-|y|^2}}\varphi(x+ty)dV_y]_{t=1}, & \\ & \text{if } n \text{ is even.}\end{cases}$$

Denote

$$v_\varphi(x):=v_\varphi(x,1), \qquad V_\varphi(x):=V_\varphi(x,1).$$

In order to write the complete asymptotic expansion of the solutions, we define the functions

$$V_\varphi^{(k)}(x)=\frac{(-1)^k}{k!}\left[\left(\frac{\partial}{\partial t}\right)^k V_\varphi(x,t)\right]_{t=1}\in C_0^\infty\big(R^n\big), \quad k=1,2,\dots.$$

Then, for every integer $N\geq 1$ we have

$$V_\varphi\big(x,1-e^{-t}\big)=\sum_{k=0}^{N-1}V_\varphi^{(k)}(x)e^{-kt}+R_{V_\varphi,N}(x,t), \quad R_{V_\varphi,N}\in C^\infty,$$

where with the constant $C(\varphi)$ the remainder $R_{V_\varphi,N}$ satisfies the inequality

$$\big|R_{V_\varphi,N}(x,t)\big|\leq C(\varphi)e^{-Nt} \quad \text{for all } x\in R^n \text{ and all } t\in[0,\infty).$$

Moreover, the support of the remainder $R_{V_\varphi,N}$ is in the cylinder

$$\operatorname{supp} R_{V_\varphi,N}\subseteq\big\{x\in R^n; \operatorname{dist}(x,\operatorname{supp}\varphi)\leq 1\big\}\times[0,\infty).$$

Analogously, we define

$$v_\varphi^{(k)}(x)=\frac{(-1)^k}{k!}\left[\left(\frac{\partial}{\partial t}\right)^k v_\varphi(x,t)\right]_{t=1}\in C_0^\infty\big(R^n\big), \quad k=1,2,\dots,$$

and the remainder $R_{v_\varphi,N}$

$$v_\varphi\big(x,1-e^{-t}\big)=\sum_{k=0}^{N-1}v_\varphi^{(k)}(x)e^{-kt}+R_{v_\varphi,N}(x,t), \quad R_{v_\varphi,N}\in C^\infty,$$

such that

$$\big|R_{v_\varphi,N}(x,t)\big|\leq C(\varphi)e^{-Nt} \quad \text{for all } x\in R^n \text{ and all } t\in[0,\infty).$$

Further, we introduce the polynomial in z with the smooth in $x\in R^n$ coefficients as follows:

$$\Phi_{asypt}^{(N)}(x,z) = z^{(n-1)/2}\left(\sum_{k=0}^{N-1} v_{\varphi_0}^{(k)}(x)z^k + \frac{n-1}{2}\sum_{k=0}^{N-1} V_{\varphi_0}^{(k)}(x)z^k\right) + z^{(n-1)/2}\sum_{k=0}^{N-1} V_{\varphi_1}^{(k)}(x)z^k,$$

where $x \in R^n$, $z \in C$. This allows us to prove the following asymptotic expansion

$$\Phi(x,t) = \Phi_{asypt}^{(N)}\big(x, e^{-t}\big) + O\big(e^{-Nt-((n-1)/2)t}\big)$$

for large t uniformly for $x \in R^n$. Thus, we have proven the next theorem.

Theorem 4 *Suppose that $m = \sqrt{n^2-1}/2$. Then, for every integer positive N the solution of (26) with the initial values $\varphi_0, \varphi_1 \in C_0^\infty(R^n)$ has the following asymptotic expansion at infinity*:

$$\Phi(x,t) \sim \Phi_{asypt}^{(N)}\big(x, e^{-t}\big),$$

in the sense that for every integer positive N the following estimate is valid:

$$\big\|\Phi(x,t) - \Phi_{asypt}^{(N)}\big(x, e^{-t}\big)\big\|_{L^\infty(R^n)} \le C(\varphi_0, \varphi_1)e^{-Nt-((n-1)/2)t} \quad \textit{for large } t.$$

Remark 1 If we take into account the relation $v_\varphi(x,t) = \frac{\partial}{\partial t} V_\varphi(x,t)$, then

$$v_\varphi^{(k)}(x) = -(k+1)V_\varphi^{(k+1)}(x)$$

and, consequently, the function $\Phi_{asypt}^{(N)}(x,z)$ can be rewritten as follows:

$$\begin{aligned}
&\Phi_{asypt}^{(N)}(x,z) \\
&\quad = z^{(n-1)/2}\left(\sum_{k=0}^{N-1} v_{\varphi_0}^{(k)}(x)z^k + \frac{n-1}{2}\sum_{k=0}^{N-1} V_{\varphi_0}^{(k)}(x)z^k\right) + z^{(n-1)/2}\sum_{k=0}^{N-1} V_{\varphi_1}^{(k)}(x)z^k \\
&\quad = z^{(n-1)/2}\left(\sum_{k=0}^{N-1}(-1)(k+1)V_{\varphi_0}^{(k+1)}(x)z^k + \frac{n-1}{2}\sum_{k=0}^{N-1} V_{\varphi_0}^{(k)}(x)z^k\right) \\
&\qquad + z^{(n-1)/2}\sum_{k=0}^{N-1} V_{\varphi_1}^{(k)}(x)z^k \\
&\quad = z^{(n-1)/2}\sum_{k=0}^{N-1}\left(\frac{n-1}{2}V_{\varphi_0}^{(k)}(x) - (k+1)V_{\varphi_0}^{(k+1)}(x) + V_{\varphi_1}^{(k)}(x)\right)z^k.
\end{aligned}$$

18.3.3 The Critical Case. L^p-L^q-Estimates

Lemma 1 *Suppose that* $m = \sqrt{n^2-1}/2$. *If* $\varphi_0 = \varphi_1 = 0$ *and* $\frac{1}{2}(n+1)(\frac{1}{p}-\frac{1}{q})-1 \le 2s \le n(\frac{1}{p}-\frac{1}{q})$, *then for the solution* $\Phi = \Phi(x,t)$ *of* (13) *the following estimate holds*

$$\left\|(-\triangle)^{-s}\Phi(x,t)\right\|_{L^q(R^n)} \le Ce^{-((n-1)/2)t}\int_0^t e^{((n+1)/2)b}\left(e^{-b}-e^{-t}\right)^{1+2s-n(1/p-1/q)} \times \left\|f(x,b)\right\|_{L^p(R^n)}db, \quad t>0.$$

For the solution $\Phi = \Phi(x,t)$ *of the Cauchy problem* (17): *if* $f \equiv 0$, $\varphi_0 = 0$, *and* $\frac{1}{2}(n+1)(\frac{1}{p}-\frac{1}{q})-1 \le 2s \le n(\frac{1}{p}-\frac{1}{q})$, *then*

$$\left\|(-\triangle)^{-s}\Phi(x,t)\right\|_{L^q(R^n)} \le Ce^{-((n-1)/2)t}\left(1-e^{-t}\right)^{1+2s-n(1/p-1/q)}\|\varphi_1\|_{L^p(R^n)}, \quad t>0,$$

while if $f \equiv 0$, $\varphi_1 = 0$, *and* $\frac{1}{2}(n+1)(\frac{1}{p}-\frac{1}{q}) \le 2s \le n(\frac{1}{p}-\frac{1}{q})$, *then*

$$\left\|(-\triangle)^{-s}\Phi(x,t)\right\|_{L^q(R^n)} \le Ce^{-((n-1)/2)t}\left(1-e^{-t}\right)^{2s-n(1/p-1/q)}\|\varphi_0\|_{L^p(R^n)}, \quad t>0.$$

Proof The following L^p-L^q decay estimates are well-known (see, e.g., [6, 22]). If $n \ge 2$, then for the solution $v = v(x,t)$ of the Cauchy problem for the wave equation in the Minkowski spacetime

$$v_{tt} - \triangle v = 0, \quad v(x,0) = 0,\ v_t(x,0) = \varphi(x),$$

with $\varphi(x) \in C_0^\infty(R^n)$ one has the following so-called L^p-L^q decay estimate

$$\left\|(-\triangle)^{-s}v(x,t)\right\|_{L^q(R^n)} \le Ct^{1+2s-n(1/p-1/q)}\|\varphi\|_{L^p(R^n)} \quad \text{for all } t>0,$$

provided that $s \ge 0$, $1 < p \le 2$, $\frac{1}{p}+\frac{1}{q} = 1$, and $\frac{1}{2}(n+1)(\frac{1}{p}-\frac{1}{q})-1 \le 2s \le n(\frac{1}{p}-\frac{1}{q})$.

Hence, if $\varphi_0 = \varphi_1 = 0$ and $\frac{1}{2}(n+1)(\frac{1}{p}-\frac{1}{q})-1 \le 2s \le n(\frac{1}{p}-\frac{1}{q})$, then

$$\begin{aligned}\left\|(-\triangle)^{-s}\Phi(x,t)\right\|_{L^q(R^n)} &\le \left\|(-\triangle)^{-s}e^{-((n-1)/2)t}\int_0^t e^{((n+1)/2)b}V_f\left(x,e^{-b}-e^{-t};b\right)db\right\|_{L^q(R^n)} \\ &\le e^{-((n-1)/2)t}\int_0^t e^{((n+1)/2)b}\left\|(-\triangle)^{-s}V_f\left(x,e^{-b}-e^{-t};b\right)\right\|_{L^q(R^n)}db\end{aligned}$$

$$\le Ce^{-((n-1)/2)t} \int_0^t e^{((n+1)/2)b} \big(e^{-b} - e^{-t}\big)^{1+2s-n(1/p-1/q)} \times \big\| f(x,b) \big\|_{L^p(R^n)} db, \quad t > 0.$$

In particular,

$$\begin{aligned}
&\big\| (-\triangle)^{-s} \Phi(x,t) \big\|_{L^q(R^n)} \\
&\le C\Big(\sup_{0\le b\le t} \big\| f(x,b) \big\|_{L^p(R^n)} \Big) e^{-((n-1)/2)t} \\
&\quad \times \int_0^t e^{((n+1)/2)b} \big(e^{-b} - e^{-t}\big)^{1+2s-n(1/p-1/q)} db \\
&\le C\Big(\sup_{0\le b\le t} \big\| f(x,b) \big\|_{L^p(R^n)} \Big) e^{-((n-1)/2)t} e^{-t(1+2s-n(1/p-1/q))} \\
&\quad \times \int_0^t e^{((n+1)/2)b} \big(e^{t-b} - 1\big)^{1+2s-n(1/p-1/q)} db, \quad t > 0.
\end{aligned}$$

For the case $s = 0$ we obtain

$$\begin{aligned}
\big\| \Phi(x,t) \big\|_{L^q(R^n)} &\le C\Big(\sup_{0\le b\le t} \big\| f(x,b) \big\|_{L^p(R^n)} \Big) e^{-((n-1)/2)t} e^{-t(1-n(1/p-1/q))} \\
&\quad \times \int_0^t e^{((n+1)/2)b} \big(e^{t-b} - 1\big)^{1-n(1/p-1/q)} db, \quad t > 0,
\end{aligned}$$

as well as

$$\begin{aligned}
&\big\| \Phi(x,t) \big\|_{L^q(R^n)} \\
&\le C\Big(\sup_{0\le b\le t} \big\| f(x,b) \big\|_{L^p(R^n)} \Big) e^{-((n-1)/2)t} \int_0^t e^{((n-1)/2)b+n(1/p-1/q)b} db,
\end{aligned}$$

for $t > 0$. In the case $p = q = 2$ and $n \ge 2$ we obtain for the L^2-norm the estimate

$$\begin{aligned}
\big\| \Phi(x,t) \big\|_{L^2(R^n)} &\le C\Big(\sup_{0\le b\le t} \big\| f(x,b) \big\|_{L^2(R^n)} \Big) e^{-((n-1)/2)t} \int_0^t e^{((n-1)/2)b} db \\
&\le C\Big(\sup_{0\le b\le t} \big\| f(x,b) \big\|_{L^2(R^n)} \Big), \quad t > 0.
\end{aligned}$$

Further, if $f \equiv 0$, $\varphi_0 = 0$, and $\frac{1}{2}(n+1)(\frac{1}{p} - \frac{1}{q}) - 1 \le 2s \le n(\frac{1}{p} - \frac{1}{q})$, then

$$\begin{aligned}
&\big\| (-\triangle)^{-s} \Phi(x,t) \big\|_{L^q(R^n)} \\
&\le Ce^{-((n-1)/2)t} \big(1 - e^{-t}\big)^{1+2s-n(1/p-1/q)} \|\varphi_1\|_{L^p(R^n)}, \quad t > 0,
\end{aligned}$$

while if $f \equiv 0$, $\varphi_1 = 0$, and $\frac{1}{2}(n+1)(\frac{1}{p}-\frac{1}{q}) \le 2s \le n(\frac{1}{p}-\frac{1}{q})$, then

$$\begin{aligned}&\left\|(-\triangle)^{-s}\Phi(x,t)\right\|_{L^q(R^n)}\\&\quad \le Ce^{-((n-1)/2)t}\left(1-e^{-t}\right)^{2s-n(1/p-1/q)}\|\varphi_0\|_{L^p(R^n)}, \quad t>0.\end{aligned}$$

The lemma is proven. □

We can check that these estimates are sharp if, for example, $f \equiv 0$. Indeed, let us consider the solution Φ of the problem (17), which is generated by smooth initial functions $\varphi_0(x)$ and $\varphi_1(x)$ with compact supports, $\varphi_0, \varphi_1 \in C_0^\infty(R^n)$. Then $\Phi \in C^\infty([0,\infty)\times R^n)$ and the support of Φ is contained in some cylinder $B_R \times [0,\infty)$, where $B_R \subset R^n$ is a ball of radius R centered at the origin, which depends on the supports of φ_0 and φ_1. We may say that the support of the solution is *permanently bounded*. That is a consequence of the finite propagation speed property of the hyperbolic equation and due to the existence of a horizon for the de Sitter spacetime. Next, we integrate the equation of (17) with respect to x and obtain the following initial value problem for the second-order ordinary differential equation,

$$I_{tt} + nI_t + m^2 I = 0, \quad I(0) = C_0,\ I_t(x,0) = C_1,$$

with the solution $I(t) := \int_{R^n}(-\triangle)^{-s}\Phi(x,t)dx$, where $C_0 = \int_{R^n}(-\triangle)^{-s}\varphi_0(x)dx$ and $C_1 = \int_{R^n}(-\triangle)^{-s}\varphi_1(x)dx$, $s \in R$. For the case of small mass m, $m \in (0, n/2)$, the last problem implies

$$I(t) = \frac{C_0\lambda_2 - C_1}{\lambda_2-\lambda_1}e^{-(n/2-M)t} + \frac{C_1 - C_0\lambda_1}{\lambda_2-\lambda_1}e^{-(n/2+M)t},$$

where $M = \sqrt{n^2/4-m^2}$, $\lambda_1 := -\frac{n}{2}+M < 0$, and $\lambda_2 := -\frac{n}{2}-M<0$ since $0 < M < \frac{n}{2}$. Hence, we have

$$\left|\int_{R^n}(-\triangle)^{-s}\Phi(x,t)dx\right| \le C(\varphi_0,\varphi_1)e^{-(n/2-M)t} \quad \text{for all } t>0.$$

The last estimate is optimal in the sense that there are $\varphi_0, \varphi_1 \in C_0^\infty(R^n)$ and $C(\varphi_0,\varphi_1) > 0$ such that

$$\left|\int_{R^n}(-\triangle)^{-s}\Phi(x,t)dx\right| \ge C(\varphi_0,\varphi_1)e^{-(n/2-M)t}.$$

Moreover, since the support of Φ is permanently bounded, we have

$$\left|\int_{R^n}(-\triangle)^{-s}\Phi(x,t)dx\right| \le C(\varphi_0,\varphi_1,q)\left\|(-\triangle)^{-s}\Phi(x,t)\right\|_{L^q(R^n)}, \quad q\in[1,\infty].$$

In the case of the dimensional mass, $n^2/4 = m^2$, the curved mass vanishes, $M = 0$, and we have

$$I(t) = C_0 e^{-(n/2)t} + \left(C_0 \frac{n}{2} + C_1 \right) t e^{-(n/2)t}.$$

Hence, there exist $\varphi_0, \varphi_1 \in C_0^\infty(R^n)$ and $C(\varphi_0, \varphi_1) > 0$ such that

$$\begin{aligned} C(\varphi_0, \varphi_1, q) \left\| (-\triangle)^{-s} \Phi(x,t) \right\|_{L^q(R^n)} &\geq \left| \int_{R^n} (-\triangle)^{-s} \Phi(x,t) dx \right| \\ &\geq C(\varphi_0, \varphi_1) t e^{-(n/2)t}, \end{aligned}$$

$q \in [1, \infty]$. In the case of the imaginary mass the corresponding Cauchy problem is

$$I_{tt} + n I_t - m^2 I = 0, \quad I(0) = C_0, \, I_t(x,0) = C_1,$$

where $M = \sqrt{\frac{n^2}{4} + m^2} > 0$ and $\lambda_1 := -\frac{n}{2} + M > 0$ while $\lambda_2 := -\frac{n}{2} - M < 0$. Consequently, there are $\varphi_0, \varphi_1 \in C_0^\infty(R^n)$ and $C(\varphi_0, \varphi_1) > 0$ such that

$$\left| \int_{R^n} (-\triangle)^{-s} \Phi(x,t) dx \right| \geq C(\varphi_0, \varphi_1) \exp\left(\sqrt{\frac{n^2}{4} + m^2} - \frac{n}{2} \right) t,$$

as well as,

$$\left\| (-\triangle)^{-s} \Phi(x,t) \right\|_{L^q(R^n)} \geq \delta \exp\left(\sqrt{\frac{n^2}{4} + m^2} - \frac{n}{2} \right) t,$$

and for all s the norms of the solution are increasing in time. Thus, we have the following statement.

Lemma 2 ([35]) *If $q \in [1, \infty]$, then for both equations, with the real small mass ($M = \sqrt{\frac{n^2}{4} - m^2} \geq 0, 0 \leq m \leq \frac{n}{2}$) and with the imaginary mass ($M = \sqrt{\frac{n^2}{4} + m^2} > \frac{n}{2}, m > 0$), there exist $\varphi_0, \varphi_1 \in C_0^\infty(R^n)$ and $\delta > 0$ such that*

$$\left\| (-\triangle)^{-s} \Phi(x,t) \right\|_{L^q(R^n)} \geq \delta t^{1 - \operatorname{sgn} M} e^{-(n/2 - M)t} \quad \textit{for all } t \in (0, \infty).$$

To complete the list of the L^p-L^q estimates we quote below results from [35] which are applicable to the scalar equation with noncritical mass. The lemma shows that the estimate of the next theorem is optimal. The bound $M = 1/2$ plays an important role in the next theorems.

Theorem 5 ([35]) *The solution $\Phi = \Phi(x,t)$ of the Cauchy problem*

$$\Phi_{tt} + n\Phi_t - e^{-2t} \triangle \Phi \pm m^2 \Phi = 0, \quad \Phi(x,0) = \varphi_0(x), \, \Phi_t(x,0) = \varphi_1(x),$$

with either $M = \sqrt{\frac{n^2}{4} - m^2}$ *and* $m < \sqrt{n^2-1}/2$ *for the case of "plus", or* $M = \sqrt{\frac{n^2}{4} + m^2}$ *for the case of "minus", satisfies the following* L^p*-*L^q *estimate*

$$\begin{aligned}\left\|(-\triangle)^{-s}\Phi(x,t)\right\|_{L^q(R^n)} &\le C_{M,n,p,q,s}\left(1-e^{-t}\right)^{2s-n(1/p-1/q)}e^{(M-n/2)t}\\ &\quad\times\left\{\|\varphi_0\|_{L^p(R^n)} + \left(1-e^{-t}\right)\|\varphi_1\|_{L^p(R^n)}\right\}\end{aligned}$$

for all $t\in(0,\infty)$, *provided that* $1<p\le 2$, $\frac{1}{p}+\frac{1}{q}=1$, $\frac{1}{2}(n+1)(\frac{1}{p}-\frac{1}{q})\le 2s\le n(\frac{1}{p}-\frac{1}{q})<2s+1$.

Theorem 6 ([35]) *Let* $\Phi=\Phi(x,t)$ *be the solution of the Cauchy problem*

$$\Phi_{tt}+n\Phi_t - e^{-2t}\triangle\Phi \pm m^2\Phi = f, \quad \Phi(x,0)=0, \Phi_t(x,0)=0, \tag{27}$$

with either $M = \sqrt{\frac{n^2}{4} - m^2}$ *and* $m < \sqrt{n^2-1}/2$ *for the case of "plus", or* $M = \sqrt{\frac{n^2}{4} + m^2}$ *for the case of "minus". Then* $\Phi=\Phi(x,t)$ *satisfies the following* L^p*-*L^q *estimate*:

$$\begin{aligned}&\left\|(-\triangle)^{-s}\Phi(x,t)\right\|_{L^q(R^n)}\\ &\quad\le C_M e^{-Mt}e^{-(n/2)t}e^{-t[2s-n(1/p-1/q)]}\\ &\qquad\times\int_0^t e^{(n/2)b}e^{Mb}\left(e^{t-b}-1\right)^{1+2s-n(1/p-1/q)}\left(e^{t-b}+1\right)^{2M-1}\\ &\qquad\times\left\|f(x,b)\right\|_{L^p(R^n)}db,\end{aligned}$$

for all $t>0$, *provided that* $1<p\le 2$, $\frac{1}{p}+\frac{1}{q}=1$, $\frac{1}{2}(n+1)(\frac{1}{p}-\frac{1}{q})\le 2s\le n(\frac{1}{p}-\frac{1}{q})<2s+1$.

Corollary 2 ([35]) *Let* $\Phi=\Phi(x,t)$ *be the solution of the Cauchy problem considered in Theorem 6. Then for* $n\ge 2$ *and* $M>1/2$ *one has the following estimate*

$$\begin{aligned}&\left\|(-\triangle)^{-s}\Phi(x,t)\right\|_{L^q(R^n)}\\ &\quad\le C_M e^{-(n/2-M)t}\int_0^t e^{(n/2-M)b}e^{-b(2s-n(1/p-1/q))}\left\|f(x,b)\right\|_{L^p(R^n)}db,\end{aligned}$$

provided that $1<p\le 2$, $\frac{1}{p}+\frac{1}{q}=1$, $\frac{1}{2}(n+1)(\frac{1}{p}-\frac{1}{q})\le 2s\le n(\frac{1}{p}-\frac{1}{q})<2s+1$.

18.4 Global Existence. Small Data Solutions

The Cauchy problem (23) for the scalar equation was studied in [33]. For the case of a nonlinearity $F(\Phi)=c|\Phi|^{\alpha+1}$, $c\ne 0$, Theorem 1.1 in [33], implies nonexistence of a global solution even for arbitrary small initial functions $\varphi_0(x)$ and $\varphi_1(x)$ under some conditions on n, α, and M. By means of the evident transformation one can apply the conclusion of Theorem 1.1 in [33] to the equation with imaginary physical mass (see (28) below) and derive the following blow up result.

Theorem 7 ([35]) *Suppose that* $F(\Phi) = c|\Phi|^{\alpha+1}$, $c \neq 0$, *and* $\alpha > 0$. *Then, for every* $\alpha > 0$, N, *and* ε, *there exist* $\varphi_0, \varphi_1 \in C_0^\infty(R^n)$ *such that*

$$\|\varphi_0\|_{C^N(R^n)} + \|\varphi_1\|_{C^N(R^n)} < \varepsilon$$

but a global in time solution $\Phi \in C^2([0,\infty); L^q(R^n))$ *of the equation*

$$\Phi_{tt} + n\Phi_t - e^{-2t}\Delta\Phi - m^2\Phi = c|\Phi|^{\alpha+1}, \tag{28}$$

with permanently bounded support does not exist for all $q \in [2,\infty)$. *More precisely, there is* $T > 0$ *such that*

$$\lim_{t\nearrow T} \int_{R^n} \Phi(x,t)dx = \infty.$$

This theorem shows that instability of the trivial solution occurs in a very strong sense, that is, an arbitrarily small perturbation of the initial data can make the perturbed solution blowing up in finite time.

If we allow large initial data, then, according to Theorem 1.2 in [33], the concentration of the mass, due to the non-dispersion property of the de Sitter spacetime, leads to the nonexistence of the global solution, which cannot be recovered even by adding an exponentially decaying factor in the nonlinear term. More precisely, the next theorem states that the solution blows up in finite time.

Theorem 8 ([35]) *Suppose that* $F(\Phi) = ce^{\gamma t}|\Phi|^{\alpha+1}$, $c \neq 0$, $\alpha > 0$, *and* $\gamma \in R$. *Then, for every* $\alpha > 0$ *and* n *there exist* $\varphi_0, \varphi_1 \in C_0^\infty(R^n)$ *such that a global in time solution* $\Phi \in C^2([0,\infty); L^q(R^n))$ *of* (28) *with permanently bounded support does not exist for all* $q \in [2,\infty)$. *More precisely, there is* $T > 0$ *such that*

$$\lim_{t\nearrow T} \int_{R^n} \Phi(x,t)dx = \infty.$$

Thus, for every $\alpha > 0$ the large energy classical solution of the Cauchy for (28) blows up.

It is evident that, if the solution is real-valued and either α is large or odd, or the nonlinear term is $\Phi^{\alpha+1}$ with an integer nonnegative α, then the support of the solution with such initial data is permanently bounded.

In this section we are going to study the global existence of solutions for the system of semilinear Klein-Gordon equations. The first step toward such result is to establish the L_p-L_q-estimates for the equation with source term. For the scalar equation this estimate is proved in [35]. Below we quote it. In fact, the results of the previous sections are valid also in more general spaces of functions. In what follows, the space $\mathcal{M}^{s,q}$ can be each of the following spaces $L^q(R^n)$, Sobolev spaces $W^{s,q}(R^n)$, $\dot{W}^{s,q}(R^n)$, or Besov spaces $B^{s,q}(R^n)$, $\dot{B}^{s,q}(R^n)$.

Lemma 3 ([35]) *Let $\Phi = \Phi(x,t)$ be a solution of the Cauchy problem (27) with either $M = \sqrt{\frac{n^2}{4} - m^2}$ and $m < \sqrt{n^2-1}/2$ for the case of "plus", or $M = \sqrt{\frac{n^2}{4} + m^2}$ for the case of "minus". Then for $n \geq 2$ one has the following estimate*

$$\left\|(-\triangle)^{l-s}\Phi(x,t)\right\|_{L^q(R^n)} \leq C_M e^{-(n/2-M)t}\int_0^t e^{(n/2-M)b}e^{-b(2s-n(1/p-1/q))} \times \left\|(-\triangle)^l f(x,b)\right\|_{L^p(R^n)} db,$$

for all $t > 0$, provided that $1 < p \leq 2$, $\frac{1}{p} + \frac{1}{q} = 1$, $\frac{1}{2}(n+1)(\frac{1}{p} - \frac{1}{q}) \leq 2s \leq n(\frac{1}{p} - \frac{1}{q}) < 2s+1$. Moreover,

$$\left\|(-\triangle)^{-s}\Phi(x,t)\right\|_{\mathcal{M}^{l,q}} \leq C_M e^{-(n/2-M)t}\int_0^t e^{(n/2-M)b}e^{-b(2s-n(1/p-1/q))}\left\|f(x,b)\right\|_{\mathcal{M}^{l,p}} db.$$

In particular,

$$\left\|\Phi(x,t)\right\|_{H_{(l)}(R^n)} \leq C_M e^{-(n/2-M)t}\int_0^t e^{(n/2-M)b}\left\|f(x,b)\right\|_{H_{(l)}(R^n)} db.$$

For the equation with "plus" and large mass, $m \geq n/2$, and with the curved mass $M = \sqrt{m^2 - n^2/4}$, one has the following estimate

$$\left\|(-\triangle)^{l-s}\Phi(x,t)\right\|_{L^q(R^n)} \leq C_M e^{-(n/2)t}\int_0^t e^{(n/2)b}e^{b}\left(e^{-b} - e^{-t}\right)^{1+2s-n(1/p-1/q)} \times (1+t-b)^{1-\operatorname{sgn} M}\left\|(-\triangle)^l f(x,b)\right\|_{L^p(R^n)} db.$$

Moreover,

$$\left\|(-\triangle)^{-s}\Phi(x,t)\right\|_{\mathcal{M}^{l,q}} \leq C_M e^{-(n/2)t}\int_0^t e^{(n/2)b}e^{b}\left(e^{-b} - e^{-t}\right)^{1+2s-n(1/p-1/q)} \times (1+t-b)^{1-\operatorname{sgn} M}\left\|f(x,b)\right\|_{\mathcal{M}^{l,p}} db.$$

In particular,

$$\left\|\Phi(x,t)\right\|_{H_{(l)}(R^n)} \leq C_M e^{-(n/2)t}\int_0^t e^{(n/2)b}(1+t-b)^{1-\operatorname{sgn} M}\left\|f(x,b)\right\|_{H_{(l)}(R^n)} db.$$

Here the rate of exponential factors is independent of the curved mass $\mathcal{M}$ and, consequently, of the mass m.

Although we want to prove a global existence for two different cases, for the system with the semi-critical mass matrix and for the system of equations with the large mass matrix, the consideration in the next subsections can be done in the single framework.

18.4.1 System of Real Scalar Fields in de Sitter Spacetime

In this subsection we reduce the Cauchy problem to the integral equation. The main tool for such reduction is the fundamental solution (the Green's function) for the interacting fields, which can be described by the system of Klein-Gordon equations containing interaction via mass matrix and the semilinear term. The model obeys the following system

$$\Phi_{tt} + nH\Phi_t - e^{-2Ht} \triangle \Phi + \mathbf{M}\Phi = F(\Phi). \tag{29}$$

Here F is a vector-valued function of the vector-valued function Φ. We assume that the matrix $\mathbf{M}$ is diagonalizable by a real-valued matrix $\mathbf{O}$, and it has eigenvalues $m_1^2, \dots, m_l^2$, $i = 1, 2, \dots, l$.

By the similarity transformation $\mathbf{O}$ the mass matrix $\mathbf{M}$ can be diagonalized, therefore we use a change of unknown function as follows:

$$\Psi = \mathbf{O}\Phi, \quad \Phi = \mathbf{O}^{-1}\Psi,$$

and arrive at

$$\Psi_{tt} + nH\Psi_t - e^{-2Ht} \triangle \Psi + \widetilde{\mathbf{M}}\Psi = \widetilde{F}(\Psi),$$

where

$$\widetilde{\mathbf{M}} := \mathbf{O}M\mathbf{O}^{-1} = \begin{pmatrix} m_1^2 & 0 & 0 & \dots & 0 \\ 0 & m_2^2 & 0 & \dots & 0 \\ & & \ddots & & \\ 0 & 0 & \ddots & \dots & 0 \\ 0 & 0 & 0 & \dots & m_l^2 \end{pmatrix}, \qquad \widetilde{F}(\Psi) := \mathbf{O}F\big(\mathbf{O}^{-1}\Psi\big).$$

Let us consider the linear diagonal system

$$\Psi_{tt} + nH\Psi_t - e^{-2Ht} \triangle \Psi + \widetilde{\mathbf{M}}\Psi = \widetilde{f}.$$

Here $\widetilde{f}$ is a vector-valued function with the components f_i, $i = 1, \dots, l$. Then, the solution of the Cauchy problem for the last system with the initial conditions

$$\Psi(x,0) = 0, \qquad \Psi_t(x,0) = 0,$$

is

$$\Psi(x,t) = 2e^{-(n/2)t} \int_0^t db \int_0^{e^{-b}-e^{-t}} dr e^{(n/2)b} \widetilde{E}(r,t;0,b)\widetilde{v}(x,r;b),$$

where the components v_i, $i=1,\dots,l$, of the vector-valued function $\widetilde{v}(x,t;b)$ are solutions to the Cauchy problem for the wave equation

$$v_{tt} - \triangle v = 0, \quad v(x,0;b) = f_i(x,b), v_t(x,0;b) = 0, i = 1,\dots,l. \tag{30}$$

The kernel $\widetilde{E}(r,t;0,b)$ is a diagonal matrix with the elements $E_i(r,t;0,b)$, $i = 1,\dots,l$, which are defined either by (8) with corresponding mass terms m_i, $i = 1,\dots,l$, or by (20), in accordance with the value of mass $m_i^2 \ge n^2/4$ or $m_i^2 < n^2/4$, respectively.

Then, the solution Ψ of the Cauchy problem for the equation

$$\Psi_{tt} + nH\Psi_t - e^{-2Ht} \triangle \Psi + \widetilde{\mathbf{M}}\Psi = 0$$

with the initial conditions

$$\Psi(x,0) = \widetilde{\psi}_0(x), \qquad \Psi_t(x,0) = \widetilde{\psi}_1(x),$$

with the vector-valued functions $\widetilde{\psi}_0, \widetilde{\psi}_1 \in C_0^\infty(R^n)$, $n \ge 2$, can be represented as follows:

$$\begin{aligned}\Psi(x,t) = {} & e^{-((n-1)/2)t}\widetilde{v}_{\widetilde{\psi}_0}\big(x,\phi(t)\big)\\ & + e^{-(n/2)t}\int_0^1 \big(2\widetilde{K}_0\big(\phi(t)s,t\big) + n\widetilde{K}_1\big(\phi(t)s,t\big)\big)\widetilde{v}_{\widetilde{\psi}_0}\big(x,\phi(t)s\big)\phi(t)ds\\ & + 2e^{-(n/2)t}\int_0^1 \widetilde{K}_1\big(\phi(t)s,t\big)\widetilde{v}_{\widetilde{\Psi}_1}\big(x,\phi(t)s\big)\phi(t)ds, \quad x \in R^n, t > 0,\end{aligned}$$

where $\phi(t) := 1 - e^{-t}$ and the kernels $\widetilde{K}_0$, $\widetilde{K}_1$, are the diagonal matrices with the elements $\widetilde{K}_{0i}(z,t)$, $i = 1,\dots,l$, and $\widetilde{K}_{1i}(z,t)$, which are defined either by (9) and (10) with the corresponding mass terms m_i, $i = 1,\dots,l$, or by the diagonal matrices with the elements $\widetilde{K}_{0i}(z,t;M)$, $i = 1,\dots,l$, and $\widetilde{K}_{1i}(z,t;M)$, which are defined by (21) and (22), in accordance with the value of mass $m_i^2 \ge n^2/4$ or $m_i^2 < n^2/4$, respectively.

Here, for the vector-valued function $\widetilde{\psi} \in C_0^\infty(R^n)$ and for $x \in R^n$, the vector-valued function $\widetilde{v}_{\widetilde{\psi}}(x,\phi(t)s)$ coincides with the value $\widetilde{v}(x,\phi(t)s)$ of the solution $\widetilde{v}(x,t)$ of the Cauchy problem

$$\widetilde{v}_{tt} - \triangle\widetilde{v} = 0, \quad \widetilde{v}(x,0) = \widetilde{\psi}(x), \widetilde{v}_t(x,0) = 0.$$

We study the Cauchy problem through the integral equation. To determine that integral equation we appeal to the operator

$$\widetilde{G} := \widetilde{\mathcal{K}} \circ \widetilde{\mathcal{W}E},$$

where the operator $\widetilde{\mathcal{W}E}$ is defined by (30), that is,

$$\widetilde{\mathcal{W}E}[f](x,t;b) = \widetilde{v}(x,t;b),$$

and the vector-valued function $\widetilde{v}(x,t;b)$ is a solution to the Cauchy problem for the wave equation, while $\widetilde{\mathcal{K}}$ is introduced either by (14),

$$\widetilde{\mathcal{K}}[v](x,t) := 2e^{-(n/2)t} \int_0^t db \int_0^{e^{-b}-e^{-t}} dr e^{(n/2)b} \widetilde{E}(r,t;0,b) \widetilde{v}(x,r;b), \quad (31)$$

for the large mass matrix, or by (24),

$$\widetilde{\mathcal{K}}[v](x,t) := 2e^{-(n/2)t} \int_0^t db \int_0^{e^{-b}-e^{-t}} dr e^{(n/2)b} \widetilde{E}(r,t;0,b;M) \widetilde{v}(x,r;b),$$

for the small mass matrix. Hence,

$$\widetilde{G}[f](x,t) = 2e^{-(n/2)t} \int_0^t db \int_0^{e^{-b}-e^{-t}} dr e^{(n/2)b} \widetilde{E}(r,t;0,b;M) \widetilde{\mathcal{W}E}[f](x,r;b).$$

Thus, the Cauchy problem (5), (6) leads to the following integral equation

$$\Psi(x,t) = \Psi_0(x,t) + \widetilde{G}\big[\widetilde{F}(\Psi)\big](x,t). \quad (32)$$

Every solution $\Phi = \Phi(x,t)$ to (5) generates the function $\Psi = \Psi(x,t)$, which solves the last integral equation with some function $\Psi_0(x,t)$, that, in fact, is generated by the solution of the Cauchy problem (17).

18.4.2 Solvability of the Integral Equation Associated with Klein-Gordon Equation

Let us consider the system of the integral equations (32), where $\Psi_0 = \Psi_0(x,t)$ is a given vector-valued function. We are going to apply Banach's fixed-point theorem. In order to estimate the nonlinear term we use the Lipschitz Condition $(\mathcal{L})$. Evidently, Condition $(\mathcal{L})$ imposes some restrictions on n, α, s. Now we consider the integral equation (32), where the vector-valued function $\Psi_0 \in C([0,\infty); L^q(R^n))$ is given. We note here that any classical solution to (5) solves also the integral equation (32) with some vector-valued function $\Psi_0(x,t)$, which is a classical solution to the Cauchy problem for the linear system (17).

The solvability of the integral equation (32) depends on the operator $\widetilde{G}$. For scalar equations with the scalar operator G which is generated by the linear part of (29), the global solvability of the scalar integral equation (32) was studied in [33]. For the case of nonlinearity $F(\Phi) = c|\Phi|^{\alpha+1}$, $c \neq 0$, the results of [33] imply the nonexistence of the global solution even for arbitrary small functions $\Phi_0(x,0)$ under some conditions on n, α, and M.

We start with the case of Sobolev space $H_{(s)}(R^n)$ with $s > n/2$, which is an algebra. In the next theorem the operator $\widetilde{\mathcal{K}}$ is generated by the linear part of (5).

Theorem 9 *Assume that $F(\Psi)$ is Lipschitz continuous in the space $H_{(s)}(R^n)$, $s > n/2$, and also that $\alpha > 0$.*

(i) *Let the spectrum of the mass matrix* $\mathbf{M}$ *be* $\{m_1^2, \dots, m_l^2\} \subset (0, (n^2-1)/4]$, *and* $m = \min\{m_1, m_2, \dots, m_l\}$. *Then for every given function* $\Psi_0(x,t) \in X(R, s, \gamma_0)$ *such that*

$$\sup_{t\in[0,\infty)} e^{\gamma_0 t}\left\|\Psi_0(x,t)\right\|_{H_{(s)}(R^n)} < \varepsilon, \quad \textit{where } \gamma_0 \le \frac{n}{2} - \sqrt{\frac{n^2}{4} - m^2},$$

and for sufficiently small ε *the integral equation* (32) *has a unique solution* $\Psi(x,t) \in X(R, s, \gamma)$ *with* $0 < \gamma < \gamma_0/(\alpha+1)$. *For the solution one has*

$$\sup_{t\in[0,\infty)} e^{\gamma t}\left\|\Psi(x,t)\right\|_{H_{(s)}(R^n)} < 2\varepsilon.$$

(ii) *If the eigenvalues of the mass matrix are large,* $\frac{n}{2} \le m_i$, $i = 1, \dots, l$, *then for every given function* $\Psi_0(x,t) \in X(R, s, 0)$ *such that*

$$\sup_{t\in[0,\infty)} \left\|\Psi_0(x,t)\right\|_{H_{(s)}(R^n)} < \varepsilon,$$

and for sufficiently small ε *the integral equation* (32) *has a unique solution* $\Psi(x,t) \in X(R, s, 0)$. *For the solution one has*

$$\sup_{t\in[0,\infty)} \left\|\Psi(x,t)\right\|_{H_{(s)}(R^n)} < 2\varepsilon.$$

Proof Consider the mapping

$$S[\Psi](x,t) := \Psi_0(x,t) + \widetilde{G}\big[\widetilde{F}(\Psi)\big](x,t).$$

We are going to prove that S maps $X(R, s, \gamma)$ into itself and is a contraction provided that ε and R are sufficiently small.

The case of a semi-critical physical mass matrix. Let the spectrum of the mass matrix $\mathbf{M}$ be $\{m_1^2, \dots, m_l^2\} \subset (0, (n^2-1)/4]$, where $m_1^2 \le m_2^2 \le \dots \le m_l^2$. Theorem 6 and Lemma 3 imply for every component $S[\Psi]_i$, $i = 1, \dots, l$, of the vector $S[\Psi]$ the following estimate:

$$\begin{aligned}
&\left\|S[\Psi]_i(x,t)\right\|_{H_{(s)}(R^n)} \\
&\quad \le \left\|\Psi_{0i}(x,t)\right\|_{H_{(s)}(R^n)} + \left\|\widetilde{G}\big[\widetilde{F}(\Psi)\big]_i(x,t)\right\|_{H_{(s)}(R^n)} \\
&\quad \le \left\|\Psi_{0i}(x,t)\right\|_{H_{(s)}(R^n)} + Ce^{-(n/2-M_i)t}\int_0^t e^{(n/2-M_i)b}\left\|\widetilde{F}(\Psi)(x,b)\right\|_{H_{(s)}(R^n)} db,
\end{aligned}$$

where $M_i = \sqrt{\frac{n^2}{4} - m_i^2} \ge 1/2$. If we denote $M = M_1$, $\gamma = \frac{1}{\alpha+1}(\frac{n}{2} - M - \delta) > 0$, and $\delta > 0$, then the last inequality leads to the estimate for the vector $S[\Psi]$:

$$
\begin{aligned}
&\left\| S[\Psi](x,t) \right\|_{H_{(s)}(R^n)} \\
&\le \left\| \Psi_0(x,t) \right\|_{H_{(s)}(R^n)} + Ce^{-(n/2-M)t} \int_0^t e^{(n/2-M)b} \left\| \widetilde{F}(\Psi)(x,b) \right\|_{H_{(s)}(R^n)} db \\
&\le \left\| \Psi_0(x,t) \right\|_{H_{(s)}(R^n)} + Ce^{-\gamma(\alpha+1)t-\delta t} \int_0^t e^{\gamma(\alpha+1)b+\delta b} \left\| \widetilde{F}(\Psi)(x,b) \right\|_{H_{(s)}(R^n)} db.
\end{aligned}
$$

Taking into account Condition $(\mathcal{L})$ we arrive at

$$
\begin{aligned}
&\left\| S[\Psi](x,t) \right\|_{H_{(s)}(R^n)} \\
&\le \left\| \Psi_0(x,t) \right\|_{H_{(s)}(R^n)} + Ce^{-\gamma(\alpha+1)t-\delta t} \int_0^t e^{\gamma(\alpha+1)b+\delta b} \left\| \Psi(x,b) \right\|^{\alpha+1}_{H_{(s)}(R^n)} db \\
&\le \left\| \Psi_0(x,t) \right\|_{H_{(s)}(R^n)} + Ce^{-\gamma(\alpha+1)t-\delta t} \int_0^t e^{\delta b} \left(e^{\gamma b} \left\| \Psi(x,b) \right\|_{H_{(s)}(R^n)} \right)^{\alpha+1} db.
\end{aligned}
$$

Then

$$
\begin{aligned}
&e^{\gamma t} \left\| S[\Psi](x,t) \right\|_{H_{(s)}(R^n)} \\
&\quad \le e^{\gamma(\alpha+1)t} \left\| S[\Psi](x,t) \right\|_{H_{(s)}(R^n)} \\
&\quad \le e^{\gamma(\alpha+1)t} \left\| \Psi_0(x,t) \right\|_{H_{(s)}(R^n)} \\
&\qquad + C\Big(\sup_{\tau\in[0,\infty)} e^{\gamma\tau} \left\| \Phi(x,\tau) \right\|_{H_{(s)}(R^n)} \Big)^{\alpha+1} e^{-\delta t} \int_0^t e^{\delta b} db \\
&\quad \le e^{\gamma_0 t} \left\| \Psi_0(x,t) \right\|_{H_{(s)}(R^n)} + C\delta^{-1} \Big(\sup_{\tau\in[0,\infty)} e^{\gamma\tau} \left\| \Psi(x,\tau) \right\|_{H_{(s)}(R^n)} \Big)^{\alpha+1},
\end{aligned}
$$

and

$$
\begin{aligned}
&\sup_{t\in[0,\infty)} e^{\gamma t} \left\| S[\Psi](x,t) \right\|_{H_{(s)}(R^n)} \\
&\quad \le \sup_{t\in[0,\infty)} e^{\gamma_0 t} \left\| \Psi_0(x,t) \right\|_{H_{(s)}(R^n)} + C\delta^{-1} \Big(\sup_{t\in[0,\infty)} e^{\gamma t} \left\| \Psi(x,\tau) \right\|_{H_{(s)}(R^n)} \Big)^{\alpha+1}.
\end{aligned} \tag{33}
$$

In particular, since $\gamma_0 = \frac{n}{2} - M > 0$, then, with $\delta > 0$ such that $\gamma(\alpha+1) = \frac{n}{2} - M - \delta < \gamma_0$, we have

$$
\begin{aligned}
&\sup_{t\in[0,\infty)} e^{\gamma t} \left\| S[\Psi](x,t) \right\|_{H_{(s)}(R^n)} \\
&\quad \le \sup_{t\in[0,\infty)} e^{(n/2-M)t} \left\| \Psi_0(x,t) \right\|_{H_{(s)}(R^n)} + C\Big(\sup_{t\in[0,\infty)} e^{\gamma t} \left\| \Psi(x,t) \right\|_{H_{(s)}(R^n)} \Big)^{\alpha+1}.
\end{aligned}
$$

Thus, the last inequality proves that the operator S maps $X(R,s,\gamma)$ into itself if ε and R are sufficiently small, namely, if $\varepsilon + CR^{\alpha+1} < R$.

It remains to prove that S is a contraction mapping. As a matter of fact, we just need to apply the estimate (4) and get the contraction property from

$$e^{\gamma t}\big\| S[\Psi_1](x,t) - S[\Psi_2](x,t)\big\|_{H_{(s)}(R^n)} \le CR(t)^\alpha d(\Psi_1,\Psi_2),$$

where

$$R(t) := \max\Big\{ \sup_{0\le\tau\le t} e^{\gamma\tau}\big\|\Psi_1(x,\tau)\big\|_{H_{(s)}(R^n)},\ \sup_{0\le\tau\le t} e^{\gamma\tau}\big\|\Psi_2(x,\tau)\big\|_{H_{(s)}(R^n)}\Big\} \le R.$$

Indeed, we have

$$\begin{aligned}
&\big\| S[\Psi_1](x,t) - S[\Psi_2](x,t)\big\|_{H_{(s)}(R^n)}\\
&\quad = \big\|\widetilde{G}\big[\big(\widetilde{F}(\Phi) - \widetilde{F}(\Psi)\big)\big](x,t)\big\|_{H_{(s)}(R^n)}\\
&\quad \le Ce^{-(n/2-M)t}\int_0^t e^{(n/2-M)b}\big\|\big(\widetilde{F}(\Psi_1) - \widetilde{F}(\Psi_2)\big)(x,b))\big\|_{H_{(s)}(R^n)} db\\
&\quad \le Ce^{-\gamma(\alpha+1)t-\delta t}\int_0^t e^{\gamma(\alpha+1)b+\delta b}\big\|\big(\widetilde{F}(\Psi_1) - \widetilde{F}(\Psi_2)\big)(x,b))\big\|_{H_{(s)}(R^n)} db\\
&\quad \le Ce^{-\gamma(\alpha+1)t-\delta t}\int_0^t e^{\gamma(\alpha+1)b+\delta b}\big\|\Psi_1(x,b) - \Psi_2(x,b)\big\|_{H_{(s)}(R^n)}\\
&\qquad \times \big(\big\|\Psi_1(x,b)\big\|^\alpha_{H_{(s)}(R^n)} + \big\|\Psi_2(x,b)\big\|^\alpha_{H_{(s)}(R^n)}\big) db.
\end{aligned}$$

Thus, taking into account the last estimate and the definition of the metric $d(\Psi_1,\Psi_2)$, we obtain

$$\begin{aligned}
&e^{\gamma(\alpha+1)t}\big\| S[\Psi_1](x,t) - S[\Psi_2](x,t)\big\|_{H_{(s)}(R^n)}\\
&\quad \le Ce^{-\delta t}\int_0^t e^{\gamma(\alpha+1)b+\delta b}\big\|\Psi_1(x,b) - \Psi_2(x,b)\big\|_{H_{(s)}(R^n)}\\
&\qquad \times \big(\big\|\Psi_1(x,b)\big\|^\alpha_{H_{(s)}(R^n)} + \big\|\Psi_2(x,b)\big\|^\alpha_{H_{(s)}(R^n)}\big) db\\
&\quad \le Ce^{-\delta t}\int_0^t e^{\delta b}\Big(\max_{0\le\tau\le b} e^{\gamma\tau}\big\|\Psi_1(x,\tau) - \Psi_2(x,\tau)\big\|_{H_{(s)}(R^n)}\Big)\\
&\qquad \times \Big(\Big(\max_{0\le\tau\le b} e^{\gamma\tau}\big\|\Psi_1(x,\tau)\big\|_{H_{(s)}(R^n)}\Big)^\alpha + \Big(\max_{0\le\tau\le b} e^{\gamma\tau}\big\|\Psi_2(x,\tau)\big\|_{H_{(s)}(R^n)}\Big)^\alpha\Big) db\\
&\quad \le C_{M,\alpha} d(\Psi_1,\Psi_2) R(t)^\alpha e^{-\delta t}\int_0^t e^{\delta b} db\\
&\quad \le C_\alpha \delta^{-1} d(\Psi_1,\Psi_2) R(t)^\alpha.
\end{aligned}$$

Consequently,

$$e^{\gamma t}\left\|S[\Psi_1](x,t)-S[\Psi_2](x,t)\right\|_{H_{(s)}(R^n)} \le C_\alpha \delta^{-1} R(t)^\alpha d(\Psi_1,\Psi_2).$$

Then we choose ε and R such that $C_\alpha \delta^{-1} R^\alpha < 1$. Banach's fixed point theorem completes the proof for the case of small physical mass matrix.

The case of a large physical mass matrix. In this case $\min\{m_i; i=1,2.\dots,l\} \ge n/2$ and the operator $\widetilde{\mathcal{K}}$ is given by (31). We set $\gamma = 0$ in the definition of the metric of the space $X(R,s,\gamma)$. Then we have

$$\begin{aligned}
&\left\|S[\Psi](x,t)\right\|_{H_{(s)}(R^n)} \\
&\le \left\|\Psi_0(x,t)\right\|_{H_{(s)}(R^n)} + \left\|\widetilde{G}\big[\widetilde{F}(\Psi)\big](x,t)\right\|_{H_{(s)}(R^n)} \\
&\le \left\|\Psi_0(x,t)\right\|_{H_{(s)}(R^n)} \\
&\quad + C_M e^{-(n/2)t}\int_0^t e^{(n/2)b}(1+t-b)\left\|\widetilde{F}(\Psi)(x,b)\right\|_{H_{(s)}(R^n)} db \\
&\le \left\|\Psi_0(x,t)\right\|_{H_{(s)}(R^n)} + C_{M,\alpha} e^{-(n/2)t}\int_0^t e^{(n/2)b}(1+t-b)\left\|\Psi(x,b)\right\|^{\alpha+1}_{H_{(s)}(R^n)} db.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\left\|S[\Psi](x,t)\right\|_{H_{(s)}(R^n)} \\
&\le \left\|\Psi_0(x,t)\right\|_{H_{(s)}(R^n)} \\
&\quad + C_{M,\alpha}\Big(\sup_{\tau\in[0,\infty)}\left\|\Psi(x,\tau)\right\|_{H_{(s)}(R^n)}\Big)^{\alpha+1} e^{-(n/2)t}\int_0^t e^{(n/2)b}(1+t-b)db \\
&\le \left\|\Psi_0(x,t)\right\|_{H_{(s)}(R^n)} + C_{M,\alpha}\frac{4}{n}\Big(\sup_{\tau\in[0,\infty)}\left\|\Psi(x,\tau)\right\|_{H_{(s)}(R^n)}\Big)^{\alpha+1}.
\end{aligned}$$

Then we choose ε and R such that $\varepsilon + 4C_{M,\alpha}R^{\alpha+1}/n < R$.

In order to prove that S is a contraction mapping, we just need to apply estimate (33) and get the contraction property from

$$\left\|S[\Psi_1](x,t)-S[\Psi](x,t)\right\|_{H_{(s)}(R^n)} \le C R(t)^\alpha d(\Psi_1,\Psi),$$

where

$$R(t) := \max\Big\{\sup_{0\le\tau\le t}\left\|\Psi_1(x,\tau)\right\|_{H_{(s)}(R^n)}, \sup_{0\le\tau\le t}\left\|\Psi(x,\tau)\right\|_{H_{(s)}(R^n)}\Big\} \le R.$$

Indeed, we have

$$
\begin{aligned}
&\big\| S[\Psi_1](x,t) - S[\Psi](x,t) \big\|_{H_{(s)}(R^n)} \\
&\quad = \big\| G\big[\big(F(\Psi_1) - F(\Psi)\big)\big](x,t) \big\|_{H_{(s)}(R^n)} \\
&\quad \le C_M e^{-(n/2)t} \int_0^t e^{(n/2)b} (1+t-b) \big\| \big(F(\Psi_1) - F(\Psi)\big)(x,b)) \big\|_{H_{(s)}(R^n)} db \\
&\quad \le C_{M,\alpha} e^{-(n/2)t} \int_0^t e^{(n/2)b} (1+t-b) \big\| \Psi_1(x,b) - \Psi(x,b) \big\|_{H_{(s)}(R^n)} \\
&\qquad \times \big(\big\| \Psi_1(x,b) \big\|^{\alpha}_{H_{(s)}(R^n)} + \big\| \Psi(x,b) \big\|^{\alpha}_{H_{(s)}(R^n)} \big) db.
\end{aligned}
$$

Thus, taking into account the last estimate and the definition of the metric, we obtain

$$
\begin{aligned}
&\big\| S[\Psi_1](x,t) - S[\Psi_2](x,t) \big\|_{H_{(s)}(R^n)} \\
&\quad \le C_{M,\alpha} e^{-(n/2)t} \int_0^t e^{(n/2)b} (1+t-b) \big\| \Psi_1(x,b) - \Psi_2(x,b) \big\|_{H_{(s)}(R^n)} \\
&\qquad \times \big(\big\| \Psi_1(x,b) \big\|^{\alpha}_{H_{(s)}(R^n)} + \big\| \Psi_2(x,b) \big\|^{\alpha}_{H_{(s)}(R^n)} \big) db \\
&\quad \le C_{M,\alpha} e^{-(n/2)t} \int_0^t e^{(n/2)b} (1+t-b) \Big(\max_{0 \le \tau \le b} \big\| \Psi_1(x,\tau) - \Psi_2(x,\tau) \big\|_{H_{(s)}(R^n)} \Big) \\
&\qquad \times \Big(\Big(\max_{0 \le \tau \le b} \big\| \Psi_1(x,\tau) \big\|_{H_{(s)}(R^n)} \Big)^{\alpha} + \Big(\max_{0 \le \tau \le b} \big\| \Psi_2(x,\tau) \big\|_{H_{(s)}(R^n)} \Big)^{\alpha} \Big) db \\
&\quad \le C_{M,\alpha} d(\Psi_1,\Psi_2) R(t)^{\alpha} e^{-(n/2)t} \int_0^t e^{(n/2)b} (1+t-b) db \\
&\quad \le C_{M,\alpha} \frac{4}{n} d(\Psi_1,\Psi_2) R(t)^{\alpha},
\end{aligned}
$$

and, consequently,

$$
\big\| S[\Psi_1](x,t) - S[\Psi_2](x,t) \big\|_{H_{(s)}(R^n)} \le C_{M,\alpha} \frac{4}{n} \delta^{-1} R(t)^{\alpha} d(\Psi_1,\Psi_2).
$$

Then we choose ε and R such that $4C_{M,\alpha}\delta^{-1}R^{\alpha}/n < 1$. The application of Banach's fixed point theorem completes the proof of theorem. □

18.4.3 Proof of Theorems 1–2

The case of a semi-critical physical mass matrix. In this case the operator $\widetilde{\mathcal{K}}$ is given by (24). Then for the function $\Psi_0 = \Psi_0(x,t)$ which is generated by the solution of the Cauchy problem (17) and for $s > \frac{n}{2}$, $p = q = 2$, $n \ge 2$, according to Theorem 5 and Lemma 1 we have the estimate

$$
\big\| \Psi_0(x,t) \big\|_{H_{(s)}(R^n)} \le C_{M,n,p,q,s} e^{(M-n/2)t} \big\{ \|\varphi_0\|_{H_{(s)}(R^n)} + \|\varphi_1\|_{H_{(s)}(R^n)} \big\}.
$$

Hence, for every initial functions φ_0 and φ_1 the function Ψ_0 belongs to the space $X(R,s,\gamma)$, where the operator S is a contraction. The considerations from Sect. 18.4.2 complete the proof of the existence of the global solution.

The case of a large physical mass matrix. In this case the operator $\widetilde{\mathcal{K}}$ is given by (14). We set $\gamma = 0$ in the definition of the metric of the space $X(R,s,\gamma)$. Then for the function Ψ_0 which is generated by the solution of the Cauchy problem (17) and for $s > \frac{n}{2}$, $p = q = 2$, $n \geq 2$, we have the estimate (18),

$$\left\| \Psi_0(x,t) \right\|_{H_{(s)}(R^n)} \leq C_M e^{-(n/2)t}(1+t)\left\{ e^{t/2} \|\varphi_0\|_{H_{(s)}(R^n)} + \|\varphi_1\|_{H_{(s)}(R^n)} \right\}$$
$$\leq C_M \left\{ \|\varphi_0\|_{H_{(s)}(R^n)} + \|\varphi_1\|_{H_{(s)}(R^n)} \right\}.$$

Thus, $\Psi_0 \in X(R,s,0)$. According to Sect. 18.4.2, Banach's fixed point theorem implies the existence of the solution $\Psi \in X(R,s,0)$ of the integral equation (33) provided that R is sufficiently small. This completes the proof of the theorem. □

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